LOCALIZED STATES NEAR A SHORT-RANGE DEFECT IN THE PRESENCE OF A HYPERBOLIC POTENTIAL FIELD

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Abstract. The exact analytical solution of the one-dimensional stationary Schrödinger equation with a spatially distributed hyperbolic potential profile ~1/x in the presence of a Dirac delta function (point) potential is found and analyzed. Localization features are described analytically in dependence on the parameters of a hyperbolic potential profile and point potential. It is found that the localization energy of the ground state monotonically decreases with an increase in defect intensity. The localized states exist near the both attractive and repulsive defects. Excited states are characterized by the formation of oscillations in the region of action of the hyperbolic field. The height of the maximum of the wave function decreases with decreasing intensity of the repulsive defect and with increasing absolute value of the defect intensity in the case of attractive defect. Localization length in the hyperbolic well is greater than in a half-space with a constant potential.

Key words: localized state; hyperbolic field; hyperbolic potential well.

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1. INTRODUCTION

During the last decades, the theoretical study of localized states of various physical natures has been actively developing [1, 2]. These include soliton-like states in optical media [3, 4], fibers [5], and Bose-Einstein condensates (BEC) [6, 7]. Such studies remain relevant from both fundamental and applied points of view [8]. Interest in theoretical studies of the features of excitation localization in various fields taking into account size-boundary effects will continue to be supported by the active development of experimental photonics [9, 10].

Localized states in photonic structure are actively studied in recent years [11–13]. In particular, much attention has been paid to the study of bound states in the continuum in photonic structures [14, 15]. Bound states are of great practical importance because they can spatially trap electromagnetic waves even if their frequency lies within the continuous spectrum [16, 17].

The search for exact solutions of the Schrödinger equation (SE) with different spatially distributed potentials remains in the field of interest of researchers [18–20]. In this paper, we investigate the features of the formation of localized states using the example of a one-dimensional hyperbolic potential depending on the spatial coordinate x as $V(x)$~$1/x$ (this name is associated with the hyperbole). The use of the term “hyperbolic” for the refractive index profile (hyperbolic graded index) adopted in nonlinear optics and photonics [21] should not be confused with that used in quantum mechanics for a potential well [22, 23]. The hyperbolic potential well are generated by hyperbolic functions as $V(x)$~$1/\cosh^n(x)$, ~$\cosh^m(x)/\cosh^n(x)$, ~$\sinh^m(x)/\cosh^n(x)$ [24, 25]. Such double hyperbolic potential wells of symmetric type find recently wide application in the context of studying quantum information entropy [26, 27], including the fractional SE [28]. A similar type of potential can also include the Konwent potential used in the theoretical description of quantum-optical effects in semiconductor optical heterostructures [29, 30].

The exact solutions of the SE considered in these papers are mainly expressed through Heun or hypergeometric functions, the analysis of which can cause significant difficulties. As a result, for the physical and visual interpretation of such solutions, there is a need to conduct additional mathematical studies of their properties and asymptotics. Another way to do this is the numerical analysis. However, for some potentials there are exact solutions expressed through simpler special functions, the analysis of which is much simpler than that of the Heun or hypergeometric functions. This class of potentials also includes $V(x)$~$1/x$.

The use of a hyperbolic profile ~$1/x$ to describe the spatial variation of the refractive index in waveguide optical structures allows one to describe theoretically a variety of effects that seem promising for the development of quantum optical device design technologies [31, 32]. Exact solutions of the equation with a hyperbolic spatial profile and their applications in photonic engineering can be found in [33, 34]. However, the study of wave properties of interfaces between materials with a hyperbolic profile remains a poorly understood problem [35]. Therefore, we take into account such a hyperbolic potential in combination with a short-range potential for the development of theoretical ideas about the localization features of elementary excitations near point defect or flat interface.

Note that the Dirac delta function potential is actively used in the SE to obtain exact solutions [36, 37]. The solutions of quantum harmonic oscillator with additional delta-function potential are studied in [38, 39]. The Dirac delta function potential is used to study soliton scattering by impurities [40, 41], localized nonlinear states in nonlinear photonic crystals [42], nonlinear lattices [43], and periodic layered medium [44, 45]. Equations similar to SEs with the Dirac delta function potential arise in the theory of electronic transitions into a soliton-lattice state [46], interface-induced states with an incommensurate spin-density wave in multilayers [47], and exciton self-trapping in BEC [48].

We also use the Dirac delta function potential to describe theoretically the effect of the interaction of excitations with the interface between different nonlinear optical medium [49–52]. In our paper [53], we described the effect of interface interaction on the field localization at the boundary between the constant-index and graded-index media. Recently, we reported a new type of the surface wave propagating along the surface of the crystal with a hyperbolic profile of the dielectric constant $\varepsilon \sim 1/x$ [54]. Now we generalize the results of [54] in the case of the local interaction of the wave and crystal surface.

In the present paper we expand the theoretical research in this direction using the hyperbolic profile. The main novelty of this paper is that we find an exact analytical solution of the one-dimensional stationary SE with a spatially distributed hyperbolic potential profile $\sim 1/x$ in the presence of a Dirac delta function (point) potential. The obtained solution describes both the ground state and the excited states localized near the defect. We focus on localization features analysis in dependence on the parameters of a hyperbolic potential profile.

This paper is organized as follows. The model equation with a point potential and a hyperbolic well and the boundary conditions are presented in Sec. 2. Exact solutions to the problem describing the localized states and dispersion equation analysis are given in Sec. 3. The conclusions are presented in Sec. 4.

2. MODEL EQUATION WITH A POINT POTENTIAL AND A HYPERBOLIC WELL

Let us consider the one-dimensional SE:

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\frac{\partial^2 \psi}{\partial x^2} + V(x)\psi \,, \tag{1}$$

where $\psi = \psi(x, t)$ is the wave function, x is the spatial coordinate, t is the time, m is an effective mass of elementary excitation, and the potential consisting of two terms and presented as

$$V(x) = U(x) + \Omega(x) \,. \tag{2}$$

The first term in Eq. (2) describes the interaction between the excitation and point defect of the medium planar interface (in the case of homogeneous medium along the interface plane) at the origin $x = 0$ and in the short-range approximation it can be written as

$$U(x) = U_0 \delta(x) \,, \tag{3}$$

where $\delta(x)$ is the Dirac delta-function, U_0 is an intensity of interaction between the excitation and defect (the case of $U_0 > 0$ corresponds to a repelling defect and the case of $U_0 < 0$ corresponds to an attracting defect) [40–45].

The second term in Eq. (2) describes the spatial dependence of the medium characteristics, for example, graded-index optical media. In this paper we suppose that it depends on the spatial coordinate accordingly with a hyperbolic law at one side from defect and it can be considered a constant one at another side from defect. The potential $\Omega(x)$ can be considered as the hyperbolic potential well or the hyperbolic field. Therefore, we propose to write the second term in Eq. (2) as follows

$$\Omega(x) = \begin{cases} -\Omega_{G0}/(1 + x/h), \ x > 0, \\ 0, \ x < 0, \end{cases} \tag{4}$$

where Ω_{G0} is a constant and h is the characteristic distance of a hyperbolic field or potential well. In the general case, we can assume that $\Omega_{G0} \neq 0$. The case of $\Omega_{G0} = 0$ corresponds to the continuous function defined by Eq. (4) and the value of Ω_{G0} corresponds to the intensity of interaction of external field with excitation at the defect (defect intensity).

We consider the stationary states only with the wave function

$$\psi(x,t) = u(x)e^{-iEt}, \tag{5}$$

where E is the energy of the stationary state and $u(x)$ is the spatial component of the wave function. Substituting Eq. (5) into Eq. (1) using Eq. (3), one can obtains

$$Eu = -u''/2m + \{U_0\delta(x) + \Omega(x)\}u = 0. \tag{6}$$

Using Eq. (4), Eq. (6) can be written as two equations at semi-axes as follows

$$u'' - q^2 u = 0, \ x < 0, \tag{7}$$

$$u'' + \{\Omega_{G0}/(1 + x/h) - q^2\}u = 0, \ x > 0, \tag{8}$$

where $q^2 = -2mE$. Therefore, the localized states are formed with $E < 0$, and then $q^2 = 2m|E| > 0$.

The spatial component of the wave function $u(x)$ also satisfies the following boundary conditions at the defect:

$$u(+0) = u(-0) = u_0, \tag{9}$$

$$u'(+0) - u'(-0) = V_0 u_0, \tag{10}$$

where u_0 is the wave amplitude at the defect, and $V_0 = 2mU_0$.

If we demand that $|u(x)| \to 0$ at $|x| \to \infty$, then such continuous and limited solution to the boundary problem (7)–(10) defines the stationary state localized near the point defect (or the medium planar interface) in the presence of the hyperbolic field.

3. LOCALIZED STATES AND DISPERSION EQUATION ANALYSIS

The solution to Eq. (7) can be written as

$$u(x) = u_0 e^{qx}, \; x < 0. \tag{11}$$

We derive the exact solution to Eq. (8) as

$$u(x) = u_0 \frac{W_{l,1/2}(2q(x+h))}{W_{l,1/2}(2qh)}, \; x > 0, \tag{12}$$

where $W_{l,1/2}(x)$ is the Whittaker function with

$$l = \Omega_{G0} h \sqrt{2m/|E|}/2.$$

Note that the solutions defined by Eqs. (11) and (12) satisfy the boundary condition (9) (they satisfy the continuity requirement).

Substituting Eq. (11) and Eq. (12) into the boundary condition (10), one can obtain the following dispersion equation:

$$q_G - q = V_0, \tag{13}$$

where

$$q_G = q\left(1 - \frac{m\Omega_{G0}}{q^2} - \frac{1}{qh}\frac{W_{l+1,1/2}(2qh)}{W_{l,1/2}(2qh)}\right), \tag{14}$$

The dispersion equation (13) can be written as

$$\frac{\Omega_{G0}}{q} + \frac{1}{mh}\frac{W_{l+1,1/2}(2qh)}{W_{l,1/2}(2qh)} + U_0 = 0 \tag{15}$$

The dispersion equation (15) defines the energy of localized (bound) states. This equation couples the localization energy with the value defect intensity and parameters of the hyperbolic potential well: $E = E(U_0, \Omega_{G0}, h)$. The solution of the dispersion equation (15) can be found using numerical methods only or its approximate solution can be found in a special limit case. Note that when the defect intensity turns to zero ($U_0 = 0$), the dispersion equation (13) is simplified and takes the simpler form as $q = q_G$.

Thus, we find the localized states defined by the wave function

$$u(x) = u_0 \begin{cases} e^{qx}, \; x < 0, \\ W_{l,1/2}(2q(x+h))/W_{l,1/2}(2qh), \; x > 0, \end{cases} \tag{16}$$

with the localization energy defined by the dispersion equation (15).

We find approximate solution to dispersion equation (15) in the limit case of $qh << 1$. In this case we can present Eq. (15) as

$$q^2 \frac{q_0}{m} + qU_0 + \frac{\Omega_{G0}}{2} = 0 \tag{17}$$

where

$$q_0 = \lim_{y \to 0} \frac{W_{l+1,1/2}(y)}{yW_{l,1/2}(y)},$$

which does not depend on energy E at small distances h. Moreover, meaning that $l \to 0$ at $h \to 0$ we derive that $q_0 \approx 1$ and Eq. (17) can be written as

$$q^2 / m + 2qU_0 + \Omega_{G0} / 2 = 0 \tag{18}$$

First we consider the limit case of $q^2 << m\Omega_{G0}$ and $q << mU_0$, which corresponds to the strong intensity of the hyperbolic field near defect (shallow potential well) and strong intensify of defect interaction with excitation. Then we find the approximate solution to the dispersion equation (18)

$$q = -\frac{\Omega_{G0}}{2U_0}, \tag{19}$$

from which we can obtain the localization energy in explicit form

$$E = -\frac{1}{2m}\left(\frac{\Omega_{G0}}{2U_0}\right)^2. \tag{20}$$

In order to get a positive value of the localization distance $l = 1/q$ from Eq. (19) it follows that localized state with energy ($E < 0$) defined by Eq. (20) exists near attractive defect with $U_0 < 0$.

Next we consider the limit case of $\Omega_{G0} << q^2/m$ and $\Omega_{G0} << qU_0$, which corresponds to the weak intensity of the hyperbolic field near defect (shallow potential well). Then we find the approximate solution to the dispersion equation (18)

$$q = -mU_0\,, \tag{21}$$

from which we can obtain the localization energy in explicit form

$$E = -\frac{mU_0^2}{2}\,. \tag{22}$$

In order to get a positive value of the localization distance $l = 1/q = m|U_0|$ from Eq. (21) it follows that the localized state with energy ($E < 0$) defined by Eq. (22) exists near attractive defect with $U_0 < 0$. It is important to note that the derived here Eqs. (21) and (22) coincide with the classical solution for the Schrödinger equation with delta-function potential [55], which corresponds to the symmetrically localized wave function

$$u(x) = u_0 \exp(-m|U_0 x|)\,.$$

Therefore, from the model we proposed we obtain the well known results for the SE with delta-function potential.

We find the exact solution to the dispersion equation (18)

$$q = -mU_0 \pm \sqrt{m(mU_0^2 - 2\Omega_{G0})}\,, \tag{23}$$

from which we can obtain the localization energy in explicit form

$$E = -\frac{1}{2m}\left(mU_0 \pm \sqrt{m(mU_0^2 - 2\Omega_{G0})}\right)^2. \tag{24}$$

The signs in Eqs. (23) and (24) should be chosen so that the length of the localization must be a positive one. In the case of $\Omega_{G0} = mU_0^2/2$, Eqs. (23) and (24) transform into Eqs. (21) and (22), respectively.

It must be noted that the wave function attenuates monotonically near the attractive defect a half-space with a constant potential when $\Omega_{G0} < mU_0^2/2$. In the opposite case of $\Omega_{G0} > mU_0^2/2$ the value of q defined by Eq. (23) becomes a complex one:

$$q = -mU_0 \pm i\gamma, \tag{25}$$

where

$$\gamma = \sqrt{m(2\Omega_{G0} - mU_0^2)}\,.$$

Therefore, substituting Eq. (25) into Eq. (11), we obtain that the wave function attenuates with oscillations a half-space with a constant potential and it can be written as

$$u(x) = u_0 e^{m|U_0|x} \cos(\gamma x - \varphi)/\cos(\varphi),\ x < 0, \tag{26}$$

where the phase φ can be found from the boundary condition (10).

We obtain numerically the solution to the dispersion equation (15) in the general case, which is plotted in Fig. 1. We call the ground state the localized state described by the wave function with the one maximum and a monotonic decreasing with the distance from the defect. The analysis of the numerical solution shows that the localization energy of the ground state monotonically decreases with an increase in defect intensity. It is important to note that the localized states exist in the cases of both signs of the defect. The localization energy near an attractive defect is higher than in the case of a repulsive defect. We find that increasing the intensity of the hyperbolic field near the defect (hyperbolic well depth) Ω_{G0} leads to the growing of the localization energy (Fig. 1*a*). The same effect is caused by increasing the characteristic distance of the hyperbolic field h (Fig. 1*b*).

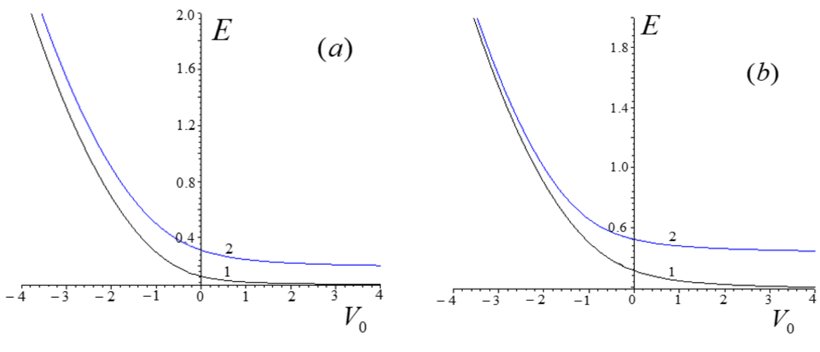

Fig. 1 – Dependence of the localization energy E on the defect intensity V_0 calculated using Eq. (15) with $m = 1$ (in dimensionless conventional units): *a*) $h = 1$, $\Omega_{G0} = 0.5$ (1), $\Omega_{G0} = 1$ (2); *b*) $\Omega_{G0} = 1$, $h = 1$, (1), $h = 3$ (2).

Figure 2 demonstrate the effect of parameters of the defect and the hyperbolic field on the spatial profiles of the wave function of the ground state. The maximum of the wave function is located in the zone of action of the hyperbolic field. However, it is located at the defect in the case of high absolute value of the defect intensity in the case of attractive defect (Fig. 2*a*). The depth of the field localization (localization length) in the hyperbolic well is greater than in a half-space with a constant potential.

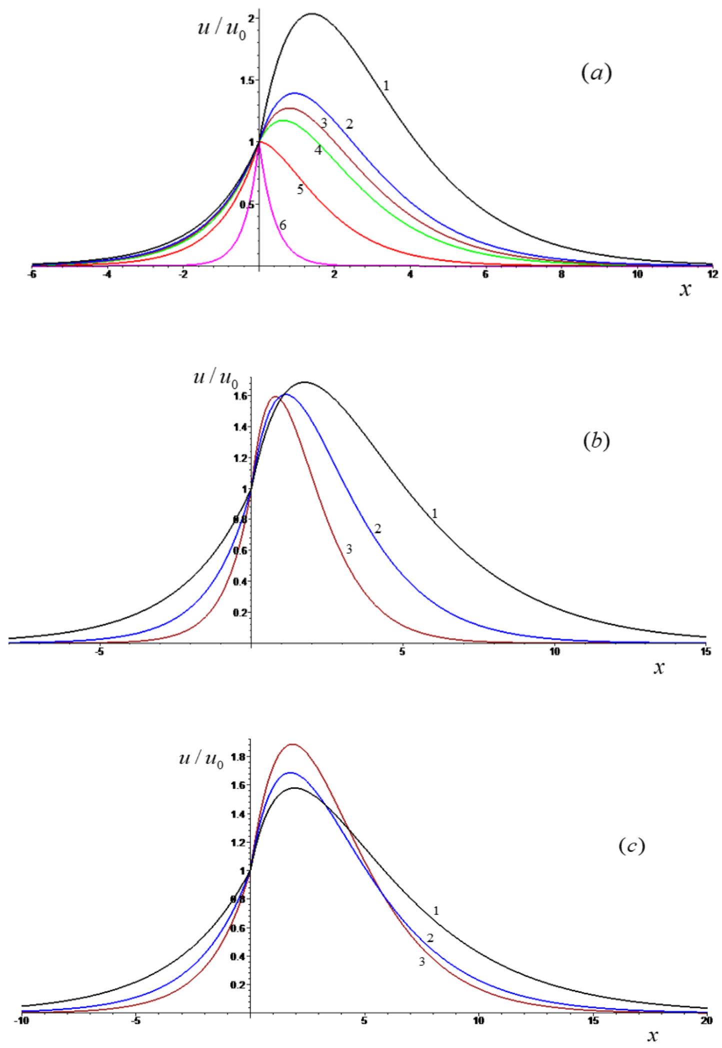

Fig. 2 – Spatial distribution of the wave function of the ground state defined by Eq. (16) with m = 1, *a*) h = 1, Ω_{G0} = 1, V_0 = 1 (1), V_0 = 0.2 (2), V_0 = 0 (3), V_0 = – 0.2 (4), V_0 = – 1 (5), V_0 = – 5 (6); *b*) h = 1, V_0 = 0.5, Ω_{G0} = 0.5 (1), Ω_{G0} = 1 (2), Ω_{G0} = 2 (3); *c*) Ω_{G0} = 0.5, V_0 = 0.5, h = 0.5 (1), h = 1 (2), h = 2 (3).

The height of the maximum of the wave function decreases with decreasing intensity of the repulsive defect U_0. The height of the maximum of the wave function decreases with increasing absolute value of the intensity of the attractive defect. The localization length is also reduced.

An increase in the intensity of the hyperbolic filed (depth of the hyperbolic potential well) Ω_{G0} leads to a slight decrease in the height of the maximum of the wave function of the ground state (Fig. 2*b*). The maximum position of the wave function is shifted closer to the defect and localization length decreases more significantly in this case.

An increase in the characteristic distance of the hyperbolic field h leads to an increase in the height of the maximum of the wave function of the ground state (Fig. 2*c*). Depth of the field localization slightly decreases and the position of the maximum of the wave function remains almost unchanged in this case.

It is important that excited localized states exist in the considered system with the energy defined by a series of solutions of the dispersion equation (15) at the same fixed values of all parameters $E_n(U_0, \Omega_{G0}, h)$, where $n = 0, 1, 2, \ldots$ is the number of excited localized state. The value of $n = 0$ corresponds to the ground localized state described below in details. The wave function of the ground state is characterized by one maximum and monotonic decaying with a distance from the defect. The values of $n = 1, 2, \ldots$ correspond the wave function with two, three, … extrema respectively.

The localization energies of the excited states E_n obtained by numerically solving the dispersion equation (15) in dependence on its number n are presented in Fig. 3. The energies of localized states with small numbers of n grow with increasing absolute value of the defect intensity. However, for large level n of excited states, their localization energies coincide.

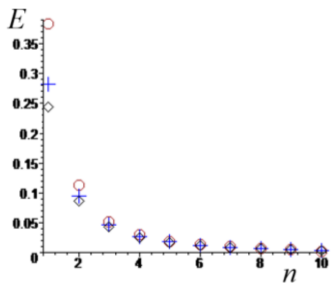

Fig. 3 – The dependence of the localization energy E_n on the level number n calculated using Eq. (15) with $m = 1$, $h = 1$, $\Omega_{G0} = 1$, $V_0 = 1$ (diamond markers), $V_0 = 0.3$ (cross markers), $V_0 = -0.5$ (circle markers).

The wave function profiles of the excited localized states are presented in Fig. 4. The excited states are characterized by the formation of oscillations in the region of action of the hyperbolic field. The number of extrema of the wave function and their intensites in this zone increase with the growth of the excited state number. The depth of the field localization in the region of the constant potential increases with the growth of the excited state number not as significantly as in the region of action of the hyperbolic field.

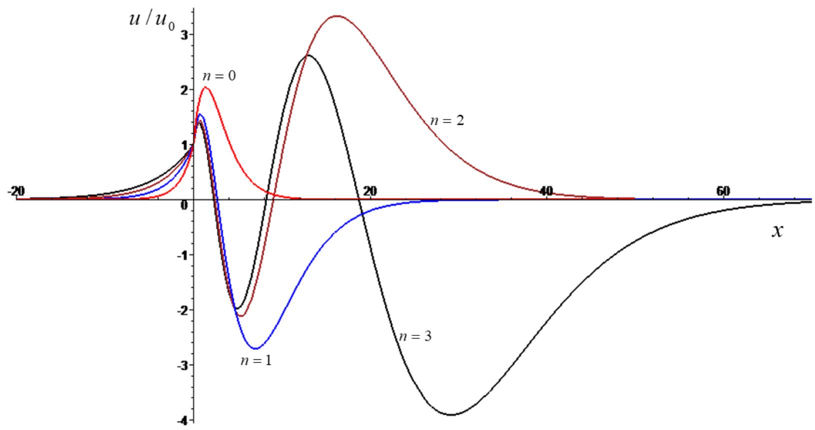

Fig. 4 – The spatial distribution of the wave function of the excited localized states defined by Eq. (16) with $m = 1$, $h = 1$, $\Omega_{G0} = 1$, $V_0 = 1$.

The excited localized states in nonlinear optics or waveguide optics correspond to guided waves propagating along the interface between two optical media [44, 45]. The ground localized state corresponds to the surface wave in this case [56, 57]. The energy of localized states corresponds to the propagation constant or the effective refractive index [58, 59].

4. CONCLUSIONS

We considered the one-dimensional model based on the stationary Schrödinger equation with the potential consisting of point interaction potential and spatially distributed hyperbolic field (or potential well). We found exact analytical solutions to the formulated problem describing the stationary states localized near the point defect in the presence of the hyperbolic field.

We analyzed dispersion equation numerically and found that the localization energy of the ground state monotonically decreases with an increase in defect intensity. The localized states exist near both attractive and repulsive defects. The localization energy near an attractive defect is higher than in the case of a repulsive defect.

We found the approximate solutions to the dispersion equation in the limit cases in explicit analytical form and showed that a localized state exists near the attractive defect. The classical solution for linear Schrödinger equation with delta-function potential was obtained in the limit case of a weak intensity of the hyperbolic field near defect. We also showed the possibility of existence of the wave function attenuating with oscillations in a half-space with a constant potential.

We analyzed the influence of parameters of the defect and the hyperbolic field on the spatial profiles of the wave function of the ground state. The analysis showed that the depth of the field localization in the hyperbolic well is greater than in a half-space with a constant potential. The height of the maximum of the wave function decreases with decreasing intensity of the repulsive defect and with increasing absolute value of the defect intensity in the case of the attractive defect.

We found that the dispersion equation has a series of solutions at fixed values of the system parameters corresponding to excited localized states. Excited states are characterized by the formation of oscillations in the region of action of the hyperbolic field. The number of oscillations corresponds to the level of excited state.

We hope that our results can be applied in the waveguide optics, photonics, quantum solid state electronics and expand our understanding of the mechanisms of formation of localized states near defects in solids.

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