STABILITY ANALYSIS IN COSMOLOGICAL MODELS USING PERTURBATIVE METHODS IN $f(R,T)$ THEORY

R. K. MISHRA[a], NAVYA JAIN[b]

Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal-148106, Punjab, India
Corresponding author[a]: ravkmishra@yahoo.co.in
E-mail [b]: jainnavya008@gmail.com

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Abstract. This study examines the stability of cosmological models through perturbation theory, focusing on different formulations of the deceleration parameter (DP). By analysing the eigenvalues of perturbations, we identify key insights into the models' long-term dynamics and stability. The results reveal the presence of stable and marginally stable modes, indicating how perturbations evolve and dissipate over cosmic time. These findings provide a deeper understanding of cosmic evolution, the formation of large-scale structures, and the behaviour of perturbations, offering significant contributions to cosmological research and theoretical modelling of the Universe.

Key words: Stability analysis, $f(R,T)$ theory, deceleration parameter, cosmological model, expansion of the Universe.

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1. INTRODUCTION

The study of cosmological models has long been a cornerstone in understanding the large-scale structure and evolution of the Universe [1]. These models aim to describe the Universe from its earliest moments to its current state and predict its future evolution. To achieve this, cosmologists employ a variety of theoretical frameworks, each offering unique perspectives and solutions to fundamental questions [2–6]. Among these frameworks, modified theories of gravity have garnered significant attention for their potential to explain phenomena that remain unresolved by General Relativity (GR). These frameworks propose alterations or extensions to GR to address its limitations and better align with observational data [7–10]. One of the most compelling pieces of evidence for the need to explore beyond GR is the discovery of the accelerated expansion of the Universe. Observations of distant Type Ia supernovae in the late 1990s [11–13] and evidence from the cosmic microwave background (CMB) radiation [14–17] challenge the conventional framework of GR, prompting the development of modified theories of gravity that can incorporate these phenomena without resorting to dark energy (DE).

One such framework is the $f(R,T)$ theory of gravity, which extends the Einstein-Hilbert action by incorporating a function dependent on both the Ricci scalar (R) and the trace of the stress-energy tensor (T). Recent research suggests that the $f(R,T)$ theory is a promising candidate for addressing key cosmological insights, such as the late-time acceleration of the Universe and the nature of DE. This theory introduces modifications that can alter the gravitational interaction at cosmological scales and provides new insights into the dynamics of the Universe. Many cosmologists have published their recent results in $f(R,T)$ theory, exploring its potential and limitations [18–30]. However, the stability of the modified cosmological models is a crucial aspect that requires thorough investigation to ensure their physical viability.

The development of perturbation theory has been instrumental in understanding the stability and evolution of cosmological models. Lifshitz [31] was one of the first to systematically develop linear perturbation theory within the framework of GR, focusing on the evolution of small perturbations in a homogeneous Universe. His foundational work laid the groundwork for modern cosmic perturbation theory. The book "The Large Scale Structure of the Universe" by P.J.E. Peebles [32] extensively uses linear perturbation theory (LPT) to describe the growth of cosmic structures, becoming a cornerstone for cosmologists studying the Universe's large-scale structure. Bernardeau *et al.* [33] advanced this work by addressing the evolution of perturbations beyond the linear regime with non-linear perturbation theory. Perturbative methods offer a powerful tool for analysing the stability of cosmological models. By introducing small perturbations to a background solution, one can examine the system's response and identify potential instabilities. This approach not only helps in understanding the behavior of the models under small deviations but also aids in exploring the parameter space that yields stable solutions. In this research, we adopt a perturbative approach to study the stability of cosmological models within the framework of $f(R,T)$ gravity. We begin by deriving the field equations specific to $f(R,T)$ theory and establish the background cosmological solution. Subsequently, we introduce perturbations to the solutions and derive the corresponding continuity equation governing the evolution. Through this analysis, we aim to identify the conditions under which the cosmological model remains stable and the implications of the findings for the broader context of cosmic dynamics.

Our work is divided into six sections. Section 2 deals with the mathematical framework of the cosmological model, whereas solutions to the field equations are derived in Section 3. Section 4 covers a detailed description of the stability analysis through perturbative methods, and detailed interpretations are made by assuming two models in Section 5. Finally, the conclusions of this research study are presented in Section 6.

2. MATHEMATICAL FRAMEWORK

$f(R,T)$ theory of gravity is a significant extension of GR. GR describes gravity as the curvature of spacetime caused by mass and energy. Despite its tremendous success in explaining a wide range of gravitational phenomena, GR faces challenges, especially in cosmology and astrophysics. These challenges include the accelerated of DE. Moreover, GR struggles to integrate smoothly with quantum mechanics. The $f(R,T)$ theory was introduced by T. Harko and collaborators [10] in 2011 to address some of these issues. This theory generalizes the Einstein-Hilbert action of GR by incorporating arbitrary functions of the Ricci scalar R and the trace of the energy-momentum tensor T. The inclusion of T allows the theory to explicitly account for the interaction between matter and geometry, offering new insights into the role of matter in the dynamics of spacetime. As we know, the action principle allows us to derive the field equations by varying an action integral with respect to te metric tensor. In $f(R,T)$ theory, the action is given by

$$S=\int\left(\frac{f(R,T)}{16\pi}+\mathcal{L}_m\right)\sqrt{-g}d^4x \tag{1}$$

where,

- g is the determinant of the metric tensor g_{jk}.
- $\mathcal{L}_m$ is the matter Lagrangian density.
- $f(R,T)$ is an arbitrary function of the Ricci scalar R and the trace of the stress-energy tensor T.
- The term $\int\frac{f(R,T)}{16\pi}\sqrt{-g}d^4x$ represents the modified gravitational action. Here, $f(R,T)$ replaces the Ricci scalar R in the standard Einstein-Hilbert action, allowing for additional degrees of freedom.
- The term $\int\mathcal{L}_m\sqrt{-g}d^4x$ represents the matter action. It describes the dynamics of matter fields in the spacetime and contributes to the overall variation of the action.

The stress-energy tensor T_{jk}, a symmetric tensor characterized by four components, is defined with indices j and k ranging from 0 to 3, representing time and three spatial dimensions.

$$T_{jk}=-\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_m}{\delta g^{jk}}, \tag{2}$$

where $T=g^{jk}T_{jk}$ is momentum tensor's trace. By integrating T_{jk} into the action of the theory, it allows for the incorporation of the effects of matter and energy on the gravitational field. In the previously mentioned equation, we assume that the matter

Lagrangian ($\mathcal{L}_m$) depends only on the components of the metric tensor g_{jk} and not on their derivatives. Consequently, we arrive at the following result:

$$T_{jk} = g_{jk} - 2\frac{\partial \mathcal{L}_m}{\partial g^{jk}} \tag{3}$$

The field equations are obtained by varying the modified action S with respect to the components of the metric tensor g^{jk}. They can be written as:

$$(8\pi - f_T(R,T))T_{jk} + \frac{1}{2}f(R,T)g_{jk} = f_R(R,T)R_{jk} + f_T(R,T)\Theta_{jk} \\ + (g_{jk}\Box - \nabla_j\nabla_k)f_R(R,T), \tag{4}$$

where ∇_k represents covariant derivative. $\Box$ is the D'Alembertian operator which is the Laplace operator of Minkowski space, has the form

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle \tag{5}$$

where $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. The value of Θ_{jk} is given by

$$\Theta_{jk} = -2\left[T_{jk} + g^{\mu\nu}\frac{\partial^2 \mathcal{L}_m}{\partial g^{jk} g^{\mu\nu}}\right] + g_{jk}\mathcal{L}_m \tag{6}$$

Since there is no universally accepted way to express the matter Lagrangian ($\mathcal{L}_m$), we address the term involving the matter Lagrangian through different approaches. In this case, we choose to use a perfect fluid matter model with $\mathcal{L}_m = -p$, leading to the following result:

$$\Theta_{jk} = -2(T_{jk} + \frac{1}{2}pg_{jk}) \tag{7}$$

Hence using equation (7), the $f(R,T)$ gravity equations (4) takes the form

$$G_{jk}f_R(R,T) = 8\pi T_{jk} + (p + T_{jk})f_T(R,T) + \frac{1}{2}(f(R,T) - Rf_R(R,T)) \\ - (g_{jk}\Box - \nabla_j\nabla_k)f_R(R,T) \tag{8}$$

The presence of $\frac{\partial f}{\partial T} \equiv f_T(R,T)$ introduces additional terms dependent on the trace of the stress-energy tensor, indicating a direct interaction between matter and spacetime curvature. This interaction can alter gravitational dynamics, providing a mechanism to address the Universe's expansion and the behaviour of dark energy. In this context, we use natural units where $G = c = 1$, setting $\kappa^2 = 8\pi$ for the Einstein gravitational constant. The field equations in $f(R,T)$ gravity are affected by the matter in the field, leading to various speculative models for different choices of f. We focus on the functional form $f(R,T) = R + 2f(T)$, as proposed by Harko *et al.*, which has been extensively studied for its cosmological implications. This paper makes a significant contribution to theoretical physics, particularly in the realm of modified gravity theory. We adopt the same assumption and consider the $f(R,T)$ function as:

$$f(R,T) = R + 2f(T), \tag{9}$$

where $f(T)$ includes T as a random function.

3. SOLUTION TO FIELD EQUATIONS

In cosmological research, the Friedmann-Lemaître-Robertson-Walker (FLRW) metric is essential, providing a mathematical framework to describe spacetime geometry in line with the cosmological principle. This principle states that, on large scales, the Universe is homogeneous and isotropic. Thus, when developing the cosmological model, we define the FLRW metric as follows:

$$ds^2 = c^2dt^2 - a^2(dx^2 + dy^2 + dz^2) \tag{10}$$

Here $a \equiv a(t)$ denotes the scale factor, which is a function of time t. Additionally, by adopting the convention $c = 1$, we can use this framework to describe the stress-energy tensor for a perfect fluid characterized by uniform pressure p, the force exerted by dust particles on their surroundings, and the energy density ρ, which quantifies the amount of energy contained within a given spatial region per unit volume.

$$T_{jk} = -pg_{jk} + (p+\rho)u_j u_k \tag{11}$$

In this context, u_j denotes the four-velocity vector, which fulfills the condition $g_{jk}u_j u_k = 1$. In a co-moving coordinate system, where the particle is at rest, we find

$$u^j = (0,0,0,1). \tag{12}$$

Thus, the $f(R,T)$ gravity equations (8), for the linear choice $f(R,T) = R + 2\gamma T$, in the context of a perfect fluid distribution within an FLRW background, considering the components of the stress-energy tensor, are:

$$T_1^1 = T_2^2 = T_3^3 = -p \text{ and } T_4^4 = \rho$$

assume the following expressions

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \gamma\rho - (1+3\gamma)p \tag{13}$$

and

$$3\left(\frac{\dot{a}}{a}\right)^2 = (1+3\gamma)\rho - \gamma p. \tag{14}$$

The Hubble parameter $H(t)$ and the deceleration parameter $q(t)$ are crucial for understanding the dynamics of the Universe, providing insights into its expansion and evolution. While $H(t)$ indicates the rate of expansion, $q(t)$ reveals how this expansion rate changes over time. To further explore this, we define these parameters as follows:

$$H(t) = \frac{\dot{a}}{a} \tag{15}$$

and

$$q(t) = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{16}$$

In our study, the deceleration parameter q is a key feature of cosmic evolution. It distinguishes between phases of the Universe's expansion: one characterized by deceleration ($q \equiv q(t) > 0$) and another by accelerated expansion ($q < 0$). Notably, observational data from Type Ia supernovae offer strong evidence for a significant cosmic transition [11–13], indicating a shift from an era of deceleration to one of accelerated expansion. This shift underscores the need to adjust the values of the deceleration parameter to accurately reflect the Universe's evolving dynamics.

4. STABILITY ANALYSIS

In cosmology, stability analysis through perturbation techniques is an indispensable tool for validating the robustness and physical viability of theoretical models, particularly in the realm of modified gravity such as $f(R,T)$ theory. This approach allows researchers to probe the resilience of cosmological models against small perturbations in the metric and matter fields, thereby determining the conditions under which these models can maintain their stability over cosmic timescales. In $f(R,T)$ gravity, perturbative analysis aids in uncovering the intricate dynamics introduced by these modifications, offering insights into how deviations from GR impact the Universe's evolution. By linearizing the field equations around a homogeneous and isotropic background and studying the resulting perturbation equations, researchers can identify potential instabilities that may lead to the non-physical growth of fluctuations or the onset of singularities.

This stability analysis is not merely a theoretical exercise; it provides critical constraints that any viable cosmological model must satisfy to be consistent with observations. Moreover, understanding the stability of these models under perturbations is essential for predicting the formation and evolution of cosmic structures, ensuring that the theoretical framework aligns with the empirical data from large-scale structure surveys, cosmic microwave background measurements, and other cosmological observations. In this section, we apply perturbation techniques to the FLRW metric to analyse the stability of the Universe, characterized by its density and pressure fields. Since the FLRW metric provides a well-defined background that represents the large-scale structure of the Universe, it sets the stage for understanding deviations from homogeneity and isotropy. From equations (10), (13), and (14), we derive our continuity equation incorporating $\rho, p,$ and $\frac{\dot{a}}{a}$ as follows

$$(1+3\gamma)\dot{\rho}-\gamma\dot{p}=-3(1+2\gamma)\frac{\dot{a}}{a}(\rho+p) \tag{17}$$

The continuity equation encapsulates the principle of energy conservation in an expanding Universe, ensuring that as the Universe expands, the energy density and pressure adjust in a manner consistent with conservation laws. Further from equation (17), we obtain:

$$(1+3\gamma)\ddot{\rho}-\gamma\ddot{p}=-3(1+2\gamma)\frac{\dot{a}}{a}(\dot{\rho}+\dot{p})+\frac{3}{2}(1+2\gamma)^2(\rho+p)^2 \tag{18}$$

Since, perturbing the system allows us to examine how small deviations from the equilibrium state evolve over time. We introduce small perturbations to the density and pressure fields

$$\rho=\rho_0+\delta\rho \tag{19}$$

and

$$p=p_0+\delta p \tag{20}$$

Here, ρ_0 and p_0 are the equilibrium values and $\delta\rho$ and δp are small perturbations. This is essential for determining the system's response to disturbances and hence its stability. Assuming the perturbation of the exponential form, *i.e.* $\delta\rho=\delta\rho_0 e^{\lambda t}$ and $\delta p=\delta p_0 e^{\lambda t}$, and substituting back the perturbed quantities into the continuous equations and linearizing them, we obtain a set of linearized equations. These can be expressed in matrix form $A.\delta=0$, where A is the perturbation matrix and δ is the vector of perturbations. So using equations (18)-(20), we obtain

$$\begin{bmatrix}(1+3\gamma)\lambda^2+X & 0\\ 0 & -\gamma\lambda^2+X\end{bmatrix}\begin{bmatrix}\delta\rho\\ \delta p\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \tag{21}$$

where, $X=3(1+2\gamma)\frac{\dot{a}}{a}\lambda-3(1+2\gamma)^2(\rho_0+p_0)$. Here, to solve for the eigenvalues λ, we consider the homogeneous system obtained by setting the right-hand side to zero. The perturbation matrix encapsulates how the perturbations interact and evolve, making it crucial in determining the stability of the system, as its eigenvalues dictate the nature of perturbation growth or decay. To find the non-trivial solutions for λ, we set the determinant of the coefficient matrix to zero. This condition ensures that the system has non-trivial solutions. This gives two separate equations:

$$(1+3\gamma)\lambda^2+3(1+2\gamma)\frac{\dot{a}}{a}\lambda-3(1+2\gamma)^2(\rho_0+p_0)=0 \tag{22}$$

and

$$-\gamma\lambda^2+3(1+2\gamma)\frac{\dot{a}}{a}\lambda-3(1+2\gamma)^2(\rho_0+p_0)=0 \tag{23}$$

Analysing the signs and magnitudes of the eigenvalues allows us to determine the stability of the Universe. This analysis allows us to categorize the Universe as stable or unstable under small perturbations. It also provides insight into the timescales of stability or instability, which is important for predicting the evolution of the Universe over time. If $\lambda > 0$, then this indicates that there is at least one mode in which perturbations will grow exponentially. This is usually a strong indicator that the overall system is unstable because even if one perturbation grows, it can dominate the system's behaviour over time. If $\lambda = 0$, then model is marginally stable. Perturbations in this mode will neither grow nor decay exponentially but could grow linearly or remain constant. If $\lambda < 0$, *i.e.* a negative eigenvalue, this indicates a damping or stable mode in the system. Perturbations associated with this eigenvalue will decay exponentially over time, leading to stability in this particular mode. Now to check the stability, we consider two models which are described in the next section.

5. DETAILED INTERPRETATION

5.1. MODEL-I

In this model, we considered an assumption of fractional quadratic deceleration parameter (FQDP) as suggested by Mishra *et al.* [19]

$$q(t) = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{\beta(1-t^2)}{1+t^2}, \tag{24}$$

where β is a positive constant and $a \equiv a(t)$ is the scale factor. The choice of β is made to show the flipping nature of cosmos. From equation (24), we obtain the value of Hubble parameter as follows

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{(1-\beta)t + 2\beta\tan^{-1}t}. \tag{25}$$

Figure 1 shows the visual representation of DP and Hubble parameter. Clearly as $t \to \infty$, DP shows the transitional phase from deceleration to the acceleration phase. On the other hand, after crossing early inflationary phase of Universe, Hubble parameter tends to the positive small value. The significance of this behaviour is profound in understanding the dynamics of the Universe. The transition of the DP from positive to negative values signifies a shift from a decelerating to an accelerating Universe, reflecting the influence of DE becoming dominant over matter. This transition phase is critical in cosmological models as it aligns with observational data indicating an accelerating Universe. Similarly, the Hubble parameter's tendency towards a small positive value after the early inflationary phase implies a sustained but slower rate of expansion, consistent with current observations of the Universe's expansion. This behaviour supports the notion that the Universe has undergone various phases

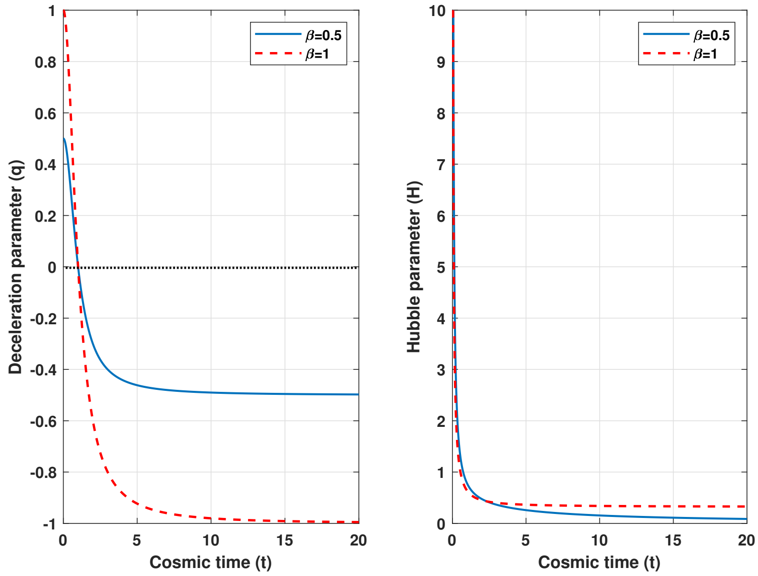

Fig. 1 – DP & Hubble parameter's visualization w.r.t cosmic time (t).

of expansion, driven by different dominant components at different times. Now substituting the value of $\frac{\dot{a}}{a}$ into the equations (22) and (23), and solving for λ, we obtain the following expressions

$$\lambda = \frac{-3(1+2\gamma)}{2(1+3\gamma)((1-\beta)t+2\beta\tan^{-1}t)} \pm \frac{1+2\gamma}{2(1+3\gamma)} \sqrt{\frac{9}{((1-\beta)t+2\beta\tan^{-1}t)^2}+12(1+3\gamma)(\rho_0+p_0)} \quad (26)$$

or

$$\lambda = \frac{-3(1+2\gamma)}{-2\gamma((1-\beta)t+2\beta\tan^{-1}t)} \pm \frac{(1+2\gamma)}{-2\gamma} \sqrt{\frac{9}{((1-\beta)t+2\beta\tan^{-1}t)^2}-12(1+3\gamma)(\rho_0+p_0)} \quad (27)$$

Since ρ_0 and p_0 are the equilibrium values, we consider $p_0 = -\rho_0$ for a cosmological constant-dominated Universe. Solving these equations give one value of λ to be zero, with the other values shown in the figures. A zero eigenvalue signifies a special case where the perturbation mode neither grows nor decays. From a stability perspective, it suggests the presence of slow, long-term dynamics, leading to significant structural

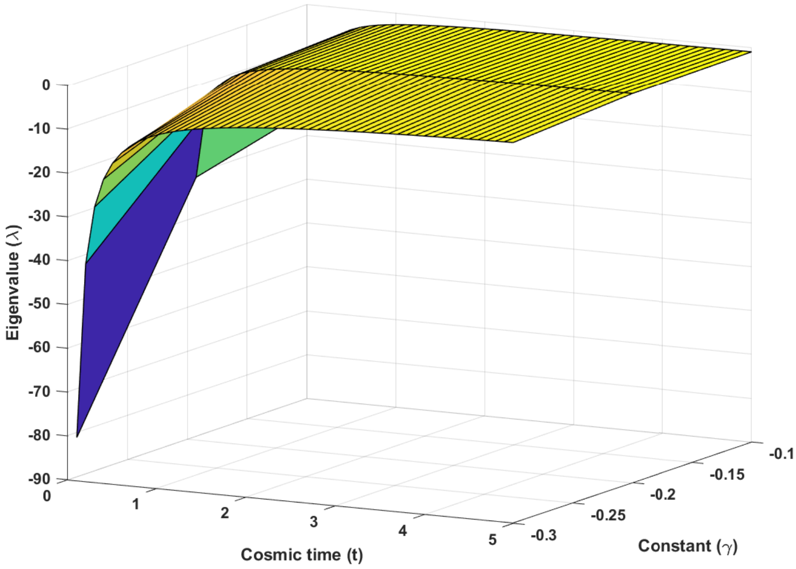

Fig. 2 – Eigenvalue's visualization w.r.t. cosmic time (t) for eq. (26).

evolution over extended timescales. Figures 2 and 3 show that the eigenvalues are negative, corresponding to stable modes where perturbations decay exponentially over time, bringing the system back toward equilibrium. The combination of one zero eigenvalue and two negative eigenvalues implies overall stability, ensuring that initial deviations from equilibrium diminish. This means that small inhomogeneities do not result in unbounded growth or collapse, thus maintaining the system's overall balance. This stability is crucial for understanding the expansion of the Universe. A stable Universe model suggests that the expansion driven by the cosmological constant (or dark energy) will continue smoothly, without leading to catastrophic instabilities.

5.2. MODEL-II

Now to check the stability, we considered the assumption of quadratic deceleration parameter as suggested by [34].

$$q(t) = -1 + \frac{d}{dt}\left(\frac{1}{H}\right) = -1 + 3(t^2 - 6\eta t + 6\eta^2), \tag{28}$$

where β is a positive constant. Now after simplifying equation (28), we obtain

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{t(3\eta - t)(6\eta - t)}. \tag{29}$$

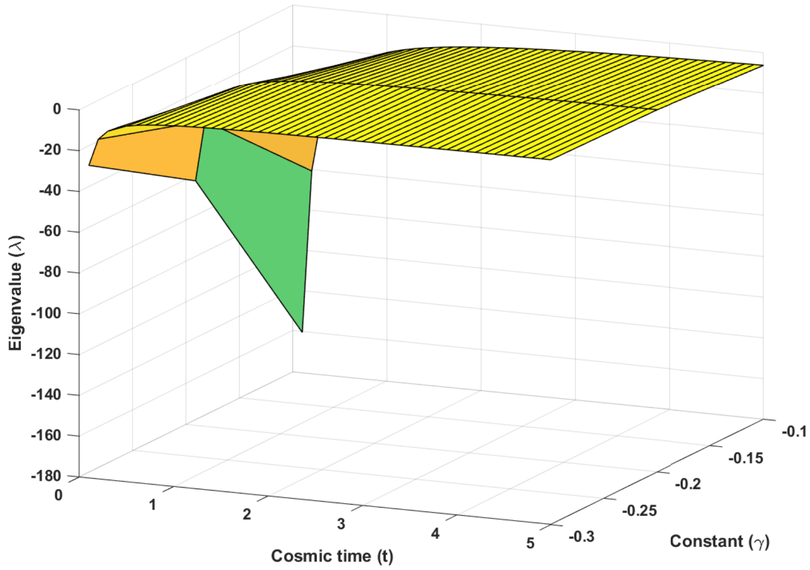

Fig. 3 – Eigenvalue's visualization w.r.t. cosmic time (t) for eq. (27).

For simplicity, we set the constant of integration to zero because at $t = 0$, $H \to \infty$ as at the early inflationary phase, the Universe's expansion rate is very large.

Figure 4 illustrates the pictorial representation of quadratic DP and corresponding Hubble parameter. Initially, the Universe decelerates ($q(t) > 0$), corresponding to the matter-dominated phase. As time progresses, the Universe transitions to an accelerating phase ($q(t) < 0$), corresponding to the DE-dominated phase. This indicates the Universe's dynamic phases from deceleration to acceleration and potential future behaviour influenced by the quadratic assumption. Similarly, the initial decrease in $H(t)$ indicates the slowing expansion rate during the decelerating phase. As the DP becomes negative, $H(t)$ starts to rise, indicating an increasing expansion rate during the accelerating phase.

Now substituting the value of $\frac{\dot{a}}{a}$ into the equations (22) and (23), and solving for λ, we obtain the following expressions

$$\lambda = \frac{-3(1+2\gamma)}{2(1+3\gamma)(t(3\eta-t)(6\eta-t))} \pm \frac{1+2\gamma}{2(1+3\gamma)} \sqrt{\frac{9}{(t(3\eta-t)(6\eta-t))^2} + 12(1+3\gamma)(\rho_0+p_0)} \quad (30)$$

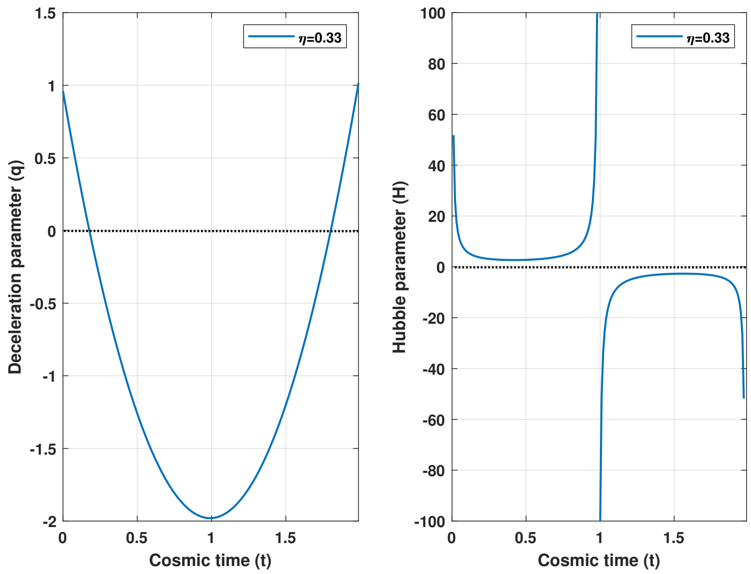

Fig. 4 – DP & Hubble parameter's visualization w.r.t cosmic time (t).

or

$$\lambda = \frac{-3(1+2\gamma)}{-2\gamma((1-\beta)t+2\beta\tan^{-1}t)} \pm \frac{(1+2\gamma)}{-2\gamma} \sqrt{\frac{9}{(t(3\eta-t)(6\eta-t))^2} - 12(1+3\gamma)(\rho_0+p_0)} \, . \quad (31)$$

After solving these equations for λ, we find one value of λ to be zero, with the other values shown in the corresponding Figs. 5 and 6. Both figures show significant variation over cosmic time indicating a dynamic and evolving system. This indicates that the parameter influences the gravitational interactions as the cosmological models are not static but change over time. Initially, at $t \to 0$, the eigenvalues indicate strong stability as they are significantly negative. As time progresses, the eigenvalues increase and tend towards zero. This suggests a transition phase where the stability of the model is in question, but the system does not become unstable as the eigenvalues remain non-positive.

As time progresses further, the eigenvalues start to decrease again, moving further into the negative region. This continued negativity of the eigenvalues throughout the time period indicates ongoing stability in the cosmological model. However, due to the quadratic nature of the DP, a similar pattern of eigenvalue behaviour appears

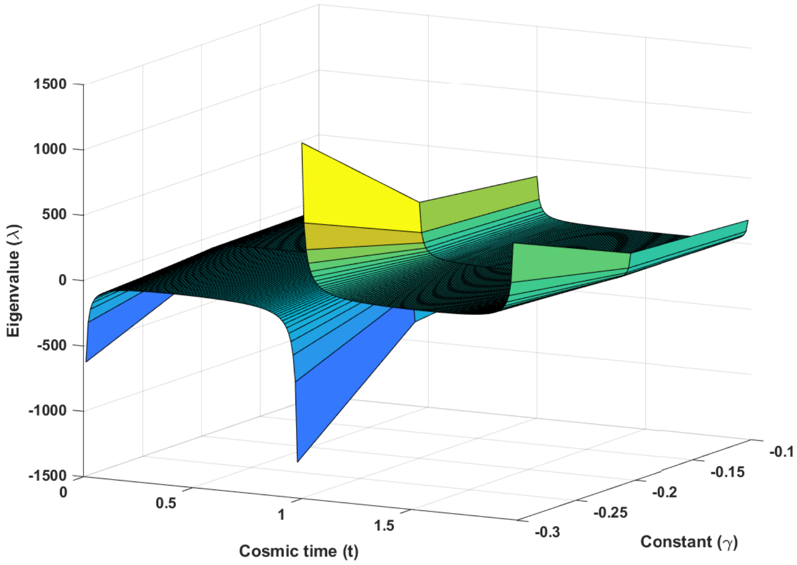

Fig. 5 – Eigenvalue's visualization w.r.t. cosmic time (t) for eq. (30).

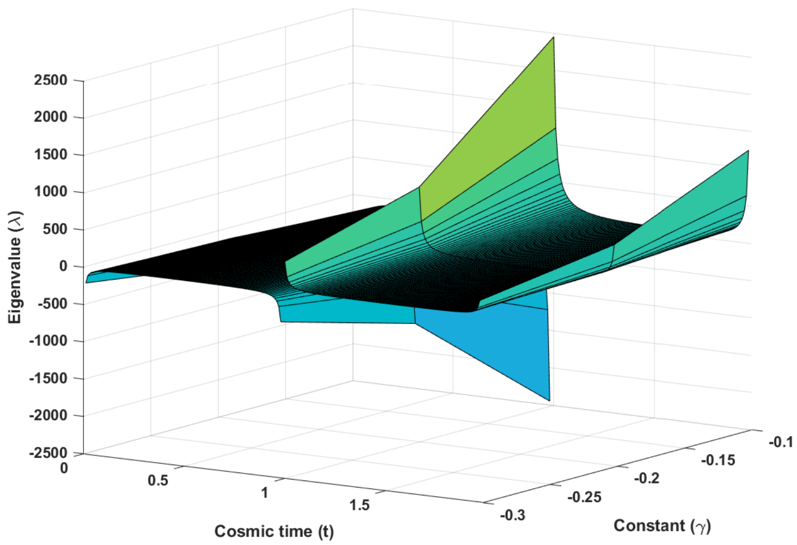

Fig. 6 – Eigenvalue's visualization w.r.t. cosmic time (t) for eq. (31).

in the positive direction, suggesting the presence of transient instabilities. These transient positive eigenvalues indicate periods of instability, but they are not dominant in the long run. The eigenvalues exhibit large fluctuations both positive and negative, suggesting the phases of stability and instability. Overall, the eigenvalues remain negative for the majority of cosmic time, ensuring stability in the model. The quadratic assumption of the DP introduces complex dynamics, including temporary phases of instability. Yet, the model stabilizes overall, reflecting the evolution of the Universe from an initial stable state, through periods of transitional instability, and back to a stable state. This behaviour aligns with the expected phases of cosmic evolution: from deceleration to acceleration, and then approaching a stable expansion driven by DE.

6. CONCLUSION

The perturbative methods has a rich history in cosmology, evolving from classical mechanics to a modern cosmology. It has enabled cosmologists to connect theoretical models with observational data, advances our understanding of the Universe's structure and evolution. In this study, we have analysed the stability of two cosmological models through perturbation theory: one assuming a fractional quadratic deceleration parameter DP and the other a quadratic DP. The eigenvalue analysis for both models provides key insights into their stability and evolution over cosmic time. For the model with a fractional quadratic DP, we observed one zero eigenvalue and two negative eigenvalues. This configuration indicates overall stability with a marginally stable mode. The zero eigenvalue represents a neutrally stable mode, suggesting long-term, slow dynamics that could influence large-scale structure formation. The negative eigenvalues ensure that most perturbations decay over time, contributing to the Universe's stability on these scales.

In contrast, the model with a quadratic DP exhibits a more complex eigenvalue behavior. Initially, the eigenvalues are highly negative, indicating strong stability. As time progresses, the eigenvalues trends towards zero around $t \to 1$, marking a transitional phase where stability is less certain but not lost. Beyond this point, the eigenvalues decrease again, reaffirming the model's stability. The transient positive eigenvalues observed in this model indicate temporary phases of instability, which are nonetheless overshadowed by the overall dominance of negative eigenvalues, ensuring long-term stability. Hence, the complex symmetry in the graphs highlights the non-linear nature of the interaction with the cosmological model. Overall, both models display stability, but with distinct dynamical behaviors. The fractional quadratic DP model maintains consistent stability with marginally stable modes, while the quadratic DP model shows complex, transient dynamics but ultimately stabilizes. These findings highlight the nuanced interplay between different forms of the deceleration parameter and the resulting cosmological stability, providing valuable insights into the Universe's evolution and the role of perturbations in shaping cosmic structures.

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