THE TOROIDAL WHEEL DYNAMICS

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Abstract. The dynamics of the toroidal wheel with loaded mass that rolls without slipping is studied. The reduced equations of motion for a dynamic system of dimension five, depending on three angular rates, the leaning angle and the rolling angle, are derived. The dynamic system is controlled by the tilt and throttle torque.

Key words: toroidal wheel, non-holonomic systems, rigid body dynamics.

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1. INTRODUCTION

The Cyr wheel [1] is a toroidal wheel that rolls and spins without slipping. Its rubber body dynamics [2] occupies an intermediate place between marble bodies that roll, but not spin and not slip and rigid bodies that roll and glide [3]. The dynamics of the toroidal wheel is not different from that of the heavy symmetrical top with one point fixed: the tip of the virtual axle of the torus precesses around the vertical axis executing also small nutation oscillations [4]. The stability analysis [5–6] shows that the circular steady motion is possible only if the rolling rate is above a critical value to keep the steady leaning angle. For every rolling rate there are two values of spinning rates, a fast one and a slow one. The dynamics of general axi-symmetric bodies of revolution [7] and of tori with general profiles [6] were studied. Due to the existence of three integrals of motion was possible to obtain the rolling rate of a disk with sharp edge [8–10] as the solution of a hyper-geometric equation. The frequency of small nutation oscillations [11] was computed for the circular rolling steady motion of the disk.

The dynamics becomes more complex when the center of mass does not coincide with the geometrical center of the rigid body. The system remains integrable when the displacement of the center of mass is along the axle of the axisymmetric body of revolution [7]. This dependence is known for the rolling disk with the displaced center of mass [7]. But it cannot be simply extended to the torus wheel, because the behaviour of the rolling disk is not the same with that of the torus wheel. For example, the trajectory of the rolling disk is an undulated helix, but the trajectory of the ring presents a retrograde turn (S-shape figure) [12]. The explanations are based on the effects of the air drag [12] or of the viscous rolling friction [13].

The control of the rolling disk can be realized with the throttle and/or the tilt torque [14–16]. It was remarked that using only the throttle torque [16], it is not clear how it is possible to change the leaning angle from positive (one side) to negative (the other side) to produce an S-trajectory, because passing through the vertical position introduces a loss of control. This property might be related with the previous mentioned behavior of the rolling disk that slows down due to the friction torque. The throttle torque and friction torque axes are parallel, the first one is along the disk axle and the second is at the ground level. The control of the torus wheel can be realized also with the throttle and the stir torque [6].

For the Cyr wheel, the control of the system is exerted by changing the position of the human body with respect to the center of the wheel. In the first approximation, the human body can be replaced by a mass attached to the wheel. The position of the attached mass can be on the virtual axle of the wheel or on the diameter of the wheel. The interesting problem is how the distance between the center of the attached mass and the center of the wheel affects the dynamics of the wheel. The dynamics of the toroidal wheel with an attached mass will be studied in this study. The problem is interesting and non-trivial for rolling the wheel without leaning, but with slipping [3, 17–18]. In our case the toroidal wheel rolls and spins without slipping and is controlled by the tilt and throttle torques.

2. THE RIGID-BODY DYNAMICS WITH NON-HOLONOMIC CONSTRAINTS

In the classical mechanics [4, 19–20] the variation in time of momentum $\boldsymbol{P}$ and the angular momentum $\boldsymbol{L}$ of a rigid body are related with the force $\boldsymbol{F}$ and respectively the torque $\boldsymbol{M}$ exerted on it: $d\boldsymbol{P}/dt = \boldsymbol{F}$, $d\boldsymbol{L}/dt = \boldsymbol{M}$. These equations of motion are valid in the fixed referential frame E_f (axes x_f, y_f, z_f) where the momentum $\boldsymbol{P}$ is proportional with the linear velocity $\boldsymbol{V}$: $\boldsymbol{P} = m\boldsymbol{V}$. The angular momentum $\boldsymbol{L}$ has no simple expression in the inertial frame. But in the mobile body frame E_b, with axes x_b, y_b, z_b along principal axes of moment of inertia $\boldsymbol{I}$, the angular momentum $\boldsymbol{L}$ is proportional with the angular velocity $\boldsymbol{\Omega}$: $\boldsymbol{L} = \boldsymbol{I}\boldsymbol{\Omega}$.

The Newton-Euler equations of motion for the fixed reference frame are:

$$d\boldsymbol{P}/dt = \boldsymbol{F},\ d\boldsymbol{L}/dt = \boldsymbol{M}. \tag{1}$$

The body position can be characterized by six generalized coordinates: the center of mass coordinates $\boldsymbol{X} = (x, y, z)$ and the Tait-Bryan angles $\boldsymbol{\Phi} = (\varphi, \theta, \psi)$. The body frame E_b can be obtained by applying successive intrinsic rotations to the mobile frame E attached to the body center of mass having the axes x, y, z parallel to those of the fixed frame E_f. The intermediate frame E_h (with axes x_h, y_h, $z_h = z$, see Fig. 1) is obtained from E by applying rotation $Rz(\psi)$ with the heading angle ψ around the axis z. The second intermediate frame E_l (with axes $x_l = x_h$, y_l, z_l, see

Fig. 1) is obtained from E_h by applying rotation $Rx(\varphi)$ with the leaning angle φ around the axis x_h. The body frame E_b (with axes x_b, $y_b = y_l$, z_b) is obtained by applying rotation $Ry(\theta)$ with the rolling angle θ around the axis y_l. The position of a body point $\boldsymbol{r_b}$ in different frames is given by the relations:

$$\boldsymbol{r} = Rz(\psi)\, \boldsymbol{r_h} = Rz(\psi)\, Rx(\varphi)\, \boldsymbol{r_l} = Rz(\psi)\, Rx(\varphi)\, Ry(\theta)\boldsymbol{r_b} \quad \text{and} \quad \boldsymbol{r_f} = \boldsymbol{X} + \boldsymbol{r}_b \tag{2}$$

with the matrix representation of rotations:

$$Rz(\psi) = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad Rx(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi & c\varphi \end{bmatrix}$$
$$Ry(\theta) = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \tag{3}$$

where $s\psi$ = sin ψ, $c\psi$ = cos ψ etc.

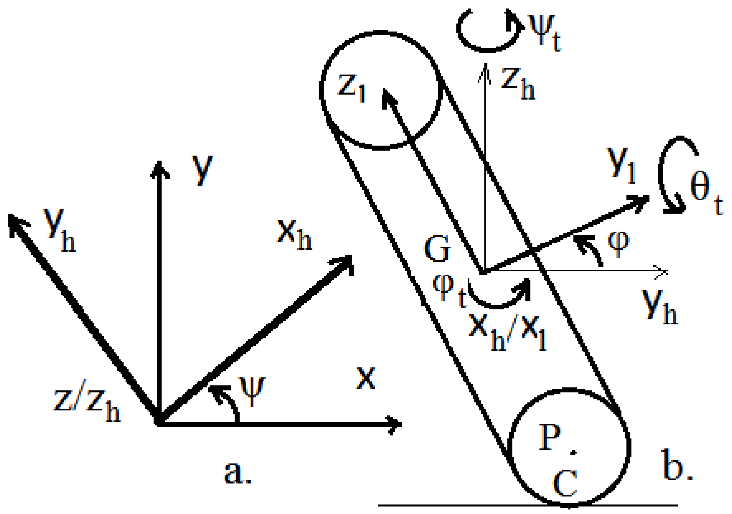

Fig. 1 – The torus wheel in the heading frame x_h, y_h, z_h: a) the view from above; b) the lateral view.

Figure 1 presents the torus wheel in the heading frame x_h, y_h, z_h: a) the view from above: rotation with the heading angle ψ from the fixed frame, b) the lateral view: rotation with the leaning angle φ from the vertical position. The torus wheel rotates with the rolling rate θ_t around the axis y_l, with the spinning rate ψ_t around the fixed vertical axis z_f and the leaning rate φ_t around the axis x_h.

Figure 1 shows the angular velocity $\boldsymbol{\Phi}_t \equiv d\boldsymbol{\Phi}/dt = (\varphi_t, \theta_t, \psi_t)^T$ around axes x_h, y_l and respectively z_h. The angular velocity $\boldsymbol{\Omega_h}$ can be obtained from $\boldsymbol{\Phi}_t$ by projecting ψ_t on axes y_h and z_h: $\boldsymbol{\Omega_h} = Jh\boldsymbol{\Phi}_t$. The angular velocities $\boldsymbol{\Omega}$, $\boldsymbol{\Omega_l}$, $\boldsymbol{\Omega_b}$ are vectors and they are related in the same way as the position of a point $\boldsymbol{r}$ in different frames in equation (2). Using the matrix representation of the rotations (3) the angular velocities $\boldsymbol{\Omega}$, $\boldsymbol{\Omega_h}$, $\boldsymbol{\Omega_l}$, $\boldsymbol{\Omega_b}$ can be written in terms of the angular velocity $\boldsymbol{\Phi}_t$:

$$\boldsymbol{\Omega} = J\boldsymbol{\Phi}_t, \quad J = \begin{bmatrix} c\psi & -s\psi \cdot c\varphi & 0 \\ s\psi & c\psi \cdot c\varphi & 0 \\ 0 & s\varphi & 1 \end{bmatrix} \quad \boldsymbol{\Omega_h} = Jh\boldsymbol{\Phi}_t, \quad Jh = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & 0 \\ 0 & s\varphi & 1 \end{bmatrix}$$

$$\boldsymbol{\Omega_l} = Jl\boldsymbol{\Phi}_t, \quad Jl = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & s\varphi \\ 0 & 0 & c\varphi \end{bmatrix} \quad \boldsymbol{\Omega_b} = Jb\boldsymbol{\Phi}_t, \quad Jb = \begin{bmatrix} c\theta & 0 & -s\theta\cdot c\varphi \\ 0 & 1 & s\varphi \\ s\theta & 0 & c\theta\cdot c\varphi \end{bmatrix} \tag{4}$$

The rotational kinetic energy T in the body frame can be expressed in terms of the angular velocity $\boldsymbol{\Phi}_t$:

$$T = \boldsymbol{\Omega_b}^T \, Ib \, \boldsymbol{\Omega_b}/2 = \boldsymbol{\Phi}_t^T \, I(\boldsymbol{\Phi}) \, \boldsymbol{\Phi}_t/2, \quad I(\boldsymbol{\Phi}) = Jb^T Ib \, Jb. \tag{5}$$

The constraints for the rotating non-holonomic systems have the form:

$$\delta\boldsymbol{X} + S \, \delta\boldsymbol{\Phi} = 0, \tag{6}$$

where the matrix S depends on the angles $\boldsymbol{\Phi}$. It is not possible to integrate the equation and to obtain the solution $\boldsymbol{X}$ in terms of the angles $\boldsymbol{\Phi}$.

The Newton-Euler equations of motion for the non-holonomic systems must be extended with the additional constraint forces $\boldsymbol{f} = (fx, fy, fz)^T$ and torques $\boldsymbol{m} = (m\varphi, m\theta, m\psi)^T$:

$$(d \, \mathbf{P}/dt) + \boldsymbol{f}(\lambda) = \boldsymbol{F}, \quad (d \, \mathbf{L}/dt) + \boldsymbol{m}(\lambda) = \boldsymbol{M}. \tag{7}$$

By taking into account that the forces $\boldsymbol{f}$ and torques $\boldsymbol{m}$ do no virtual work δW for virtual displacements of generalized coordinates:

$$\delta W = 0 = fx \, \delta x + fy \, \delta y + fz \, \delta z + m\varphi \, \delta\varphi + m\theta \, \delta\theta + m\psi \, \delta\psi, \tag{8}$$

it is possible to express $\boldsymbol{f}(\lambda)$ and $\boldsymbol{m}(\lambda)$ in terms of the unknown Lagrangian multipliers λ. Due to no work condition (8) and the non-holonomic constraint (6) the additional constraint forces $\boldsymbol{f}$ and torques $\boldsymbol{m}$ satisfy the equation:

$$\delta\boldsymbol{X}^T \boldsymbol{f} + \delta\boldsymbol{\Phi}^T \boldsymbol{m} = 0 = \delta\boldsymbol{X}^T \boldsymbol{\lambda} + \delta\boldsymbol{\Phi}^T S^T \boldsymbol{\lambda} \tag{9}$$

with the solutions:

$$\boldsymbol{f} = \boldsymbol{\lambda},\ \boldsymbol{m} = S^T \boldsymbol{\lambda}. \tag{10}$$

Inserting the solution (10) in the equation (7) and taking into account that the moment of inertia is given by the equation (5), it is possible to rewrite the Newton-Euler equations of motion in terms of the generalized coordinates $\boldsymbol{X}$ and $\boldsymbol{\Phi}$:

$$m\,\boldsymbol{X}_{tt} + I\boldsymbol{\lambda} = \boldsymbol{F} \tag{11a}$$

$$d\,(I(\boldsymbol{\Phi})\,\boldsymbol{\Phi}_t)/dt + S^T \boldsymbol{\lambda} = \boldsymbol{M}. \tag{11b}$$

These equations can be supplemented with the constraint (6) differentiated with respect to time:

$$\boldsymbol{X}_{tt} + S\boldsymbol{\Phi}_{tt} + (dS/dt)\,\boldsymbol{\Phi}_t = 0. \tag{11c}$$

Introducing the expression (11a) and (11c) of the Lagrangian multipliers in the equation (11b) the reduced equations of motion depend only on $\boldsymbol{\Phi}$:

$$(I(\boldsymbol{\Phi}) + mS^T S)\,\boldsymbol{\Phi}_{tt} + [dI(\boldsymbol{\Phi})/dt + mS^T (dS/dt)]\,\boldsymbol{\Phi}_t = \boldsymbol{M} - S^T \boldsymbol{F}. \tag{12}$$

Until this point the discussion was quite general and the theory can be applied to a generic rigid body with non-holonomic constraints. The reduced equations of motion (12) depend on the constraint matrix S and the moment of inertia $I(\boldsymbol{\Phi})$ (5). In the next section is obtained the explicit form of S and $I(\boldsymbol{\Phi})$ for the specific case of the wheel torus.

3. THE REDUCED EQUATIONS OF MOTION FOR TOROIDAL WHEEL

For the wheel torus rolling without sliping (with the big radius $R = |PG|$ and the small radius $r = |PC|$ – see Fig. 1) the position of the contact point C of the torus with the plane in the fixed frame $\boldsymbol{r_{fC}} = \boldsymbol{X} + Rz(\psi)\,\boldsymbol{r_{hC}}$ depends on the position in the heading frame (see Fig. 1): $\boldsymbol{r_{hC}} = (0, Rs\varphi, -Rc\varphi - r)$. The non-holonomic constraint requires that the horizontal velocity of the contact point in the fixed frame is zero: $d\boldsymbol{r_{fC}}/dt = \boldsymbol{X}_t + \boldsymbol{\Omega}\ x\ Rz(\psi)\,\boldsymbol{r_{hC}}$. Using the relation: $\boldsymbol{\Omega} = Rz(\psi)\,\boldsymbol{\Omega_h}$ and the expression (4) for the angular velocity $\Omega h = Jh\,\boldsymbol{\Phi}_t$, the constraint $d\boldsymbol{r_{fC}}/dt = \boldsymbol{X}_t - Rz(\psi)\,(\boldsymbol{r_{hC}} x\ Jh\,\boldsymbol{\Phi}_t)$ gives rise for axes x and y to two constraints in the fixed frame:

$$\delta\boldsymbol{X} + S\,\delta\boldsymbol{\Phi} = 0, \tag{13}$$

$$S = \begin{bmatrix} -(R\,c\varphi + r)\,s\psi & -(R + r\,c\varphi)\,c\psi & -R\,s\varphi\,c\psi \\ (R\,c\varphi + r)\,c\psi & -(R + r\,c\varphi)\,s\psi & -R\,s\varphi\,s\psi \\ R\,s\varphi & 0 & 0 \end{bmatrix}$$

The moment of inertia Ib for the torus wheel in the body frame has the diagonal non-zero components:

$$Ib = \mathrm{diag}\,(I_T, I_A, I_T)$$
$$I_T = (m/8)\,(5r^2 + 4R^2), \quad I_A = (m/4)\,(5r^2 + 4R^2), \tag{14}$$

where the axial component I_A is along the virtual wheel axle y_b and the transversal component I_T is along the diameter of the wheel. The expression (5) of the moment of inertia $I(\Phi)$ depends only on the leaning angle φ:

$$I(\Phi) = \begin{bmatrix} I_T & 0 & 0 \\ 0 & I_A & I_A\, s \\ 0 & I_A\, s & I_T\, c^2 + I_A\, s^2 \end{bmatrix} \tag{15}$$

where $s = s\varphi$, $c = c\varphi$.

The reduced equations of motion (12) have the inertial term $J(\Phi)$:

$$J(\Phi) = I(\Phi) + mS^TS = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & I_{23} \\ 0 & I_{23} & I_{33} \end{bmatrix} \tag{16}$$

$I_{11} = I_T + m(R^2 + 2crR + r^2)$, $I_{22} = I_A + m(R + rc)^2$,
$I_{23} = I_A\, s + mR(R + rc)\, s$, $I_{33} = I_T\, c^2 + (I_A + mR^2)\, s^2$

and the quadratic velocity term $C(\Phi)$:

$$C(\Phi) = dI(\Phi)/dt + mS^T\, dS/dt = \begin{bmatrix} C_{11}\, \varphi_t & C_{12}\, \psi_t & C_{13}\, \theta_t + C_{14}\psi_t \\ C_{21}\, \psi_t & C_{22}\, \varphi_t & C_{23}\, \varphi_t \\ C_{31}\, \psi_t & C_{32}\, \varphi_t & C_{33}\, \varphi_t \end{bmatrix} \tag{17}$$

$C_{11} = -mRrs$, $C_{12} = -I_Ac/2 - m\,(Rc + r)\,(R + rc)$, $C_{13} = -I_Ac/2$, $C_{14} = -(I_A - I_T)\,sc - mRs(Rc + r)$, $C_{21} = m(Rc + r)\,(R + rc)$, $C_{22} = -mrs(R + rc)$, $C_{23} = I_Ac + mRc(R + rc)$, $C_{31} = mRs(Rc + r)$, $C_{32} = I_Ac - mrRs^2$, $C_{33} = [2(I_A - I_T) + mR^2]\, sc$.

Using the relations (16) and (17) and the gravity force $Fz = -mg$, the reduced equations of motion (12) can be written:

$$\begin{gathered} I_{11}\, \varphi_{tt} + C_{11}\, \varphi_t^2 + (C_{12} + C_{13})\, \psi_t\, \theta_t + C_{14}\, \psi_t^2 = mgRs \\ I_{22}\, \theta_{tt} + I_{23}\, \psi_{tt} + \varphi_t\, [C_{22}\, \theta_t + (C_{21} + C_{23})\, \psi_t] = 0 \\ I_{23}\, \theta_{tt} + I_{33}\, \psi_{tt} + \varphi_t\, [C_{32}\, \theta_t + (C_{31} + C_{33})\, \psi_t] = 0 \end{gathered} \tag{18}$$

In the limit $r = 0$ the reduced equations of motion for the hoop are obtained:

$$\begin{gathered}(I_T + mR^2)\,\varphi_{tt} - (I_A + mR^2)\,c\,\psi_t\,\theta_t - (I_A - I_T + mR^2)\,sc\,\psi_t^2 = mgRs \\ (I_A + mR^2)\,\theta_{tt} + (I_A + mR^2)\,s\,\psi_{tt} + (I_A + 2mR^2)\,c\,\varphi_t\,\psi_t = 0 \\ I_T\,c\,\psi_{tt} + I_A\,\varphi_t\,\theta_t + (I_A - 2I_T)\,s\,\varphi_t\,\psi_t = 0.\end{gathered} \tag{19}$$

These equations can be compared with the reduced equations of motion for the rolling disk [11]. They are practically identical, apart signs due to different orientation of angles. It is also relevant to compare these equations with the heavy symmetric top with the Tait-Bryan angles:

$$\begin{gathered}I_T\,\varphi_{tt} - I_A\,c\,\psi_t\,\theta_t - (I_A - I_T)\,sc\,\psi_t^2 = mgRs \\ I_A\,\theta_{tt} + I_A\,s\,\psi_{tt} + I_A\,c\,\varphi_t\,\psi_t = 0 \\ I_T\,c\,\psi_{tt} + I_A\,\varphi_t\,\theta_t + (I_A - 2\,I_T)\,s\,\varphi_t\,\psi_t = 0.\end{gathered} \tag{20}$$

In the equations (20) the non-holonomic constraints add a supplementary moment of inertia mR^2 of the center of mass with respect to the contact point. Due to this similarity of equations, the dynamics of the hoop is not different from that of the heavy symmetrical top [4]. But the equations (18) of the toroidal wheel have a more complex coupling between the angular velocities. Even in this case it is possible to find a non-trivial solution. The simplest solution of equations (18) is the steady circular motion with constant leaning angle, rolling and spinning rates.

In the steady circular motion the tip of the virtual wheel axle executes a precession with uniform rate. The small oscillations around the steady leaning angle (nutation) generate small radius sub-loops when the wheel still moves on average on a circular path.

4. THE TOROIDAL WHEEL WITH LOADED MASS

As was said in introduction, it is interesting to see how the dynamics of the toroidal wheel is modified by attaching a mass at some distance from the centre of the wheel. We study three cases when the position of the loaded mass is: A) on the wheel axle y_b; B) on the diameter x_l of the wheel center (maintaining the same altitude with respect the ground); C) on the diameter z_b of the wheel center. All three cases can be studied by considering a wheel with different principal components for the moment of inertia:

$$I(\Phi) = \begin{bmatrix} I_1\,c_\theta^2 + I_3\,s_\theta^2 & 0 & 0 \\ 0 & I_2 & I_2\,s \\ 0 & I_2\,s & (I_1\,s_\theta^2 + I_3\,c_\theta^2)c^2 + I_2 s^2 \end{bmatrix} \tag{21}$$

The expression (21) of the moment of inertia $I(\Phi)$ depends on the leaning angle φ and the rolling angle θ with the notations $s = \sin\varphi$, $c = \cos\varphi$, $s_\theta = \sin\theta$, $c_\theta = \cos\theta$.

The reduced equations of motion (12) have the inertial term $J(\Phi)$:

$$J(\Phi) = I(\Phi) + mS^TS = \begin{bmatrix} I_{11} & 0 & I_{13} \\ 0 & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix} \tag{22}$$

$I_{11} = I_1\, c_\theta^2 + I_3\, s_\theta^2 + m(R^2 + 2crR + r^2)$, $I_{13} = (I_3 - I_1)\, c\, s_\theta c_\theta$, $I_{22} = I_2 + m(R + rc)^2$,

$I_{23} = I_2 s + mR(R + rc)\, s$, $I_{33} = (I_1\, s_\theta^2 + I_3\, c_\theta^2)\, c^2 + (I_2 + mR^2)\, s^2$

and the quadratic velocity term $C(\Phi)$:

$$C(\Phi) = dI(\Phi)/dt + mS^T\, dS/dt =$$
$$= \begin{bmatrix} C_{11}\, \varphi_t + 2C_1\theta_t + sC_1/2\, \psi_t & C_{12}\, \psi_t & -sC_1/2\, \varphi_t + C_{13}\, \theta_t + C_{14}\, \psi_t \\ -C_1\, \varphi_t + C_{21}\, \psi_t & C_{22}\, \varphi_t & C_{23}\, \varphi_t + C_1 c^2\, \psi_t \\ -sC_1\, \varphi_t + C_2\, \theta_t + C_{31}\, \psi_t & C_{32}\, \varphi_t & C_{33}\, \varphi_t - 2\, C_1 c^2\, \theta_t \end{bmatrix} \tag{23}$$

$C_1 = (I_3 - I_1)\, s_\theta c_\theta$, $C_2 = (I_3 - I_1)\, c\, (c_\theta^2 - s_\theta^2)$, $C_3 = sc\, (I_2 - I_1\, s_\theta^2 - I_3\, c_\theta^2)$, $C_{11} = -\, mRrs$,

$C_{12} = -\, I_2 c/2 - m\, (Rc + r)\, (R + rc)$, $C_{13} = C_2 - I_2 c/2$, $C_{14} = -\, mRs\, (Rc + r) - C_3$,

$C_{21} = m\, (Rc + r)\, (R + rc) - C_2/2$, $C_{22} = -\, mrs\, (R + rc)$, $C_{23} = I_2 c - C_2/2 + mRc\, (R + rc)$,

$C_{31} = mRs\, (Rc + r)$, $C_{32} = I_2\, c - mRrs^2$, $C_{33} = 2C_3 + mR^2 sc$

Using the moment of inertia $J(\Phi)$ (22) and the quadratic velocity term $C(\Phi)$ (23) the reduced equations of motion (12) can be written:

$$\begin{gathered} I_{11}\, \varphi_{tt} + I_{13}\, \psi_{tt} + C_{11}\, \varphi_t^2 + 2C_1\, \varphi_t\, \theta_t + (C_{12} + C_{13})\, \psi_t\, \theta_t + C_{14}\, \psi_t^2 = M_1 \\ I_{22}\, \theta_{tt} + I_{23}\, \psi_{tt} + C_1\, (c^2\, \psi_t^2 - \varphi_t^2) + \varphi_t\, [C_{22}\, \theta_t + (C_{21} + C_{23})\, \psi_t] = M_2 \\ I_{13}\, \varphi_{tt} + I_{23}\, \theta_{tt} + I_{33}\, \psi_{tt} - sC_1\, \varphi_t^2 + \varphi_t\, [(C_2 + C_{32})\, \theta_t + \\ + (C_{31} + C_{33})\, \psi_t] - 2C_1\, c^2\, \theta_t\, \psi_t = 0 \end{gathered} \tag{24}$$

The control of the Cyr wheel is realized in our case by the throttle and the tilt torque. The principal components of the moment of inertia I_1, I_2, I_3 and the torques for the three cases are:

A) When the loaded mass M is on the virtual axle y_b at distance a from the wheel center, the control is exerted by the tilt torque M_1:

$$M_1 = g\, (m + M)\, Rs - g\, Mac; \quad I_1 = I_T + Ma^2, \quad I_2 = I_A, \quad I_3 = I_T + Ma^2; \tag{25a}$$

B) When the loaded mass M is maintained at the same altitude as the wheel center on the diameter x_l at distance a from the wheel center (it mimics the human motion – a person walking on the wheel) the control is exerted by the tilt torque M_1 and the throttle torque M_2:

$$M_1 = g\,(m + M)\,Rs, \quad M_2 = g\,Mac; \quad I_1 = I_T + M\,(ac_\theta)^2, \\ I_2 = I_A + M\,(ac_\theta)^2, \quad I_3 = I_T; \tag{25b}$$

C) When the loaded mass M is on the diameter z_b at distance a from the wheel center and rotates together with the wheel:

$$M_1 = g\,(m + M)\,Rs + g\,Mas\,c_\theta, \quad M_2 = g\,Mac\,s_\theta; \\ I_1 = I_T + Ma^2, \quad I_2 = I_A, \quad I_3 = I_T. \tag{25c}$$

In case A) the wheel with the loaded mass is an axi-symmetric body of revolution and a steady circular motion exists. From the first equation (24) results that it is possible to maintain a constant leaning angle φ if $\varphi_t = 0$ and the condition holds:

$$A\psi_t^2 + B\,\psi_t\,\theta_t + M_1 = 0 \\ A = mRs\,(Rc + r) + sc(I_A - I_T - Ma^2), \quad B = I_Ac + m(Rc + r)\,(R + rc), \\ M_1 = g(m + M)\,Rs - g\,Mac. \tag{26}$$

The critical rolling rate is given by $\theta_{t\,c}^2 = 4AM_1/B^2$ with the corresponding critical spinning rate $\psi_{t\,c}^2 = M_1/A$. When the rolling rate is higher than the critical value, the spinning rate can take two values for fast and slow spinning motion.

The energy of the dynamic system in the three cases is:

$$E = (I_{11}\,\varphi_t^2 + I_{22}\,\theta_t^2 + I_{33}\,\psi_t^2)/2 + \psi_t\,(I_{13}\,\varphi_t + I_{23}\,\theta_t) + V. \tag{27}$$

A) for the mass fixed on the axle y_b: $V = (m + M)\,g(Rc + r) + Mgas$; B) for the mass fixed on the diameter x_l: $V = (m + M)\,g(Rc + r)$; C) for the mass fixed on the diameter z_b: $V = (m + M)\,g(Rc + r) + Mgac\,c_\theta$.

The variation of energy in time follows from the expression of energy (27) and the relations (24):

$$dE/dt = dV\,dt + (M_1\,\varphi_t + M_2\,\theta_t). \tag{28}$$

In cases A) and B) the energy of the system is conserved. The tilt and throttle torque redistributes the energy between its potential and kinetic angular form. In case B) the loaded mass increases the energy of the system: $dE/dt = g\,Mac\,\theta_t$. The wheel energy does not conserve because the loaded mass is not fixed on the wheel and the entire system is not a rigid body. The moment of inertia is not constant as that of a rigid body, but changes with the rolling angle as that of a deformable body. The loaded mass takes energy from outside to maintain the same altitude as the wheel center and does not roll with the wheel. When the loaded mass rolls with the wheel, the system corresponds to that of the case C), it is a rigid body and the system energy is conserved. In case B) it is possible to add to energy the additional potential $V = -g\, Mac\, \theta$, but again the energy is not conserved: $dE/dt = g\, Mas\, \theta\, \varphi_t$ because there is no contra-balancing angular momentum of the form $M_1 = -\, g\, Mas\, \theta$.

5. CONCLUSIONS

The dynamics of the toroidal wheel with loaded mass was studied. The reduced equations of motion were derived. Three cases were studied when the position of the loaded mass is on the wheel axle and on the two wheel diameters.

In the first case when the loaded mass is on the virtual axle a steady circular motion exists. The relation between the rolling and spinning rate for a constant leaning angle was deduced. In the second case when the loaded mass is maintained at the same altitude as the center of mass on the wheel diameter, the system is no longer an axi-symmetric body of revolution and the nutation rate depends not only on the leaning angle, but also on the rolling angle. In this case we have the rotation of a deformable body that takes the energy from outside to increase the rolling rate. In the last case the loaded mass is fixed on the wheel diameter and moves with the wheel.

The dynamic system was studied in the absence of the friction torque. An interesting problem is to study how the tilt and throttle torque can pump energy in this dissipative system and how the energy is distributed among the degrees of freedom of the system.

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