ON EXACT SOLUTIONS FOR THE VIBRATIONS OF A NONLINEAR CONTINUOUS MODEL STRING

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Abstract. In this paper, we consider a nonlinear boundary value problem for the vibrations of a nonlinear continuous model string, which describes the standing vibrations of a finite, continuous, and nonlinear string. Under the assumption that the nonlinear string is proportional to the derivative of the stress with respect to the strain, exact solutions of this problem are provided. To obtain the solution, the given nonlinear partial differential equation is transformed into a first-order nonlinear system by introducing the dependent variables transformation $u=y_x$, $v=y_t$. The modified method of separation of variables is applied to the system and the solutions for t and x are written as general implicit formulas. It is shown that the technique used here offers advantages in finding solutions of such problems.

Key words: Boundary value problem, vibrations of a nonlinear continuous model string, method of separation of variables, exact solutions.

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1. INTRODUCTION

In [1], the author studied the following nonlinear boundary value problem:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2}-\left[1+\epsilon\frac{\partial y}{\partial x}\right]^{\alpha}\frac{\partial^2 y}{\partial x^2} &= 0,\ 0<x<1,\\ y(x,0)=y_0(x),\ \frac{\partial y}{\partial t}(x,0) &= 0,\\ y(0,t)=y(1,t) &= 0, \end{cases} \tag{1}$$

which describes the standing vibrations of a finite, continuous, and nonlinear string. The nonlinearity studied, $\left[1+\epsilon\frac{\partial y}{\partial x}\right]^{\alpha}$, was motivated by the work of Fermi, Pasta, and Ulam (1955) [2], where they reported on numerical studies of the "equipartition of energy" in nonlinear systems. Zabusky [1] has given an accurate estimate of the time interval in which solutions of the nonlinear string equation $\frac{\partial^2 y}{\partial t^2}-\left[1+\epsilon\frac{\partial y}{\partial x}\right]^{\alpha}\frac{\partial^2 y}{\partial x^2}=0$ exist. The same result based on an estimate provided in Zabusky's research has been proved by Lax [3].

Due to the longitudinal vibrations of a nonlinear string, Zabusky [1] introduced the finite difference method to show that the numerical solution generated by a finite difference method converges to the analytical solution as the grid size of the finite difference method converges to zero.

The nonlinear models related to the vibrating string have also been proposed in the following forms:

1. The nonlinear model that proposed by Riesz and Nagy [4]:

$$\frac{M}{l}\frac{\partial^2 u}{\partial t^2}-T_0\frac{\partial}{\partial x}\left(\frac{\frac{\partial u}{\partial x}}{\sqrt{1+(\frac{\partial u}{\partial x})^2}}\right)=f, \tag{2}$$

   where the tension T_0 of the string is constant and does not depend on the strain and l_0 is the proper length of the string with mass M.

2. The nonlinear model which has been introduced in [5, 6]:

$$\frac{M}{l}\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left[\varphi\left(l\sqrt{1+(\frac{\partial u}{\partial x})^2}-l_0\right)\frac{\frac{\partial u}{\partial x}}{\sqrt{1+(\frac{\partial u}{\partial x})^2}}\right]=f, \tag{3}$$

   where the function φ defines the the stress-strain law $S=\varphi(\tau),\tau\geq 0$.

A special case of this second model is

$$\frac{\partial^2 u}{\partial t^2}-\sigma l\frac{\partial^2 u}{\partial x^2}+\sigma l_0\left(1+(\frac{\partial u}{\partial x})^2\right)^{-\frac{3}{2}}\frac{\partial^2 u}{\partial x^2}=f. \tag{4}$$

The existence and uniqueness Theorems of Eq. (3) subject to the Cauchy-Dirichlet conditions have been proved, and some numerical experiments are also described in [7].

As highlighted in [8], the mathematical discretization is often not satisfactory from an analytical perspective, and does not provide explicit solutions to the problems. Additionally, we note that in [1, 3] and in many such works, the solutions of such problems are not expressible explicitly in terms of elementary functions.

This paper is devoted to applying an adaptation of the method of separation of variables for finding the exact solution of the nonlinear boundary value problem (1). It will be shown that the given nonlinear equation $\frac{\partial^2 y}{\partial t^2}-\left[1+\epsilon\frac{\partial y}{\partial x}\right]^{\alpha}\frac{\partial^2 y}{\partial x^2}=0$ can always be decoupled into a first-order nonlinear system without increase the order of the differential equation by introducing the dependent variables transformation $u=y_x,\ v=y_t$. The resulting equations may be solved.

Our approach can be applied to address nonlinear models of the vibrating string like those mentioned above [4–6], as well as various problems, including the $(2+1)-$ dimensional generalized modified dispersive water-wave system [9]. Furthermore, given the widespread use of the implicit finite difference method in fluid dynamics for solving boundary layer equations [10] and finding, for example, the numerical solution for the Brusselator reaction diffusion model by finite-difference (FD) scheme

[11], we anticipate that this technique, when properly implemented, could prove highly valuable to physicists when dealing with such problems.

2. EXACT SOLUTIONS OF PROBLEM (1)

If we define the following dependent variables transformation

$$u = \frac{\partial y}{\partial x} \text{ and } v = \frac{\partial y}{\partial t}. \tag{5}$$

Then the given nonlinear partial differential equation

$$\frac{\partial^2 y}{\partial t^2} - \left[1 + \epsilon \frac{\partial y}{\partial x}\right]^{\alpha} \frac{\partial^2 y}{\partial x^2} = 0 \tag{6}$$

can be written in an equivalent form as

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} - [1 + \epsilon u]^{\alpha} \frac{\partial u}{\partial x} &= 0. \end{cases} \tag{7}$$

Letting $w(x,t) = 1 + \epsilon u$. Thus, Eqs. (7) become

$$\begin{cases} \frac{1}{\epsilon} \frac{\partial w}{\partial t} - \frac{\partial v}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} - \frac{1}{\epsilon} w^{\alpha} \frac{\partial w}{\partial x} &= 0. \end{cases} \tag{8}$$

We shall present an adaptation of the method of separation of variables for solving these coupled nonlinear partial differential equations. We begin our analysis by assuming that the functions $v(x,t)$ and $w(x,t)$ are identified as the product of two distinct functions, *i.e.* $v(x,t) = R(t)S(x)$ and $w(x,t) = T(t)X(x)$, where $T(t)$ and $R(t)$ depend on the time variable t and $S(x)$ and $X(x)$ depend on the space variable x. Assume that T, R, S, and X are twicely continuously differentiable. Thus the substitution $v(x,t) = R(t)S(x)$ and $w(x,t) = T(t)X(x)$ into system (8) gives

$$\begin{cases} \frac{1}{\epsilon} \frac{T'(t)}{R(t)} &= \frac{S'(x)}{X(x)}, \\ \epsilon \frac{R'(t)}{T^{\alpha+1}(t)} &= \frac{X^{\alpha}(x) X'(x)}{S(x)}. \end{cases} \tag{9}$$

Since the left hand sides of Eqs. (9) depend only on t and the right hand sides depend only on x, we have

$$\begin{cases} \frac{1}{\epsilon} \frac{T'(t)}{R(t)} = \frac{S'(x)}{X(x)} &= \lambda_1, \\ \epsilon \frac{R'(t)}{T^{\alpha+1}(t)} = \frac{X^{\alpha}(x) X'(x)}{S(x)} &= \lambda_2, \end{cases} \tag{10}$$

where $\lambda_i, i = 1, 2$ are two constants.
This gives us two distinct coupled problems of nonlinear ordinary differential equations given by

$$\begin{cases} T'(t) &= \lambda_1 \epsilon R(t), \\ \epsilon R'(t) &= \lambda_2 T^{\alpha+1}(t). \end{cases} \tag{11}$$

and

$$\begin{cases} S'(x) &= \lambda_1 X(x), \\ X'(x)X^{\alpha}(x) &= \lambda_2 S(x). \end{cases} \tag{12}$$

Multiplying the first equation of system (11) by $R'(t)$, we obtain

$$R'(t)T'(t) = \lambda_1 \epsilon R(t)R'(t), \tag{13}$$

and inserting the second equation of (11) into the left hand side of (13), we obtain

$$\frac{\lambda_2}{\epsilon} T'(t)T^{\alpha+1}(t) = \lambda_1 \epsilon R(t)R'(t). \tag{14}$$

Hence,

$$\frac{1}{\alpha+2}\frac{d}{dt}\left[T^{\alpha+2}(t)\right] = \frac{\lambda_1}{\lambda_2}\frac{\epsilon^2}{2}\frac{d}{dt}\left[R^2(t)\right]. \tag{15}$$

Integration yields

$$T^{\alpha+2}(t) = \frac{\lambda_1}{\lambda_2}\frac{\epsilon^2(\alpha+2)}{2}R^2(t) + C_1 \tag{16}$$

or

$$T^{\alpha+2}(t) = \epsilon^2 A_\alpha R^2(t) + C_1, \tag{17}$$

where C_1 is an unknown constant and $A_\alpha = \frac{\lambda_1}{\lambda_2}\frac{\alpha+2}{2}$.
Substituting now Eq. (17) into the right hand side of the second equation of (11) we obtain

$$R'(t) = \frac{\lambda_2}{\epsilon}\left(\epsilon^2 A_\alpha R^2(t) + C_1\right)^{\frac{\alpha+1}{\alpha+2}}, \tag{18}$$

that is

$$\frac{dR}{\left(\epsilon^2 A_\alpha R^2 + C_1\right)^{\frac{\alpha+1}{\alpha+2}}} = \frac{\lambda_2}{\epsilon}dt. \tag{19}$$

Consequently,

$$\int \frac{1}{\left(\epsilon^2 A_\alpha R^2 + C_1\right)^{\frac{\alpha+1}{\alpha+2}}}dR = \frac{\lambda_2}{\epsilon}t + D_1, \tag{20}$$

where D_1 is a constant of integration.
In order to find the functions $S(x)$ and $X(x)$, proceeding as before, multiplying both sides of the first equation of system (12) by $S'(x)$, we obtain

$$S'(x)S(x) = \lambda_1 X(x)S(x). \tag{21}$$

Substituting now the second equation of system (12) into the right hand side of Eq. (21) to obtain

$$S'(x)S(x) = \frac{\lambda_1}{\lambda_2} X(x)X^{\alpha+1}(x). \tag{22}$$

So that

$$\frac{1}{2}\frac{d}{dx}\left[S^2(x)\right] = \frac{\lambda_1}{\lambda_2}\frac{1}{\alpha+2}\frac{d}{dx}\left[X^{\alpha+2}(x)\right]. \tag{23}$$

Thus,

$$S^2(x) = \frac{\lambda_1}{\lambda_2}\frac{2}{\alpha+2}X^{\alpha+2}(x) + C_2 \tag{24}$$

or

$$S^2(x) = B_\alpha X^{\alpha+2}(x) + C_2, \tag{25}$$

where C_2 is a constant of integration and $B_\alpha = \frac{\lambda_1}{\lambda_2}\frac{2}{\alpha+2}$.
A simple substitution of (25) into the second equation of system (12) leads to

$$X'(x)X^\alpha(x) = \lambda_2\sqrt{B_\alpha X^{\alpha+2}(x) + C_2}. \tag{26}$$

Hence,

$$\int \frac{X^\alpha}{\sqrt{E_\alpha X^{\alpha+2} + C_3}}dX = x + D_2, \tag{27}$$

where $E_\alpha = \lambda_2^2 B_\alpha$, $C_3 = \lambda_2^2 C_2$, and D_2 is a constant of integration. Therefore, the functions $R(t)$, $T(t)$, $X(x)$, and $S(x)$ can be obtained from the following general implicit formulas which contain integrals:

$$\begin{cases} \int \frac{1}{(\epsilon^2 A_\alpha R^2 + C_1)^{\frac{\alpha+1}{\alpha+2}}}dR &= \frac{\lambda_2}{\epsilon}t + D_1, \\ \\ T^{\alpha+2}(t) &= \epsilon^2 A_\alpha R^2(t) + C_1. \end{cases} \tag{28}$$

and

$$\begin{cases} \int \frac{X^\alpha}{\sqrt{E_\alpha X^{\alpha+2} + C_3}}dX &= x + D_2, \\ \\ S^2(x) &= B_\alpha X^{\alpha+2}(x) + C_2. \end{cases} \tag{29}$$

These give the set of solutions:

$$u(x,t) = \frac{1}{\epsilon}T(t)X(x) - \frac{1}{\epsilon} \text{ and } v(x,t) = R(t)S(x). \tag{30}$$

We note here that such general implicit formulas encompassing related integrals appear similarly in many traveling wave solutions of diffusion equations, see *e.g.*, [12]. For the clarity of our approach, let us consider $\alpha = -1$. Thus, a simple substitution into (28) and (29) gives the closed form solutions:

$$R(t) = \frac{\lambda_2}{\epsilon}t + D_1,\ T(t) = \epsilon^2 A_{-1}\left(\frac{\lambda_2}{\epsilon}t + D_1\right)^2 + C_1. \tag{31}$$

Since

$$\int \frac{dX}{X\sqrt{E_{-1}X+C_3}} = \begin{cases} -\frac{2}{\sqrt{C_3}} \coth^{-1}\left(\frac{\sqrt{E_{-1}X+C_3}}{\sqrt{C_3}}\right), & \text{if } C_3>0,\ E_{-1}X>0, \\ -\frac{-2}{\sqrt{C_3}} \tanh^{-1}\left(\frac{\sqrt{E_{-1}X+C_3}}{\sqrt{C_3}}\right), & \text{if } C_3>0,\ E_{-1}X<0, \\ \frac{2}{\sqrt{-C_3}} \arctan\left(\frac{\sqrt{E_{-1}X+C_3}}{\sqrt{-C_3}}\right), & \text{if } C_3<0,\ E_{-1}X<0, \end{cases} \tag{32}$$

if we choose, for example, $C_3>0$ and $E_{-1}X>0$, then

$$-\frac{2}{\sqrt{C_3}} \coth^{-1}\left(\frac{\sqrt{E_{-1}X+C_3}}{\sqrt{C_3}}\right) = x+D_2. \tag{33}$$

It follows that

$$X(x) = \frac{C_3}{E_{-1}} \coth^2\left(-\frac{C_3}{E_{-1}}(x+D_2)\right) - \frac{C_3}{E_{-1}} \tag{34}$$

and

$$S(x) = \sqrt{\frac{C_3B_{-1}}{E_{-1}} \coth^2\left(-\frac{C_3}{E_{-1}}(x+D_2)\right) - \frac{C_3B_{-1}}{E_{-1}} + C_2}, \tag{35}$$

where $C_2>0$ and $B_{-1}X>0$.
Hence

$$u(x,t) = \left(\epsilon A_{-1}(\frac{\lambda_2}{\epsilon}t+D_1)^2 + \frac{C_1}{\epsilon}\right)\left(\frac{C_3}{E_{-1}} \coth^2\left(-\frac{C_3}{E_{-1}}(x+D_2)\right) - \frac{C_3}{E_{-1}} - \frac{1}{\epsilon}\right) \tag{36}$$

and

$$v(x,t) = \left(\frac{\lambda_2}{\epsilon}t+D_1\right)\sqrt{\frac{C_3B_{-1}}{E_{-1}} \coth^2\left(-\frac{C_3}{E_{-1}}(x+D_2)\right) - \frac{C_3B_{-1}}{E_{-1}} + C_2}. \tag{37}$$

Returning to the original dependent variable $v=\frac{\partial y}{\partial t}$ we obtain the general solution of the Problem (1) when $\alpha=-1$:

$$y(x,t) = \left(\frac{\lambda_2}{2\epsilon}t^2+D_1t\right)\sqrt{\frac{C_3B_{-1}}{E_{-1}} \coth^2\left(-\frac{C_3}{E_{-1}}(x+D_2)\right) - \frac{C_3B_{-1}}{E_{-1}} + C_2} + K(x), \tag{38}$$

where $K(x)$ is an arbitrary function. Employing now the given initial-boundary conditions, we obtain

$$K(x)=0,\ D_1=0 \tag{39}$$

and

$$\begin{cases} \sqrt{\frac{C_3B_{-1}}{E_{-1}}\coth^2\left(-\frac{C_3}{E_{-1}}D_2\right)-\frac{C_3B_{-1}}{E_{-1}}+C_2} &= 0, \\ \sqrt{\frac{C_3B_{-1}}{E_{-1}}\coth^2\left(-\frac{C_3}{E_{-1}}(1+D_2)\right)-\frac{C_3B_{-1}}{E_{-1}}+C_2} &= 0. \end{cases} \tag{40}$$

The particular solution of the Problem (1) when $\alpha=-1$ is then given by

$$y(x,t)=\frac{1}{2\epsilon}t^2\sqrt{C_3\coth^2\left(-\frac{C_3}{E_{-1}}(x+D_2)\right)+\frac{C_2E_{-1}}{B_{-1}}-C_3}, \tag{41}$$

where the constants $C_2,\ C_3,\ D_2,\ B_{-1},\ E_{-1}$ satisfy the conditions (40).

3. CONCLUSION

In conclusion, as a complement of the numerical results obtained in the cited references, we have successfully presented some explicit solutions to the nonlinear boundary value problem for the vibrations of a continuous model string.

The basic key of this method is to transform the given nonlinear equation $\frac{\partial^2 y}{\partial t^2}-\left[1+\epsilon\frac{\partial y}{\partial x}\right]^{\alpha}\frac{\partial^2 y}{\partial x^2}=0$ into coupled nonlinear partial differential equations of first-order, by introducing the dependent variables transform $u=y_x,\ v=y_t$, which can be solved by a reliable technique.

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