NEW (3+1)-DIMENSIONAL PAINLEVÉ INTEGRABLE EXTENSIONS OF
MIKHAILOV-NOVIKOV-WANG EQUATION: VARIETY OF MULTIPLE
SOLITON SOLUTIONS

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For the first time, the new (3+1)-dimensional Painlevé integrable extensions to the Mikhailov-Novikov-Wang equation (MNWE) are constructed. The new extended model is obtained by adding a set of linear terms to the standard (1+1)-dimensional MNWE. These additional terms do not kill the integrability of the new models. However, for checking the integrability of the new models, the Painlevé analysis is employed for this purpose. Using the simplified Hirota’s scheme, a variety of multiple soliton solutions is obtained by imposing appropriate parameter selections. The extended models to the MNWE provide new insights of the integrable equations.

Key words: Mikhailov-Novikov-Wang equation; Painlevé analysis; multiple-soliton solutions; kink waves.

1. INTRODUCTION

Integrable nonlinear equations play an important role in a variety of scientific fields. Solitary waves theory and integrable systems are the most famous features of nonlinear phenomena, and have attracted a vast number of intensive research work with promising findings in science and engineering fields [1-12]. Studies on the (1+1)-dimensional integrable equations, such as the Korteweg-de Vries (KdV) equation, the modified KdV equation, the Gardner equation, the Boussinesq equation (BE), and many others, have led to several remarkable achievements and contributed to explanations of propagation phenomena of waves in many physical settings. Many studies on multiple soliton solutions (MSSs), freak waves (FWs), breathers solutions, and different types of acoustic waves (AWs) in plasma physics, nonlinear lattices, and nonlinear optics and photonics have been published in the scientific literature; see Refs. [13-23].

The higher-dimensional integrable equations have gained a huge size of
interest, and this led to explaining several interesting nonlinear physical phenomena in nonlinear optics and photonics, plasma physics, fluid flows, chemical reactions, turbulent flows, and oceanography; see Refs. [24-53].

Regarding the (1+1)-dimensional systems, the following integrable Mikhailov-Novikov-Wang equation (MNWE) was proposed in Ref. [1]

$$\mathcal{R} ≡ u_{tt} - u_{xxxt} - 8u_x u_{xt} - 4u_{xx} u_t + 2u_x u_{xxxx} + 4u_{xxx} u_{xx} + 24u_x^2 u_x = 0,$$  \hspace{1cm} (1)

where $u ≡ u(x,t)$. Equation (1) includes two-linear terms and five-nonlinear terms. The nonlinear terms include spatial derivatives of first, second, third, and fourth order in addition to other derivatives.

Equation (1) is a completely integrable system, that provides MSSs, derived by using the perturbative symmetry approach [2], is a time second-order nonlinear equation similar to the BE. Also, it belongs to a symmetries hierarchy with an infinite number of higher symmetries [1-8], where constructive results were obtained.

In Ref. [3], we extended the MNWE (1) to the following new extended (3+1)-dimensional MNWE (eMNWE)

$$\mathcal{R} + u_{xxxx} + u_{xx} + u_{yy} + u_{zz} + u_{xy} + u_{xz} = 0,$$  \hspace{1cm} (2)

In the current work, we plan to construct new (3+1)-dimensional MNW-like equations, given as

$$\mathcal{R} + \alpha_1 u_x + \alpha_2 u_y + \alpha_3 u_z = 0,$$  \hspace{1cm} (3)

$$\mathcal{R} + \beta_1 u_{xt} + \beta_2 u_{yt} + \beta_3 u_{zt} = 0,$$  \hspace{1cm} (4)

$$\mathcal{R} + \alpha_1 u_x + \alpha_2 u_y + \alpha_3 u_z + \beta_1 u_{xt} + \beta_2 u_{yt} + \beta_3 u_{zt} = 0,$$  \hspace{1cm} (5)

which will be named MNWE1, MNWE2, and MNWE3, respectively. The extended MNWE1, MNWE2, and MNWE3 are developed by adding the sets of partial derivatives $\{u_x, u_y, u_z\}$, $\{u_{xt}, u_{yt}, u_{zt}\}$, and $\{u_x, u_y, u_z, u_{xt}, u_{yt}, u_{zt}\}$, respectively, to the standard MNWE (1).

We plan to run this work in two different streams. First, we will employ the Painlevé test to show that each extended models (3), (4), and (5) are completely Painlevé integrable, by retaining the same resonance points for all extended models. Regarding this fact, we try to find criteria that relate the coefficients $(k_i, r_i, s_i)$ of $(x, y, z)$, respectively, to get MSSs for all suggested new models. By assuming suitable parameter constraints, using the simplified Hirota’s method, then a variety of MSSs can be obtained.
We begin with the first extended MNWE1
\[ R + \alpha_1 u_x + \alpha_2 u_y + \alpha_3 u_z = 0, \] (6)

We will prove that Eq. (6) passes the Painlevé test (PT). Next, we will examine the necessary conditions of the coefficients of the spatial variables that will guarantee MSSs for this model.

2.1. PAINLEVÉ ANALYSIS TO EXTENDED MNWE1

To investigate the integrability of the extended MNWE1 (6), we employ the PT [1-10] to confirm it. Following [3], we assume that Eq. (6) has a solution as a Laurent expansion about a singular manifold \( \Phi \equiv \Phi(x, y, z, t) \) as
\[ u = \sum_{k=0}^{\infty} u_k \Phi^{k-\gamma}, \] (7)
where \( u_k \equiv u_k(x, y, z, t) \).

A characteristic equation (CE) with one branch for resonances at \( k = -1, 1, 6, \) and 8 can be determined by following the same analysis as in Refs. [1-10]. The obtained resonance points are compatible with the resonance points in Ref. [1] of the standard MNWE (1), and also identical to the resonance points of the extended MNWE developed in Ref. [3]. This demonstrates that resonances are preserved despite the addition of three linear terms. For \( k = -1 \), the arbitrariness of the singular manifold \( \Phi = 0 \) is recovered, and the compatibility conditions demonstrated the existence of free functions for \((u_1, u_6, u_8)\), whereas \((u_2, u_3, u_4, u_5)\) turn out to have explicit expressions. The MNW1 passes the Painlevé integrability test (PIT) in a like manner to our work in [3].

2.2. THE FIRST EXTENDED MNWE1 AND MSSS

To get the multiple soliton solutions (MSSs) to Eq. (6), we set
\[ u = e^{\Theta_i}, \] (8)
where the dispersion variables may be set as
\[ \Theta_i \equiv \Theta_i(x, y, z, t) = K_i x + L_i y + M_i z - \omega_i t \forall 1 \leq i \leq 3 \] (9)
while the parameters \((K_i, L_i, M_i)\) indicate the coefficients of \((x, y, z)\), respectively, and will be determined later. After that the following conversion to \( u \) is used
\[ u = 2 (\ln f)_x, \] (10)
where the auxiliary function (AF) \( f \equiv f(x, y, z, t) \) is defined as
\[
f = 1 + e^{\Theta_1}.
\] (11)

Inserting Eq. (11) into Eq. (10), and using Eq. (6), the soliton solutions exist only under the following conditions
\[
\begin{align*}
\omega_i &= -K_i^3, \\
M_i &= -\frac{\alpha_1 K_i + \alpha_2 L_i}{\alpha_3}, \alpha_3 \neq 0,
\end{align*}
\] (12)

where the coefficients \((K_i, L_i)\) are left as free parameters and \(i = 1, 2, \cdots, N\). The case for \(\alpha_3 = 0\), will be examined later.

It is shown from Eq. (12) that the coefficients \(M_i\) depend only on the coefficients \((K_i, L_i)\) and on the parameters \((\alpha_1, \alpha_2, \alpha_3)\) of the additional linear terms. As a result, it will produce a set of solutions with a variety of distinct selection coefficients values.

Based on the obtained conditions, the dispersion variables become
\[
\Theta_i = K_i x + L_i y - \frac{\alpha_1 K_i + \alpha_2 L_i}{\alpha_3} z + K_i^3 t, \quad 1 \leq i \leq N.
\] (13)

Using Eq. (10) and Eq. (11), for \(i = 1\), the single soliton solution is furnished as
\[
u = \frac{2k_1}{f_1} e^{K_1 x + L_1 y - \frac{\alpha_1 K_1 + \alpha_2 L_1}{\alpha_3} z + K_1^3 t},
\] (14)

For determining the two soliton solutions, the following AF is introduced
\[
f_2 = 1 + e^{\Theta_1 + e^{\Theta_2} + P_{12} e^{\Theta_1 + \Theta_2}},
\] (15)

where \((\Theta_1, \Theta_2)\) are defined in Eq. (13) and \(P_{12}\) represents the phase shift between the two solitons. Inserting, Eq. (10) and Eq. (15) into Eq. (6), we finally obtain \(P_{12}\) as follows
\[
P_{12} = \frac{(K_1 - K_2)^2}{(K_1 + K_2)^2},
\] (16)

The general formula for the phase shifts can be classified as the KdV-type phase shifts as follows
\[
P_{ij} = \frac{(K_i - K_j)^2}{(K_i + K_j)^2},
\] (17)

for \(1 \leq i < j \leq 3\).

Now, the two-soliton solutions to Eq. (6) can be determined by introducing Eq. (15) in Eq. (10).

Following the same steps, we get the three-soliton solutions by using the fol-
Following AF
\[ f_3 = 1 + \sum_{i=1}^{3} e^{\Theta_i} + P_{12} e^{\Theta_1 + \Theta_2} + P_{23} e^{\Theta_2 + \Theta_3} + P_{13} e^{\Theta_1 + \Theta_3} + P_{12} a_{13} e^{\Theta_1 + \Theta_2 + \Theta_3}. \]  
(18)

where the phase shifts \( P_{ij} \) are identified in Eq. (17). Consequently, proceeding as in the case of the double-solitons we easily derive the three soliton solution.

It was stated earlier, that the MNWE1 (6) is Painlevé integrable for all real values of the parameters \((\alpha_1, \alpha_2, \alpha_3)\). In what follows, we set specific selections of these parameters, and examine the dispersion relation for each case and the single soliton solution as well.

**Case 1.**

We first select \( \alpha_1 = 0 \), where this selection retains the same (3+1) dimensions for the MNWE1. This in turn will give us

\[
\begin{aligned}
M_i &= -\frac{\alpha_2 L_i}{\alpha_3}, \alpha_3 \neq 0, \\
\omega_1 &= -K_1^3, \\
f_1 &= 1 + e^{(K_1 x + L_1 y - \frac{\alpha_2 L_1}{\alpha_3} z + K_1^3 t)}.
\end{aligned}
\]  
(19)

Thus, the one-soliton solution of the kink type reads

\[
u = 2K_1 e^{(K_1 x + L_1 y - \frac{\alpha_2 L_1}{\alpha_3} z + K_1^3 t)} \frac{1 + e^{(K_1 x + L_1 y - \frac{\alpha_2 L_1}{\alpha_3} z + K_1^3 t)}}{1 + e^{(K_1 x + L_1 y - \frac{\alpha_2 L_1}{\alpha_3} z + K_1^3 t)}},
\]  
(20)

Following the same earlier analysis, the two- and three-soliton solutions can be obtained, while the phase shifts are the same as in Eq. (17).

**Case 2.**

We next select \( \alpha_2 = 0 \), to convert the (3+1) dimensions for the MNWE1 to a (2+1) dimensions in \((x, z)\). This in turn will give us

\[
\begin{aligned}
M_i &= -\frac{\alpha_1 K_i}{\alpha_3}, \alpha_3 \neq 0, \\
\omega_1 &= -K_1^3, \\
f_1 &= 1 + e^{(K_1 x - \frac{\alpha_1 K_1}{\alpha_3} z + K_1^3 t)},
\end{aligned}
\]  
(21)

and consequently, the one-soliton solution of the kink type

\[
u(x, z, t) = 2K_1 e^{(K_1 x - \frac{\alpha_1 K_1}{\alpha_3} z + K_1^3 t)} \frac{1 + e^{(K_1 x - \frac{\alpha_1 K_1}{\alpha_3} z + K_1^3 t)}}{1 + e^{(K_1 x - \frac{\alpha_1 K_1}{\alpha_3} z + K_1^3 t)}},
\]  
(22)

follows immediately. Following the same earlier analysis, the two- and three-soliton solutions can be obtained, where the phase shifts remain the same as those of the KdV type.

Case 3.

We finally select $\alpha_3 = 0$, where this selection converts the (3+1)-dimensional MNWE1 to a (2+1)-dimensional model with $u \equiv u(x, y, t)$. This in turn will give us

$$
\begin{align*}
L_i &= -\frac{\alpha_1 K_i}{\alpha_2}, \alpha_2 \neq 0, \\
\omega_i &= -K_i, \\
f_1 &= 1 + e^{(K_i x - \frac{\alpha_1 K_i}{\alpha_2} y + K_i^3 t)}.
\end{align*}
$$

(23)

Accordingly, the following one-soliton solution of the kink type is obtained

$$
u = \frac{2K_1 e^{(K_1 x - \frac{\alpha_1 K_1}{\alpha_2} y + K_1^3 t)}}{1 + e^{(K_1 x - \frac{\alpha_1 K_1}{\alpha_2} y + K_1^3 t)}}.
$$

(24)

This also means that we can obtain the two- and three-soliton solutions noting that the phase shifts keep their structures as those of the KdV type.

3. THE SECOND EXTENDED MNWE2

We next study the second extended MNWE2

$$
\mathbb{R} + \beta_1 u_{xt} + \beta_2 u_{yt} + \beta_3 u_{zt} = 0,
$$

(25)

We will illustrate that the MNWE2 (25) passes the Painlevé test. Next, we will examine the necessary conditions of the coefficients of the spatial variables that will guarantee the existence of MSSs for this model.

3.1. PAINLEVÉ ANALYSIS TO MNWE2

To investigate the integrability of the MNWE2 (25), we follow the PT that was applied in the previous Section. As a result, the CE with one branch for resonances at $k = -1, 1, 6, 8$ is obtained. The MNWE2 passes the PIT in a like manner to the above Section although the three additional terms are the partial derivatives $u_{xt}, u_{yt},$ and $u_{zt}$.

3.2. THE SECOND EXTENDED MNWE2 AND MSSS

In order to derive the MSSs to the MNWE2 (25), we set

$$
u = e^{\Theta_i},
$$

(26)

where $\Theta_i$ is given by

$$
\Theta_i = K_i x + L_i y + M_i z - \omega_i t \quad \forall \ 1 \leq i \leq 3
$$

(27)
We next use the following conversion of $u$:

$$u = 2(\ln f)_x,$$

while the AF $f$ is given by

$$f = 1 + e^{\Theta_1}.$$  \hfill (28)

Inserting Eq. (29) into Eq. (28), and using (25), the following conditions for the existence of soliton solutions are obtained

$$\begin{align*}
\omega_i &= -K_3^i, \\
M_i &= -\frac{\beta_1 K_i + \beta_2 L_i}{\beta_3}, \beta_3 \neq 0,
\end{align*}$$

(30)

where the coefficients $K_i$ and $L_i$ are left as free parameters, for $i = 1, 2, \cdots, N$. The case for $\beta_3 = 0$ will be examined later.

It is clear that the coefficients $M_i$ depend mainly on the coefficients $K_i$ and $L_i$ as well as on the coefficients $(\beta_1, \beta_2, \beta_3)$ of the additional linear terms. As a result, it will produce a set of solutions with a variety of distinct selection coefficients values.

Based on the obtained conditions, the dispersion variables become

$$\Theta_i = K_i x + L_i y - \frac{\beta_1 K_i + \beta_2 L_i}{\beta_3} z + K_3^i t, 1 \leq i \leq N.$$  \hfill (31)

Now, for $i = 1$, the following single soliton solution is obtained

$$u = \frac{2K_1 e^{K_1 x + L_1 y - \frac{\beta_1 K_1 + \beta_2 L_1}{\beta_3} z + K_3^1 t}}{1 + e^{K_1 x + L_1 y - \frac{\beta_1 K_1 + \beta_2 L_1}{\beta_3} z + K_3^1 t}},$$

(32)

Moreover, for determining two-soliton solutions, the following AF is introduced

$$f_2 = 1 + e^{\Theta_1} + e^{\Theta_2} + P_{12} e^{\Theta_1 + \Theta_2},$$

(33)

where $(\Theta_1, \Theta_2)$ are defined in Eq. (31).

Inserting, Eqs. (28) and (33) into Eq. (25), the phase shift $P_{12}$ and the generalized phase shifts $P_{ij}$ (1 $\leq i < j \leq 3$) have the same forms as given in Eq. (17). Moreover, the two- and three-solitons of Eq. (25) can be determined in a similar way as presented earlier.

It was stated earlier, that the MNWE2 (25) is Painlevé integrable for all real values of the parameters $(\beta_1, \beta_2, \beta_3)$. In what follows, we set specific selections of these parameters, and examine the dispersion relation for each case and the single soliton solution as well.

**Case 1.**

We second select $\beta_1 = 0$, where this selection retains the same (3+1) dimen-
sions of the MNWE2. This in turn will give us

\[
\begin{align*}
M_i &= -\frac{\beta_3 L_1}{\beta_3}, \beta_3 \neq 0, \\
\omega_1 &= -K_1^3, \\
f_1 &= 1 + e^{(K_1 x + L_1 y - \frac{\beta_3 L_1}{\beta_3} z + K_1^3 t)},
\end{align*}
\]

Thus, the one-soliton solution of the kink type

\[
u = \frac{2K_1 e^{(K_1 x + L_1 y - \frac{\beta_3 L_1}{\beta_3} z + K_1^3 t)}}{1 + e^{(K_1 x + L_1 y - \frac{\beta_3 L_1}{\beta_3} z + K_1^3 t)}},
\]

is readily obtained. Following our analysis presented earlier, we obtain two- and three-soliton solutions, noting that the phase shifts are the same as in Eq. (17).

**Case 2.**

We next select \(\beta_2 = 0\), to convert the (3+1) dimensions for the MNW2 equation to (2+1) dimensions with \(u \equiv u(x, z, t)\). This in turn will give us

\[
\begin{align*}
M_i &= -\frac{\beta_3 K_1}{\beta_3}, \beta_3 \neq 0, \\
\omega_i &= -K_1^3, \\
f_1 &= 1 + e^{(K_1 x - \frac{\beta_3 K_1}{\beta_3} y + K_1^3 t)},
\end{align*}
\]

and consequently, the one-soliton solution of the kink type

\[
u = \frac{2K_1 e^{(K_1 x - \frac{\beta_3 K_1}{\beta_3} y + K_1^3 t)}}{1 + e^{(K_1 x - \frac{\beta_3 K_1}{\beta_3} y + K_1^3 t)}},
\]

follows immediately. Following our analysis presented earlier, we obtain two- and three-soliton solutions, noting that the phase shifts remain the same.

**Case 3.**

We select \(\beta_3 = 0\), where this selection converts the (3+1)-dimensional MNWE2 to a (2+1)-dimensional model with \(u \equiv u(x, y, t)\). This in turn will give us

\[
\begin{align*}
L_i &= -\frac{\beta_1 K_1}{\beta_2}, \beta_2 \neq 0, \\
\omega_i &= -K_1^3, \\
f_1 &= 1 + e^{(K_1 x - \frac{\beta_1 K_1}{\beta_2} y + K_1^3 t)},
\end{align*}
\]

Accordingly, the one-soliton solution of the kink type is obtained as follows

\[
u = \frac{2k_1 e^{(K_1 x - \frac{\beta_1 K_1}{\beta_2} y + K_1^3 t)}}{1 + e^{(K_1 x - \frac{\beta_1 K_1}{\beta_2} y + K_1^3 t)}},
\]

This also means that we can obtain two- and three-soliton solutions, noting that the phase shifts keep their structures as those of the KdV type.
4. THE THIRD EXTENDED MNWE3

We close this study by examining the third extended MNWE3

\[ R + \alpha_1 u_x + \alpha_2 u_y + \alpha_3 u_z + \beta_1 u_{xt} + \beta_2 u_{yt} + \beta_3 u_{zt} = 0, \]  

(40)

where six linear terms are added to the standard MNW equation. We will show that this extended equation passes the Painlevé test. Next, we will examine the necessary conditions of the coefficients of the spatial variables that will guarantee MSSs for this model.

4.1. PAINLEVÉ ANALYSIS TO MNWE3

Employing the PT in a like manner to the preceding Sections gives the CE with one branch for resonances at \( k = -1, 1, 6, \) and 8. Proceeding as before confirms the PIT of the MNWE3, hence we skip the details.

4.2. THE THIRD EXTENDED MNWE3 AND MSSS

In order to determine the MSSs to Eq. (40), the following set is introduced

\[ u = e^{\Theta_i}, \]  

(41)

where the dispersion variables may be set as

\[ \Theta_i = K_i x + L_i y + M_i z - \omega_i t \quad \forall \ 1 \leq i \leq 3. \]  

(42)

We next use the following conversion of \( u \)

\[ u = 2(\ln f)_x, \]  

(43)

with the AF \( f \) reads

\[ f = 1 + e^{\Theta_1}. \]  

(44)

Inserting Eq. (44) into Eq. (43), and using (40), the soliton solutions can be obtained only under the following conditions

\[ \begin{cases}
  L_i = -\frac{M_{13}}{M_{23}} K_i, M_{23} \neq 0, \\
  M_i = -\frac{M_{13}}{M_{23}} K_i, M_{23} \neq 0, \\
  \omega_i = -K_i^3, \\
  f_1 = 1 + e^{(K_1 x - \frac{M_{13}}{M_{23}} K_1 y - \frac{M_{12}}{M_{23}} K_1 z + K_1^3 t)},
\end{cases} \]  

(45)

where the determinants \( M_{12}, M_{13}, \) and \( M_{23} \) are defined as follows:
\[
M_{12} = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \alpha_1 \beta_2 - \alpha_2 \beta_1,
\]
\[
M_{13} = \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_3 & \beta_3 \end{vmatrix} = \alpha_1 \beta_3 - \alpha_3 \beta_1,
\]
\[
M_{23} = \begin{vmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{vmatrix} = \alpha_2 \beta_3 - \alpha_3 \beta_2,
\]
and the coefficients \( K_i \) are left as free parameters, for \( i = 1, 2, \cdots, N \).

It is shown that the coefficients \((L_i, M_i)\) depend mainly on the coefficients \( K_i \) as well as on the coefficients \( \alpha_i, \beta_i, i = 1, 2, 3 \) of the additional linear terms. As a result, it will produce a set of solutions with a variety of distinct selection coefficients values.

Based on the obtained conditions, the dispersion variables become
\[
\Theta_i = K_i x - \frac{M_{13} M_{23}}{M_{23}} K_i y - \frac{M_{12}}{M_{23}} K_i z + K_i^3 t \forall 1 \leq i \leq N. \quad (46)
\]
As a result, the single soliton solution can be obtained for \( i = 1 \) as follows
\[
u = \frac{2K_1 e^{\left(K_1 x - \frac{M_{13}}{M_{23}} K_1 y - \frac{M_{12}}{M_{23}} K_1 z + K_1^3 t\right)}}{1 + e^{\left(K_1 x - \frac{M_{13}}{M_{23}} K_1 y - \frac{M_{12}}{M_{23}} K_1 z + K_1^3 t\right)}}. \quad (47)
\]
For determining the two-soliton solutions, the following AF is introduced
\[
f_2 = 1 + e^{\Theta_1} + e^{\Theta_2} + P_{12} e^{\Theta_1 + \Theta_2}, \quad (48)
\]
where \((\Theta_1, \Theta_2)\) are defined in Eq. (46). Substituting Eq. (48) into Eq. (40), the phase shift \(P_{12}\) is obtained as defined in Eq. (16). Also, the general form for the phase shifts \(P_{ij} \) \((1 \leq i < j \leq 3)\) can be obtained as defined in Eq. (17).

The two-soliton solutions to Eq. (40) can be determined by inserting (48) in (43).

In a similar manner and using the following AF, we can get the three-soliton solutions
\[
f_3 = 1 + \sum_{i=1}^{3} e^{\Theta_i} + P_{12} e^{\Theta_1 + \Theta_2} + P_{23} e^{\Theta_2 + \Theta_3} + P_{13} e^{\Theta_1 + \Theta_3} + P_{12} a_{13} e^{\Theta_1 + \Theta_2 + \Theta_3}.
\]

5. CONCLUSIONS

We formally introduced three extended integrable (3+1)-dimensional Mikhailov-Novikov-Wang (eMNW) equations. The complete integrability of each developed
model was emphasized using the Painlevé test. To guarantee the existence of multiple-soliton solutions (MSSs), we obtained the appropriate coefficients constraints. The simplified Hirota’s method has been employed to obtain MSSs for the three extended integrable models. Furthermore, we demonstrated that the wave variable and the dispersion relation depend on the coefficients of spatial dimensions.

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