SPATIOTEMPORAL DYNAMICS OF BOSE-EINSTEIN CONDENSATES WITH SPATIOTEMPORALLY VARYING TWO- AND THREE-BODY INTERACTIONS

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Abstract. We study the spatiotemporal dynamics of Bose-Einstein condensates (BECs) with spatiotemporally varying two- and three-body interactions and without an external trapping potential. The cubic and quintic terms in the Gross-Pitaevskii (GP) equation play the roles of external trapping potentials. Spatiotemporal modulation of the two- and three-body interactions can be of great importance in BEC field. But it is the subject of relatively fewer studies. Through theoretical analyses we obtain the equilibrium points (centers and saddles in phase space) of the unperturbed repulsive and attractive cases. With the heteroclinic solution of the unperturbed repulsive system, we theoretically construct the general solution of the 1st-order perturbed equation of the system. By using the boundedness conditions of the general solution we obtain the Melnikov chaos criterion predicting the existence of Smale-horseshoe chaos in the system. With a set of system parameters satisfied the Melnikov chaos criterion, numerical simulations show that the system is in a chaotic state. For BECs that do not satisfy the Melnikov chaos criterion, numerical simulations demonstrate that the modulating frequency can be an effective controlling parameter for the spatiotemporal dynamics of the BECs.

Key words: Bose-Einstein condensates, Spatiotemporal dynamics, Chaos.

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1. INTRODUCTION

During the past decade of Bose-Einstein condensate (BEC) research, various methods have been developed to control and manipulate BECs. One of the most important and effective methods is the so-called Feshbach resonance (FR) technique [1–6], through which researchers can effectively control the s-wave scattering length a_s (atom-atom interaction) and then the sign and value of the nonlinear parameter according to demand. A varying s-wave scattering length can exert a significant impact on the dynamical behaviors of BECs and will inevitably trigger various interesting physical phenomena [1–6]. As is well known, in early stage the FR was usually used to realize a time-dependent s-wave scattering length to control the temporal dynamical behaviors of BECs [7–12]. Spin-aligned hydrogen and deuterium were generally predicted to be stable at very low temperature, but the FR provide exceptions to this general rule with a temporally modulating scattering length [7]. By a suitable choice

of the magnetic-field strength inducing the FR the interatomic interaction can be made effectively repulsive and for a large range of temperatures the thermalizing elastic collisions are much more frequent than inelastic two- and three-body collisions [8]. In presence of a temporally modulating scattering length, the macroscopic quantum tunneling [9–12], generation of soliton trains [13], and collapse in BECs were theoretically and numerically investigated [14].

Later, advances in experimental technique make it possible to tune the scattering length of BEC atoms through an optical way. Inducing an optical FR *via* stimulated Raman [15] and tuning the scattering length with the optically induced FR were reported [16]. The importance of optical FR lies in the fact that it can produce a spatially varying s-wave scattering length and then a spatially modulated atom-atom interaction, with which the dynamics of matter-wave solitons in BECs is investigated [17–21]. When the spatial modulation strength of the interatomic interaction is strong enough, the BEC loses it superfluidity completely [22]. In our previous papers, spatial chaos in BECs with spatially modulated atom-atom interaction was analytically and numerically explored [23, 24].

It is worth noting that the success of modulating the atom-atom interactions in time or space separately provides the possibility of simultaneously modulating the atom-atom interactions in time and space through technological integration. Simultaneous spatiotemporal modulation of the atom-atom interactions can provide more options for controlling and manipulating BECs and will inevitably have unique advantages. BECs with time- and space-dependent atom-atom interactions may have very interesting properties but have been the subject of relatively fewer studies. So far, we have only found a very small number of papers that have conducted researches on BECs with time- and space-dependent two-body interactions [25–29]. And we have not found any papers in which the two- and three-body interactions are spatiotemporally modulated simultaneously. Therefore, there are still many open problems to be considered in BEC field. Here in this letter we will introduce spatiotemporally modulated two- and three-body interactions to the BEC and check the spatiotemporal dynamics of the system.

2. THEORETICAL ANALYSIS

We start by considering the normalized one-dimensional (1D) mean-field Gross-Pitaevskii (GP) equation [30–33]

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + V_{ext}\Psi + g_1\left|\Psi\right|^2\Psi + g_2\left|\Psi\right|^4\Psi, \tag{1}$$

where V_{ext} is the external trapping potential, and g_1 and g_2 are the coefficients of the cubic and quintic terms corresponding to the two- and three-body interactions,

respectively. The cubic term with a coefficient periodically modulated as a function of the time coordinate t may, in a certain sense, play the role of an external trapping potential [14]. Ref. [17] also points out that the spatial modulation of the cubic coefficient, combining the dc and ac parts, may act as a nonlinear optical lattice (NOL). In this letter we will consider the case in which the two- and three-body nonlinear coefficients are periodically modulated as functions of the spatial coordinate x and temporal coordinate t. Undoubtedly, the cubic and quintic terms can play the roles of NOLs. We mainly check the effect of the two- and three-body interactions on the spatiotemporal dynamics of the BEC. Therefore, in the subsequent research of this work we will remove the external potential V_{ext} and the GP equation is reduced to

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2}\frac{\partial^2\Psi}{\partial x^2} + g_1\left|\Psi\right|^2\Psi + g_2\left|\Psi\right|^4\Psi. \tag{2}$$

The main purpose of this work is to study the spatiotemporal dynamics of the BEC (namely, the spatiotemporal distribution of BEC atoms), so we take the wave function in the following form

$$\Psi = A(\zeta)e^{iB(\zeta)} \tag{3}$$

with a spatiotemporal variable $\zeta = ax + bt$. a and b are two undetermined real constants. Inserting Eq. (3) into Eq. (2) leads to the following ordinary differential equation

$$\frac{d^2A}{d\zeta^2} - A(\frac{dB}{d\zeta})^2 - \frac{2Ab}{a^2}\frac{dB}{d\zeta} - \frac{2g_1}{a^2}A^3 - \frac{2g_2}{a^2}A^5 + i[\frac{2b}{a^2}\frac{dA}{d\zeta} + 2\frac{dA}{d\zeta}\frac{dB}{d\zeta} + A\frac{d^2B}{d\zeta^2}] = 0. \tag{4}$$

The above equation is very complicated and it is difficult to seek its exact solution. We consider the case in which the imaginary part of Eq. (4) is deleted. For this purpose, we set

$$\frac{2b}{a^2}\frac{dA}{d\zeta} + 2\frac{dA}{d\zeta}\frac{dB}{d\zeta} + A\frac{d^2B}{d\zeta^2} = 0. \tag{5}$$

Because $|A|^2$ is the atom number density usually varying with the space and time, which means that A is not a constant and $\frac{dA}{d\zeta} \neq 0$. After carefully inspecting Eq. (5), we find that as long as the condition $(b + a^2\frac{dB}{d\zeta}) = 0$ is satisfied, Eq. (5) can be satisfied too. And then the imaginary part of Eq. (4) can be eliminated. Subsequently, Eq. (4) can be reduced to

$$\frac{d^2A}{d\zeta^2} + \frac{b^2}{a^4}A - \frac{2g_1}{a^2}A^3 - \frac{2g_2}{a^2}A^5 = 0. \tag{6}$$

As mentioned above, we consider the temporally and spatially modulated two- and three-body interactions and the nonlinear coefficients of the cubic and quintic terms are periodically modulated as functions of the spatiotemporal variable ζ. We select

the nonlinear coefficients bearing the common form

$$g_1 = g_{10} + g_{11}\sin(\omega\zeta),\ \ g_2 = g_{20} + g_{21}\sin(\omega\zeta) \tag{7}$$

Here g_{10} and g_{20} are the nonlinear parameters without modulations. g_{10} and g_{20} represent the intensities of the modulations. ω is the frequency of modulation. In experiments, nonlinear coefficients periodically modulated as functions of the spatial coordinate x [34] or temporal coordinate t [14] through sine or cosine function can be easily achieved with the FR technique. Applying Eq. (7) to Eq. (6) lead to

$$\frac{d^2A}{d\zeta^2} + \frac{b^2}{a^4}A - \frac{2}{a^2}[g_{10} + g_{11}\sin(\omega\zeta)]A^3 - \frac{2}{a^2}[g_{20} + g_{21}\sin(\omega\zeta)]A^5 = 0. \tag{8}$$

To perform perturbation analyses, we set $g_{10} > g_{11} \sim g_{20} \sim g_{21}$ and then Eq. (8) can be rewritten as

$$\frac{d^2A}{d\zeta^2} + \frac{b^2}{a^4}A - \frac{2}{a^2}g_{10}A^3 = \frac{2}{a^2}g_{11}\sin(\omega\zeta)A^3 + \frac{2}{a^2}[g_{20} + g_{21}\sin(\omega\zeta)]A^5. \tag{9}$$

The above equation is a typical Duffing equation describing a driven Duffing oscillator. Its unperturbed Hamiltonian is

$$H(A,\dot{A}) = \frac{1}{2}(\dot{A})^2 + \frac{b^2}{2a^4}A^2 - \frac{g_{10}}{2a^2}A^4. \tag{10}$$

When $H(A,\dot{A}) = 0$, one can obtain the equilibrium points of the unperturbed Hamiltonian system. Let

$$f(A) = \frac{d^2A}{d\zeta^2} = -\frac{b^2}{a^4}A + \frac{2g_{10}}{a^2}A^3. \tag{11}$$

(1) For a repulsive BEC with $g_{10} > 0$, Eq. (10) tells that the unperturbed system has three equilibrium points: $(A,\dot{A}) = (0,0)$ and $(A,\dot{A}) = (\pm\frac{b}{a\sqrt{2g_{10}}},0)$. When $(A,\dot{A}) = (0,0)$, $\frac{df(A)}{dA} = -\frac{b^2}{a^4} < 0$, denoting that $(A,\dot{A}) = (0,0)$ is a center; [35] while $(A,\dot{A}) = (\pm\frac{b}{a\sqrt{2g_{10}}},0)$, $\frac{df(A)}{dA} = \frac{2b^2}{a^4} > 0$, denoting that $(A,\dot{A}) = (\pm\frac{b}{a\sqrt{2g_{10}}},0)$ are two saddles [35]. Theoretical analyses reveal that the unperturbed repulsive system bears the following solutions [36]

$$A = \pm\Gamma\mathrm{sn}[\kappa(\zeta-\zeta_0),m],\ \Gamma = \frac{akm}{\sqrt{g_{10}}},\ \kappa = \frac{b}{a^2\sqrt{1+m^2}},\ 0 \le m \le 1. \tag{12}$$

Here sn is a Jacobian elliptic function with a modulus m and ζ_0 is the initial spatiotemporal coordinate. When $m = 1$, solutions (12) can be simplified to heteroclinic ones expressed as

$$A = A_0 = \pm\frac{b}{a\sqrt{2g_{10}}}\tanh[\frac{b}{\sqrt{2}a^2}(\zeta-\zeta_0)]. \tag{13}$$

In Fig. 1 we plot the phase portrait of the unperturbed repulsive system. The two phase orbits passing through the two saddles correspond to the heteroclinic solutions

and form the separatrix. In our perturbation analysis, we will pour our attention on the system dynamical behaviors close to the separatrix.

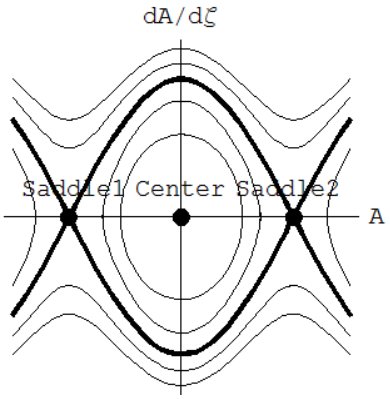

Fig. 1 – Phase portrait of the unperturbed repulsive system with one center $(A,\dot{A})=(0,0)$ and two saddles $(A,\dot{A})=(\pm\frac{b}{a\sqrt{2g_{10}}},0)$.

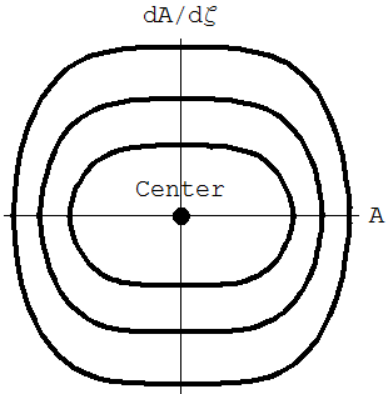

Fig. 2 – Phase portrait of the unperturbed attractive system with only one center $(A,\dot{A})=(0,0)$.

(2) For an attractive BEC with $g_{10}<0$, the unperturbed system has only one equilibrium point: $(A,\dot{A})=(0,0)$. When $(A,\dot{A})=(0,0)$, $\frac{df(A)}{dA}=-\frac{b^2}{a^4}<0$, denoting that $(A,\dot{A})=(0,0)$ is a center [35]. And no saddles exist for $g_{10}<0$. In Fig. 2 we plot the phase portrait of the unperturbed attractive system. Figure 2 shows that only one center appears in the phase space. The solutions of the unperturbed attractive system can be expressed as [36]

$$A=\pm m|\frac{b}{a^2}|\sqrt{-\frac{a^2}{g_{10}(1-2m^2)}}\,\mathrm{cn}[\frac{|\frac{b}{a^2}|}{\sqrt{1-2m^2}}(\zeta-\zeta_0),m]. \tag{14}$$

Here cn is a Jacobian elliptic function with a modulus $m \in (0, 1/\sqrt{2})$. Because the modulus $m \neq 1$, the above solutions cannot be further simplified into explicit forms. Subsequently, it is difficult to perform perturbation analysis on the attractive system.

Next, we will perform perturbation analysis on the repulsive BEC to find the Melnikov chaos criterion determining the existence of chaos in the system. According to the direct perturbation method [37, 38], we expand the general solution of Eq. (9) to the first order

$$A = A_0 + A_1, \ \ |A_0| \gg |A_1|. \tag{15}$$

After applying Eq. (15) to Eq. (9), we obtain the zeroth- and 1st-order equations. The zeroth-order equation is just the unperturbed equation of the repulsive system and the 1st-order equation can be expressed as

$$\frac{d^2 A_1}{d\zeta^2} + \frac{b^2}{a^4} A_1 - \frac{6}{a^2} g_{10} A_0^2 A_1 = \varepsilon, \tag{16}$$

$$\varepsilon = \frac{2}{a^2} g_{11} \sin(\omega\zeta) A_0^3 + \frac{2}{a^2} [g_{20} + g_{21} \sin(\omega\zeta)] A_0^5.$$

The general solution of the 1st-order equation bears the following form [37, 38]

$$A_1 = G_2 \int_{C_1}^{\zeta} G_1 \varepsilon d\zeta - G_1 \int_{C_2}^{\zeta} G_2 \varepsilon d\zeta, \tag{17}$$

$$G_1 = \frac{dA_0}{d\zeta}, \ \ G_2 = G_1 \int (G_1)^2 d\zeta.$$

Here G_1 and G_2 are two linearly independent solutions of Eq. (16) when $\varepsilon = 0$. C_1 and C_2 are two arbitrary constants determined by initial conditions. Careful calculations reveal that A_1 will tend towards infinity as $\zeta \to \pm\infty$, which denotes that A_1 is Lyapunov unstable [37]. But this kind of instability can be well controlled, as long as the conditions

$$\Pi_{\pm} = \lim_{\zeta \to \infty} \int_{C_1}^{\zeta} G_1 \varepsilon d\zeta = 0 \tag{18}$$

are satisfied [37, 38]. It is interestingly found that the above conditions contain the famous Melnikov function

$$M(\zeta_0) = \Pi_+ - \Pi_- = \int_{-\infty}^{+} G_1 \varepsilon d\zeta = 0. \tag{19}$$

Substituting G_1 and ε into Eq. (19) and performing the integration, we reach

$$M(\zeta_0) = J \cos(\omega\zeta_0) = 0, \tag{20}$$

$$J = \left[-a g_{11} (\omega^2 a^4 - 4b^2) + \frac{g_{21} (46b^4 - 20b^2\omega^2 a^4 + \omega^4 a^8)}{30 a g_{10}} \right].$$

As is well known, as long as the Melnikov function $M(\zeta_0)$ has a simple zero at some point, then the system has chaotic solutions. For the system considered in this letter, a simple zero of $M(\zeta_0)$ means the existence of spatiotemporal chaos in the system. To ensure that the Melnikov function $M(\zeta_0)$ has at least one simple zero, the following condition

$$\frac{dM(\zeta_0)}{d\zeta_0} = -J\sin(\omega\zeta_0) \neq 0 \tag{21}$$

must be satisfied. Usually, conditions (20) and (21) are called Melnikov chaotic criterion predicting the onset of Smale-horseshoe chaos. By comparing the above conditions, it can be seen that as long as $J \neq 0$ and $\zeta_0 = \frac{2n+1}{2\omega}\pi$ with n being an integer, the Melnikov chaotic criterion can be strictly satisfied. For a set of parameters $n = 0,\ a = 1.9, \mathrm{b} = 1.5, \mathrm{g}_{10} = 3.5, \mathrm{g}_{11} = \mathrm{g}_{20} = g_{21} = 0.12, \omega = 0.55$, we obtain $J = 1.19589$, $A(0) = -0.204502,\ \dot{A}(0) = 0.0464923$ through calculations. What should be pointed out is that $g_{10} = 29\frac{1}{6}g_{11}$ and $g_{11} = g_{20} = g_{21}$. These parameters and initial conditions can ensure that both the perturbation conditions and Melnikov chaotic criterion are satisfied. With these parameters, by using MATHEMATICA, we numerically solve Eq. (9) and plot a phase portrait in Fig. 3. It is clear that Fig. 3 demonstrates a chaotic phase portrait with self-similar structures, which is also one of the typical characteristics of chaos.

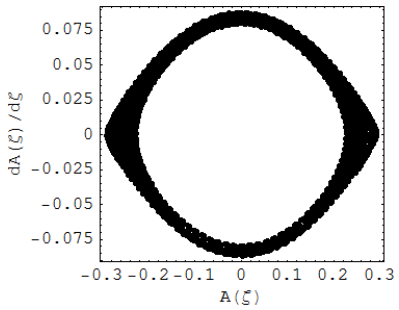

Fig. 3 – Phase portrait of the perturbed attractive system with $n = 0,\ a = 1.9, \mathrm{b} = 1.5, \mathrm{g}_{10} = 3.5, \mathrm{g}_{11} = \mathrm{g}_{21} = \mathrm{g}_{20} = 0.12, \omega = 0.55$, $J = 1.19589$, $A(0) = -0.204502,\ \dot{A}(0) = 0.0464923$.

If the dc and ac components of the two- and three-body nonlinear interactions are of the same order of magnitude, the perturbation conditions will not be satisfied and the above theoretical methods cannot be used to study the spatiotemporal dynamical behaviors of the BEC. That is to say, the Melnikov chaos criterion will be invalid. In this case, we can study the spatiotemporal dynamical behaviors of the system through numerical simulation method. Considering a repulsive BEC with the system parameters and initial conditions being set as $a = 2.3,\ b = 2.2,\ g_{10} = g_{11} = 0.5,\ g_{21} =$

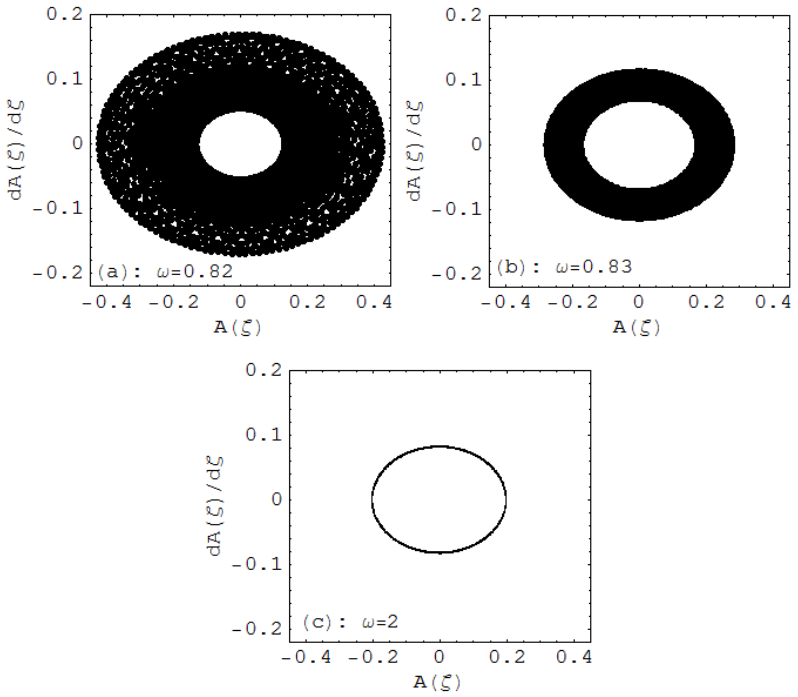

Fig. 4 – Phase portraits of the perturbed attractive system with $a = 2.3$, $b = 2.2$, $g_{10} = g_{11} = 0.5$, $g_{21} = g_{20} = 0.4$, $\omega = 0.82$ and different values of the modulating frequency ω.

$g_{20} = 0.4, A(0) = 0.2,\ \dot{A}(0) = 0$, we plot a series of phase portraits in Fig. 4 for different values of ω. From Fig. 4(a) with $\omega = 0.82$, one can see a typical chaotic phase portrait. This indicates that the BEC is in a chaotic spatiotemporal state. When the value of ω is increased to 0.83, Fig. 4(b) shows that the system is still in a chaotic state, but the phase orbits are densely distributed on the phase plane and the area occupied by the phase orbits is significantly reduced. In the process of numerical simulations, we find that with the increase of ω the area occupied by the phase orbits on the phase plane continues to decrease, showing a trend of phase space motion evolving towards a single-periodic one. Finally, when the value of ω is increased to 2, only one closed orbit appears on the phase plane, indicating that the chaos has completely disappeared and the BEC enters a single-periodic spatiotemporal state. Further numerical simulations reveal that when ω is an integer great than 2, the BEC is always in a series of single-periodic states under the above selected parameters and initial conditions.

It is interesting that, keeping the absolute values of $(g_{10}, g_{11}, g_{20}, g_{21})$ unchangeable and with their signs being the following choices $(-,-,-,-)$, $(+,+,-,-)$, $(-,-,+,+)$, as long as the modulating frequency is greater than 2, the chaos will be

completely suppressed and the BEC will always be in a series of single-periodic states under the above selected parameters and initial conditions.

It has been demonstrated that the modulating frequency ω can be an effective controlling parameter for the spatiotemporal dynamics of the BEC with spatiotemporally varying two- and three-body interactions in certain ranges of parameters and initial conditions. That is to say, in experiments one can use the modulating frequency to avoid or generate chaos according to demand.

3. CONCLUSION

In summary, we have studied the spatiotemporal dynamics of BECs with spatiotemporally varying two- and three-body interactions and without an external trapping potential. One can treat such BECs as free ones. But for BECs only involving two-body interaction, the cubic term with a coefficient periodically modulated as a function of the time t [14] or the spatial coordinate x [17] actually plays the role of a NOL, namely the external trapping potential. In this letter, the two- and three-body interactions are considered and represented by the cubic and quintic terms with coefficients periodically modulated as functions of the spatiotemporal coordinate, respectively. Undoubtedly, such cubic and quintic terms can also play the roles of external potentials. The spatiotemporally varying interactions provide more ways for studying BECs. It will inevitably trigger some new research hot spots and promote the research and application of BECs. But at present there are very few studies on BECs with spatiotemporally varying interactions. We firmly believe that this must be a new hot topic in the BEC field, which is the motivation for our research. The equilibrium points (centers and saddles in phase space) of the unperturbed repulsive and attractive BECs are theoretically obtained. Through perturbation analysis on the perturbed repulsive BEC, the Melnikov chaos criterion predicting the existence of Smale-horseshoe chaos is found. For a set of parameters that satisfy the Melnikov chaos criterion, numerical simulations indicate that the system is in a state of spatiotemporal chaos. For BECs that do not satisfy the Melnikov chaos criterion, no matter which combination of the signs of the two- and three-body nonlinear parameters is taken, as long as the modulating frequency is greater than 2, the BECs will always enter single-periodic states from chaotic ones under the selected parameters and initial conditions. This indicates that the modulating frequency can be an effective controlling parameter for the spatiotemporal dynamics of BECs.

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