LOW ENERGY ANALYSIS OF PHASE SHIFTS AND CROSS-SECTION FOR $n-^3He$ SYSTEM

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Abstract. Considering the Hulthén potential as the nuclear part of interaction regular solution is derived by exploiting the Frobenius method. The Jost function is constructed by utilizing its integral representation in terms of regular solution. The scattering phase shifts and cross-sections are calculated for the n-3He system from the on-shell Jost function. The results obtained are in close conformity with the previous works that exist in the literature.

Key words: Hulthén potential, Frobenius method, Jost Function, phase shifts, scattering cross-sections.

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1. INTRODUCTION

In low-energy nuclear physics hadron-hadron elastic scattering can be well estimated in terms of a two-particle central potential model to understand the nuclear structure. The scattering phase shifts and cross-sections play a crucial role in this direction. Various numerical techniques are used on realistic potentials to accurately represent few-nucleon systems (A<6). Their validity and accuracy can be tested compared with existing experimental data. Several groups [1-5] have treated different A=4 systems, calculating various scattering observables which provide us with a large number of phase shift and cross-section data. The Hulthén potential serves as a model for two-nucleon and nucleon-nucleus interaction and has the ability to reproduce low-energy scattering parameters [6-8]. For the Hulthén potential Schrodinger wave equation is exactly solvable only for the partial wave. However, using different mathematical methods approximated higher partial wave solutions can be constructed from the ground state (S wave) [9-13]. The present text deals with the construction of regular solutions for motion in nuclear Hulthén potential and the calculation of phase parameters, cross-section, etc. by applying the Jost formalism [14]. The Jost/irregular solution $f_l(k,r)$ of the radial Schrödinger equation has immense importance in scattering theory as its near-the-origin $(r \to 0)$ behavior represents the Jost function. The Jost function has a dual character as it has the ability to extract all the relevant information relating to the bound and scattering states of a system. The bound and resonating state energies can be directly calculated by analytic continuation of the Jost function [14,15] into the complex k-plane; as the zeros of this function in the upper and lower half of the k-plane reproduce bound and resonant state energies respectively. It also plays an important role in examining the analytical properties of partial wave scattering amplitude (the S-matrix). The scattering phase shift can be calculated from the real and imaginary parts of the Jost function, which in turn also relates to the scattering cross-section. In section 2 we describe our methodology. Section 3 is devoted to results and discussion where we compute the scattering phase shifts and differential cross-section and compare them with standard data [16,17]. Finally, we conclude in section 4.

2. METHODOLOGY

The Schrödinger equation for the Hulthén potential is written as

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{\delta^2 l(l+1)}{(1-e^{-\delta r})^2} e^{-\delta r} - V_0 \frac{e^{-\delta r}}{1-e^{-\delta r}}\right] \psi_l(k,r) = 0. \tag{1}$$

Here $V_0 \frac{e^{-\delta r}}{(1-e^{-\delta r})}$ is the Hulthén potential taken to be the nuclear interaction, and $\frac{\delta^2 l(l+1) e^{-\delta r}}{(1-e^{-\delta r})^2}$ is the centrifugal barrier term. $V0 = -(\alpha^2 - \beta^2)$ and $\delta = (\beta - \alpha)$ represents the strength and inverse scattering radius of the potential where $\beta(fm^{-1})$ and $\alpha(fm^{-1})$ are two adjustable parameters. Using the trial solution

$$\psi_l(k,r) = \frac{e^{ikr}(1-e^{-\delta r})^{l+1}}{\delta^{l+1}} F_l(k,r) \tag{2}$$

in eqn. (1) we get

$$\begin{aligned}
&\frac{e^{ikr}(1-e^{-\delta r})^{l+1}}{\delta^{l+1}} \frac{d^2 F_l(k,r)}{dr^2} + \frac{2(l+1)e^{ikr}e^{-\delta r}(1-e^{-\delta r})\delta}{\delta^{l+1}} \frac{dF_l(k,r)}{dr} + \\
&\frac{2ike^{ikr}(1-e^{-\delta r})^{l+1}}{\delta^{l+1}} \frac{dF_l(k,r)}{dr} + \frac{2ik(l+1)e^{ikr}e^{-\delta r}(1-e^{-\delta r})^{l})\delta}{\delta^{l+1}} \frac{dF_l(k,r)}{dr} \\
&+ \frac{\delta^2 e^{ikr}(l+1)}{\delta^{l+1}} \left[l e^{-2\delta r}(1-e^{-\delta r})^{l-1} - e^{-\delta r}(1-e^{-\delta r})^{l}\right] F_l(k,r) = \\
&\left(V_0 \frac{e^{-\delta r}}{(1-e^{-\delta r})} + \frac{\delta^2 e^{-\delta r}}{(1-e^{-\delta r})^2}\right) \frac{e^{ikr}(1-e^{-\delta r})^{l+1}}{\delta^{l+1}} F_l(k,r).
\end{aligned} \tag{3}$$

Substituting $1-e^{-\delta r}=z$; eqn. (3) becomes

$$z(1-z)\frac{d^2F_l(k,z)}{dz^2}+\left[2(l+1)+\left(\frac{2ik}{\delta}-2l-3\right)z\right]\frac{dF_l(k,z)}{dz}+ \left(\frac{2ik}{\delta}(l+1)-(l+1)^2-\frac{V_0}{\delta^2}\right)F_l(k,z)=0. \tag{4}$$

Applying the Frobenius method at $z=0$ to solve eqn. (4) we write

$$F_l(k,z)=\sum_{n=0}^{\infty}a_n z^{n+\lambda}, \tag{5}$$

where $a_0\neq 0$. The recurrence relation for $\lambda=0$ is obtained as

$$a_{j+2}=\frac{(j+1)\left[j+2l+3-\frac{2ik}{\delta}\right]+Y}{(j+2)(j+2l+3)}a_j+1; \tag{6}$$

where

$$Y=(l+1)^2+\frac{V_0}{\delta^2}-\frac{2ik}{\delta}(l+1). \tag{7}$$

The coefficients are

$$a_1=\frac{Y}{2(l+1)}a_0; \tag{8}$$

$$a_2=\left[\frac{(2l+3-\frac{2ik}{\delta}+Y)}{2(2l+3)}\frac{Y}{2(l+1)}\right]a_0; \tag{9}$$

$$a_3=\left[\frac{2(2l+4-\frac{2ik}{\delta}+Y)}{3(2l+4)}\right]\times\left[\frac{(2l+3-\frac{2ik}{\delta}+Y)}{2(2l+3)}\frac{Y}{2(l+1)}\right]a_0; \tag{10}$$

$$a_4=\left[\frac{3(2l+5-\frac{2ik}{\delta}+Y)}{4(2l+5)}\right]\times\left[\frac{2(2l+4-\frac{2ik}{\delta}+Y)}{3(2l+4)}\right]\times \left[\frac{(2l+3-\frac{2ik}{\delta}+Y)}{2(2l+3)}\frac{Y}{2(l+1)}\right]a_0. \tag{11}$$

Substitution of the coefficients in eqn. (2) one gets

$$\psi_l(k,r)=\frac{e^{ikr}(1-e^{-\delta r})^{l+1}}{\delta^{l+1}}\times \left[a_0+a_1(1-e^{-\delta r})+a_2(1-e^{-\delta r})^2+a_3(1-e^{-\delta r})^3+a_4(1-e^{-\delta r})^4+...\right]. \tag{12}$$

There exists a relation between the Jost function and the regular solution [18-20], which reads as

$$f_l(k)=1+\frac{k^l}{(2l+1)!!}\int_0^{\infty}V(r)\psi_l(k,r)\hat{h_l}^{+}(k,r)dr, \tag{13}$$

where $\hat{h}_l^+(kr)$ is the Riccati-Hankel function, given by

$$\hat{h}_l^+(kr) = \sum_{L=0}^{\infty} \frac{(i)^{2L-1}(l+L)!e^{ikr}\gamma^L}{(2ik)^L L!(1-e^{-\gamma r})^L (l-L)!}. \tag{14}$$

From eqns. (12) and (13) with the effective potential

$$V(r) = V_0 \frac{e^{-\delta r}}{1-e^{-\delta r}} + \frac{\delta^2 l(l+1)e^{-\delta r}}{(1-e^{-\delta r})^2} \tag{15}$$

one obtains

$$f_l(k) = 1 + \frac{k^l}{(2l+1)!!}\sum i^{2L-1}\frac{(l+L)!\delta^L}{(2ik)^L L!(l-L)!}\times \left[\int_0^\infty \frac{V_0 e^{-\delta r}}{(1-e^{-\delta r})^{L+1}} e^{ikr}\psi_l(k,r)dr + \int_0^\infty \frac{\delta^2 l(l+1)e^{-\delta r}}{(1-e^{-\delta r})^{L+2}} e^{ikr}\right]. \tag{16}$$

Integrating term by term we get

$$f_l(k) = 1 + \frac{k^l}{(2l+1)!!}\sum i^{2L-1}\frac{(l+L)!\delta^{L-l-1}}{(2ik)^L L!(l-L)!}\times \left(\frac{V_0}{\delta}(A1+A2+A3+A4+A5) + \delta l(l+1)(B1+B2+B3+B4+B5)\right), \tag{17}$$

where

$$A1 = a_0 \frac{\Gamma(1-\frac{2ik}{\delta})\Gamma(l-L+1)}{\Gamma(1-\frac{2ik}{\delta}+l-L+1)}, \tag{18}$$

$$A2 = a_1 \frac{\Gamma(1-\frac{2ik}{\delta})\Gamma(l-L+2)}{\Gamma(1-\frac{2ik}{\delta}+l-L+2)}, \tag{19}$$

$$A3 = a_2 \frac{\Gamma(1-\frac{2ik}{\delta})\Gamma(l-L+3)}{\Gamma(1-\frac{2ik}{\delta}+l-L+3)}, \tag{20}$$

$$A4 = a_3 \frac{\Gamma(1-\frac{2ik}{\delta})\Gamma(l-L+4)}{\Gamma(1-\frac{2ik}{\delta}+l-L+4)}, \tag{21}$$

and

$$A5 = a_4 \frac{\Gamma(1-\frac{2ik}{\delta})\Gamma(l-L+5)}{\Gamma(1-\frac{2ik}{\delta}+l-L+5)}. \tag{22}$$

In deriving the above equations, we have made use of [21]

$$\int_0^1 (1-x)^{\mu-1}x^{\nu-1}dx = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}. \tag{23}$$

Similarly, the terms read as

$$B1 = a_0\left(\frac{1}{(1-\frac{2ik}{\delta})} - \frac{1}{(2-\frac{2ik}{\delta})}\right), \tag{24}$$

$$B2 = a_1\frac{\Gamma\left(1-\frac{2ik}{\delta}\right)\Gamma(l-L+1)}{\Gamma(1-\frac{2ik}{\delta}+l-L+1)}, \tag{25}$$

$$B3 = a_2\frac{\Gamma\left(1-\frac{2ik}{\delta}\right)\Gamma(l-L+2)}{\Gamma(1-\frac{2ik}{\delta}+l-L+2)}, \tag{26}$$

$$B4 = a_3\frac{\Gamma\left(1-\frac{2ik}{\delta}\right)\Gamma(l-L+3)}{\Gamma(1-\frac{2ik}{\delta}+l-L+3)}, \tag{27}$$

and

$$B5 = a_4\frac{\Gamma\left(1-\frac{2ik}{\delta}\right)\Gamma(l-L+4)}{\Gamma(1-\frac{2ik}{\delta}+l-L+4)}. \tag{28}$$

As the Jost function is a complex quantity, the phase shift can be calculated as

$$\delta = -tan^{-1}\left(\frac{Imf_l(k)}{Ref_l(k)}\right). \tag{29}$$

The differential scattering cross-section is given by [22]

$$\frac{d\sigma}{d\omega} = |F(\theta)|^2 = |\frac{1}{k}\sum_{l=0}^{\infty}(2l+1)P_l(cos\theta)exp(i\delta_l)sin(\delta_l)|^2, \tag{30}$$

where $F(\theta)$ is the scattering amplitude and $P_l(cos\theta)$ is the Legendre polynomial.

3. RESULTS AND DISCUSSION

For numerical computation of the scattering phase shifts we have used $\frac{\hbar^2}{m_p} = 41.47MeVfm^2$. The best-fitted parameters are obtained using Chi-square method and are presented in Table 1. The scattering phase shifts for different states are

Table 1

Parameters for Hulthén potential

State:	$^1S_0$	$^3S_1$	$^1P_1$	$^3P_0$
$\alpha(fm^{-1})$	0.125	0.33	0.002	0.02
$\beta(fm^{-1})$	1.02	0.8	2.4	0.195

portrayed in figures 1 and 2. Figure 1. shows that our computed scattering phase shifts $\delta^{0+}(^1S_0)$ and $\delta_{01}^{1+}(^3S_1)$ for $l=0$ states match almost perfectly with Ref. [16].

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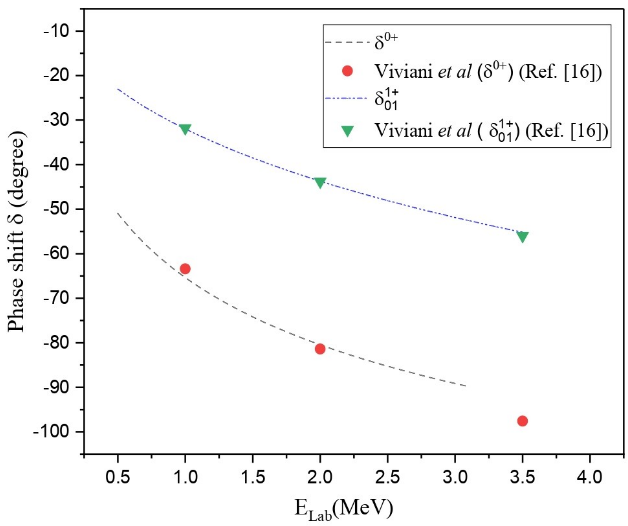

Fig. 1 – n-^3He S wave phase shifts with standard data.

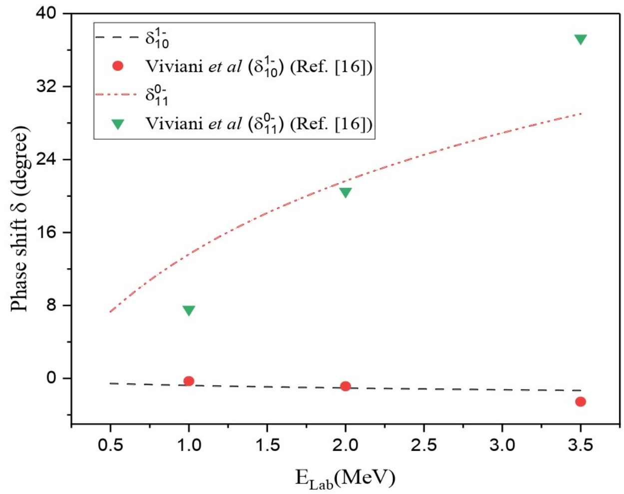

Fig. 2 – n-^3He P wave phase shifts with standard data.

For $l = 1$, scattering phase shifts δ_{10}^{1-} for $^1P_1$ state are also in good agreement with Viviani et al. [16], whereas $^3P_0$ scattering phase shift δ_{11}^{0-} data show slight deviations for lower and higher energies within our energy range. Further, we investigate to what extent our phase parameters will be able to reproduce cross-section data in light of the observable disparities between the results of these existing phase shift data [16] and our calculations. In Figure 3 we plot our differential scattering cross-section

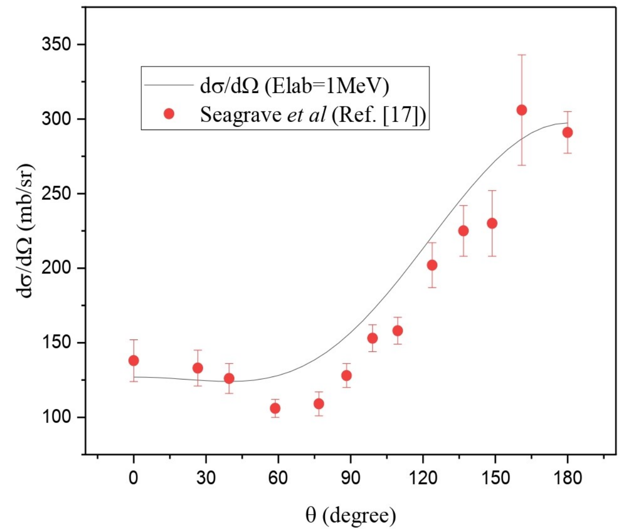

Fig. 3 – Differential cross-section for n-3He elastic scattering at $E_{lab} = 1$ MeV with standard data.

data with those of Seagrave *et al.* [17] for lab energy 1 MeV. The calculated cross-section data points follow a similar trend of progression as Ref. [17], though slight deviation is observed within scattering angles 450 to 1500. The probable reason for the deviation is due to the contribution of $^3P_0$ scattering phase shift in the calculation of the scattering cross-section formula.

4. CONCLUSIONS

We have studied the low energy $n-^3He$ scattering using a simple phenomenological potential. Considering the single-channel elastic scattering, the on-shell Jost function is calculated from its integral representation using the all-partial wave regular solutions. The scattering phase shifts agree quite well with the standard data, except for the $^3P_0$ state. Further, the differential cross-section data follow a similar trend as those of Seagrave *et al.* [17] with slight discrepancies in numerical values. We remark that a possible reason for these discrepancies in phase parameters as well as in cross-section is attributed to spin-orbit force due to the nucleon magnetic moment that plays a significant role in the difference. Our single-channel nuclear model, in spite of being the simplest form, produces correct trends for both phase shifts and cross sections of the $n-^3He$ system. The effects of charged hadron scattering can be further observed with the addition of appropriate electromagnetic interaction.

REFERENCES

1. P. Heiss, H. H. Hackenbroich, Z. Physik. **251**, 167-172 (1972).
2. S. B. Doma, Helvetica Physica Acta. **58**(6), 1072-1077 (1985).
3. P. Jany, W. Heeringa, H. O. Klages, B. Zeitnitz, Nucl. Phy. A **483**, 269-281 (1988).
4. A. Deltuva, A. C. Fonseca, Phys. Rev. C **76**, 021001 (2007).
5. S. B. Dubovichenko, Ch. T. Omarov, N. A. Burkova, Russ. Phy. Jour. **59**(6), 884-891(2016).
6. U. Laha, C. Bhattacharyya, K. Roy, B. Talukdar, Phys. Rev. C **38**, 558-560 (1988).
7. J. Bhoi, U. Laha, Theor. Math. Phys. **190**, 69-76 (2017).
8. U. Laha, M. Majumder, J. Bhoi, Pramana-J. Phys. **90**, Art. 48 (2018).
9. J. Bhoi, M. Majumder, U. Laha, Ind. Jour. of Pure & Appl. Phys. **56**, 538-543 (2018).
10. P. Sarkar, B. Khirali, U. Laha, P. Sahoo, Int. J. Mod. Phys. E **30**, 2150066 (2021).
11. P. Sahoo, U. Laha, Phys. Scr. **96**, 095302 (2022).
12. J. Bhoi, U. Laha, J. Phys. G: Nucl. Part. **40**, 045107 (2013).
13. U. Laha, J. Bhoi, Phys. Rev. C **91**, 034614 (2015).
14. R. G. Newton, "*Scattering theory of Waves and Particles* ", (McGraw-Hill, New York, 1982).
15. S. K. Adhikari, K. L. Kowalski, "*Dynamical Collision Theory and Its Applications* ", (Academic Press, New York, 1991).
16. M. Viviani, A Deltuva, R Lazauskas, A.C. Fonseca, A. Kievsky, L.E. Marcucci, Phys. Rev. C **95**, 034003 (2017).
17. J. D Seagrave, L. Cranberg, J.E. Simmons, Phys. Rev. **119**, 1981-1991 (1960).
18. B. Talukdar, U. Laha, U. Das, Phys. Rev. A **43**, 1183-1185 (1991).
19. U. Laha, Pramana-J Phys. **67**, 357-362 (2006).
20. U. Laha, J. Bhoi, Pramana - J Phys. **86**, 947-956 (2016).
21. I. S. Gradshteyn, I. M. Ryzhik, "*Table of Integrals, Series, and Products*", (Academic Press, London, 2000).
22. J. R Taylor, "*Scattering Theory: The Quantum Theory of Nonrelativistic Collisions*", (Dover Publications INC, New York, 2006).