CONNECTION MATRICES ON THE SIEGEL-JACOBI UPPER HALF SPACE AND EXTENDED SIEGEL-JACOBI UPPER HALF SPACE

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Abstract. The inverse of the metric matrices on the Siegel-Jacobi upper half space $\mathcal{X}_n^J$, invariant to the restricted real Jacobi group $G_n^J(\mathbb{R})_0$ and extended Siegel-Jacobi $\tilde{\mathcal{X}}_n^J$ upper half space, invariant to the action of the real Jacobi $G_n^J(\mathbb{R})$, are presented. The results are relevant for Berezin quantization of the manifolds $\mathcal{X}_n^J$ and $\tilde{\mathcal{X}}_n^J$. Explicit calculations in the case $n=2$ are given.

Key words: Berry phase, Berry connection, Connection matrix, Jacobi group, invariant metric, Siegel-Jacobi disk, Siegel-Jacobi upper half plane, extended Siegel-Jacobi upper half plane, almost cosymplectic manifold.

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1. INTRODUCTION

The real Jacobi group of degree n is defined as $G_n^J(\mathbb{R}) := \mathrm{H}_n(\mathbb{R}) \rtimes \mathrm{Sp}(n,\mathbb{R})$, where $\mathrm{H}_n(\mathbb{R})$ denotes the real Heisenberg group [1–3]. The Siegel-Jacobi upper half space is the $G_n^J(\mathbb{R})$-homogeneous manifold $\mathcal{X}_n^J := \frac{G_n^J(\mathbb{R})}{\mathrm{U(n)}\times\mathbb{R}} \approx \mathcal{X}_n \times \mathbb{R}^{2n}$, where $\mathcal{X}_n$ denotes the Siegel upper half space realized as $\frac{\mathrm{Sp}(n,\mathbb{R})}{\mathrm{U(n)}}$ [4, page 398]. Also $\mathcal{X}_n^J = \frac{G_n^J(\mathbb{R})_0}{\mathrm{U(n)}}$, where $G_n^J(\mathbb{R})_0$ denotes the restricted real Jacobi group [5, §5].

The Jacobi group $G_n^J := \mathrm{H}_n \rtimes \mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$, where $\mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}} := \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(n,n)$, is also studied in Mathematics, Mathematical Physics and Theoretical Physics, together with the G_n^J-homogeneous Siegel-Jacobi ball $\mathcal{D}_n^J \approx \mathbb{C}^n \times \mathcal{D}_n$, where $\mathcal{D}_n$ denotes the Siegel ball realized as $\frac{\mathrm{Sp}(n,\mathbb{C})}{\mathrm{U}(n)}$ [4, page 399].

The extended Siegel-Jacobi upper half space is $\tilde{\mathcal{X}}_n^J := \frac{G_n^J(\mathbb{R})}{\mathrm{U(n)}} \approx \mathcal{X}_n \times \mathbb{R}^{2n+1}$ [6].

It is well known that $G_n^J(\mathbb{R})$, $\mathrm{Sp}(n,\mathbb{R})$, $\mathrm{H}_n(\mathbb{R})$, $\mathcal{X}_n^J$ and $\mathcal{X}_n$ are isomorphic with G_n^J, $\mathrm{Sp}(n,\mathbb{R})_{\mathbb{C}}$, H_n, $\mathcal{D}_n^J$, respectively $\mathcal{D}_n$ [6].

The dimensions of the enumerated manifolds are: $\dim \mathrm{Sp}(n,\mathbb{R}) = 2n^2+n$, $\dim \mathrm{H}_n(\mathbb{R}) = 2n+1$, $\dim G_n^J(\mathbb{R}) = (2n+1)(n+1)$, $\dim G_n^J(\mathbb{R})_0 = \dim G_n^J(\mathbb{R}) - 1$, $\dim \mathrm{U}(n) = n$, $\dim \mathcal{X}_n^J = n(n+3)$, $\dim \tilde{\mathcal{X}}_n^J = \dim \mathcal{X}_n^J + 1$, $\dim \mathcal{X}_n = n(n+1)$. In particular, $\dim \mathcal{X}_1^J = 4$, $\dim \tilde{\mathcal{X}}_1^J = 5$, $\dim \mathcal{X}_2^J = 10$, $\dim \tilde{\mathcal{X}}_1^J = 11$.

The Jacobi group, as a unimodular group, is a non-reductive, algebraic group of Harish-Chandra type. $\mathcal{D}_n^J$ is a partially bounded domain, non-symmetric, a Lu Qi-Keng manifold, a projectively induced quantizable Kähler manifold [7].

In our papers [7–11] we studied invariant metrics to the action of the Jacobi group and Berezin quantisation [12–17] on Siegel-Jacobi upper half-plane $\mathcal{X}_1^J$, extended Siegel-Jacobi upper half-plane $\tilde{\mathcal{X}}_1^J$, Siegel-Jacobi space $\mathcal{X}_n^J$, Siegel-Jacobi ball $\mathcal{D}_n^J$. One important step in the investigation of Berezin quantisation on $\mathcal{X}_n^J$ and $\tilde{\mathcal{X}}_n^J$ is the calculation of the inverse matrix of invariant metric to the real Jacobi group on the mentioned manifolds. The aim of the present investigation of the inverse of the invariant metric on $\tilde{\mathcal{X}}_n^J$. Even in case $n=2$ the formulas presented here are quite complicated.

The new results contained in this paper concern the matrix of the inverse metric on the Siegel-Jacobi space $g^{-1}_{\mathcal{X}_n^J}(x,y,p,q)$ in Proposition 1 and extended Siegel-Jacobi space $g^{-1}_{\tilde{\mathcal{X}}_n^J}(x,y,p,q,\kappa)$ in Proposition 2, which are particularised in Proposition 4 to the case $n=2$. Also in Proposition 5 we calculate $\det g'_{\tilde{\mathcal{X}}_2^J}(x,y,q,p,\kappa)$ and the inverse of the same matrix for the case $n=2$. Remarks 1 and 2 systematize the results of Proposition 5.

Notation. We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}$ the field of real numbers, the field of complex numbers, the ring of integers, and the set of non-negative integers, respectively. We denote the imaginary unit $\sqrt{-1}$ by i, the real and imaginary parts of a complex number $z\in\mathbb{C}$ by $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ respectively, and the complex conjugate of z by $\bar{z}$. We denote by d the differential. We use Einstein's summation convention, *i.e.* repeated indices are implicitly summed over. The set of vector fields (1-forms) on real manifolds is denoted by $\mathfrak{D}^1$ (respectively $\mathfrak{D}_1$). We denote a mixed tensor contravariant of degree r and covariant of degree s by $\mathfrak{D}_s^r=\mathfrak{D}^r\times\mathfrak{D}_s$, where $\mathfrak{D}^r=\underbrace{\mathfrak{D}^1\times\cdots\times\mathfrak{D}^1}_{r}$ and $\mathfrak{D}_s=\underbrace{\mathfrak{D}_1\times\cdots\times\mathfrak{D}_1}_{s}$ [4, pages 13-17]. If we denote with Roman capital letteres the Lie groups, then their associated Lie algebras are denoted with the corresponding lower-case letteres. If $\mathfrak{H}$ is a Hilbert space, then we adopt the convention that the scalar product $(\cdot,\cdot)$ on $\mathfrak{H}\times\mathfrak{H}$ is antilinear in the first factor $(\lambda a,b)=\bar{\lambda}(a,b),\quad \lambda\in\mathbb{C}\setminus\{0\}$. If π is a representation of a Lie group G on the Hilbert $\mathfrak{H}$ and $X\in\mathfrak{g}$, then we denote $\mathbf{X}:=\mathrm{d}\pi(\mathbf{X})$ [18–20]. The interior product $i_X\omega$ (interior multiplication or contraction) of the differential form ω with $X\in\mathfrak{D}^1$ is denoted $X\lrcorner\omega$. We denote by $M(n,m,\mathbb{F})$ the set of $n\times m$ matrices with elements in the field $\mathbb{F}$ and $M(n,\mathbb{F})$ denotes $M(n,n,\mathbb{F})$. If $X\in M(n,m,\mathbb{F})$, then X^t denotes the transpose of X. We denote by $MS(n,\mathbb{F})=\{X\in M(n,\mathbb{F})|X=X^t\}$. The conjugate transpose (or hermitian transpose) of $A\in M(q,\mathbb{C})$ is $A^H:=\bar{A}^t$, also denoted A^*, $A^\dagger$, A^+. If f is a function on $\mathbb{C}^n$, we write the total differential of f as $\mathrm{d}f=\partial f+\bar{\partial}f$, with $\partial f=\sum_1^n\partial_\alpha f\mathrm{d}z_\alpha$, where $\partial_\alpha f=\frac{\partial f}{\partial z_\alpha}$ [21, page 6]. If $F=F_{ij}\in M(n,\mathbb{F})$, where

$\mathbb{F} = \mathbb{R},\ \mathbb{C}$, we denote by F^{ij} the elements of the inverse F^{-1}, *i.e.* $F_{ik}F^{kj} = \delta_{ij}$. If $A,\ B$ are matrices, then $A \otimes B$ denotes the Kronecker product of A, B [22, page 3].

2. PREPARATION

Following [5, §5], we consider the restricted real group $G_n^J(\mathbb{R})_0$ consisting of elements of the form $g = (M, X)$, where $M \in \mathrm{Sp(n,\mathbb{R})}, \mathrm{X} = (\lambda, \mu),\ \lambda,\ \mu \in M(1, n, \mathbb{R})$.

We consider the Siegel-Jacobi upper half space $\mathcal{X}_n$ realized as in (2.1).

We introduce for $\mathcal{X}_n^J$ the analog parametrization used in [23, page 7], [1, page 11], [24, § 38] for $\mathcal{X}_1^J$

$$u := pv + q,\ v := x + \mathrm{i}\, y,\ v = v^t,\ y > 0,\ p, q \in M(1, n, \mathbb{R}). \tag{2.1}$$

It should be noted that there is an isomorphism $G_n^J(\mathbb{R}) \ni (M, X, K) \to (M, X) \in G_n^J(\mathbb{R})_0$ through which the action of $G_n^J(\mathbb{R})_0$ on $\mathcal{X}_n^J$ can be defined as in [5, Proposition 2].

It is easy to prove that [11, Lemma 8]

Lemma 1. *a) If $\mathcal{X}_n \ni v = x + \mathrm{i}\, y$, then the action of $G_n^J(\mathbb{R})_0$ on $\mathcal{X}_n^J$: $(M, X) \times (v', u') \to (v_1, u_1)$, where $M \in \mathrm{Sp}(n, \mathbb{R})$ has the expression*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in M(n, \mathbb{R}), \tag{2.2}$$

is given by the formulas

$$v_1 = (av' + b)(cv' + d)^{-1} = (v'c^t + d^t)^{-1}(v'a^t + b^t), \tag{2.3a}$$

$$u_1 = (u' + \lambda v' + \mu)(cv' + d)^{-1}. \tag{2.3b}$$

b) For $\lambda, \mu \in M(1, n, \mathbb{R})$, let us consider (p, q) such that

$$(p, q) = (\lambda, \mu)M^{-1} = (\lambda d^t - \mu c^t, -\lambda b^t + \mu a^t), \tag{2.4a}$$

$$(\lambda, \mu) = (p, q)M = (pa + qc, pb + qd), \quad p, q, \lambda, \mu \in M(1, n, \mathbb{R}). \tag{2.4b}$$

Then the action of $G_n^J(\mathbb{R})_0$ on $\mathcal{X}_n^J$: $(M, X) \times (x', y', p', q') \to (x_1, y_1, p_1, q_1)$ is given by (2.3a)*, while*

$$(p_1, q_1) = (p, q) + (p', q') \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = (p + p'd^t - q'c^t, q - p'b^t + q'a^t). \tag{2.5}$$

c) The action of $G_n^J(\mathbb{R})$ on $\tilde{\mathcal{X}}_n^J \approx \mathcal{X}_n^J \times \mathbb{R}$:

$$(M, (\lambda, \mu), \kappa) \times (v', u', \kappa') \to (v_1, u_1, \kappa_1), \tag{2.6a}$$

$$(M, (\lambda, \mu), \kappa) \times (x', y', p', q', \kappa') \to (x_1, y_1, p_1, q_1, \kappa_1) \tag{2.6b}$$

is given by (2.3)*,* (2.5) *and*

$$\kappa_1 = \kappa + \kappa' + \lambda q'^t - \mu p'^t.$$

d) The 1*-form*

$$\lambda^R = \mathrm{d}\kappa - p\mathrm{d}q^t + q\mathrm{d}p^t \tag{2.7}$$

is invariant to the action (2.6) *of* $G_n^J(\mathbb{R})$ *on* $\tilde{\mathcal{X}}_n^J$.

e) The action of $G_n^J(\mathbb{R})$ *on* $G_n^J(\mathbb{R})$

$$(M,(\lambda,\mu),\kappa)\times(S_n)' \to (S_n)_1,$$

is given in (2.8) *for* $X', Y' \in (S_n)'$,

$$X_1 - \mathrm{i}Y_1 = (y_1)^{\frac{1}{2}}\{(cx'+d)(y')^{-\frac{1}{2}}X' + c(y')^{\frac{1}{2}}Y' + \mathrm{i}[cy'^{\frac{1}{2}}X' - (cx'+d)(y')^{-\frac{1}{2}}Y']\}, \tag{2.8}$$

while the other actions are given in a)-d) *of the present Lemma.*

If parameters $\alpha,\ \gamma,\ \delta > 0$, we have proved [11, Theorem 1]:

Theorem 1. *The metric on* $\mathcal{X}_n^J$, $G_n^J(\mathbb{R})_0$*-invariant to the action in* Lemma 1*, has the expressions*

$$\begin{aligned}
\mathrm{d}s^2_{\mathcal{X}_n^J}(x,y,q,p) =& \alpha\mathrm{Tr}[(y^{-1}\mathrm{d}x)^2 + (y^{-1}\mathrm{d}y)^2] \\
&+\gamma[\mathrm{d}p(xy^{-1}x + yy^{-1}y)\mathrm{d}p^t + \mathrm{d}qy^{-1}\mathrm{d}q^t + 2\mathrm{d}pxy^{-1}\mathrm{d}q^t]; \quad (2.9a)\\
\mathrm{d}s^2_{\mathcal{X}_n^J}(x,y,\chi,\psi) =& \alpha\mathrm{Tr}[(y^{-1}\mathrm{d}x)^2 + (y^{-1}\mathrm{d}y)^2] \\
&+\gamma[\mathrm{d}\psi^t(xy^{-1}x + yy^{-1}y)\mathrm{d}\psi + \mathrm{d}\chi^t y^{-1}\mathrm{d}\chi + 2\mathrm{d}\psi^t xy^{-1}\mathrm{d}\chi]; \quad (2.9b)\\
\mathrm{d}s^2_{\mathcal{X}_n^J}(x,y,\xi,\rho) =& \alpha\mathrm{Tr}[(y^{-1}\mathrm{d}x)^2 + (y^{-1}\mathrm{d}y)^2] \\
&+\gamma[\mathrm{d}\xi y^{-1}\mathrm{d}\xi^t + \mathrm{d}\rho y^{-1}\mathrm{d}\rho^t + \rho y^{-1}\mathrm{d}xy^{-1}(\rho y^{-1}\mathrm{d}x)^t \\
&+\rho y^{-1}\mathrm{d}yy^{-1}(\rho y^{-1}\mathrm{d}y)^t - 2\rho y^{-1}\mathrm{d}xy^{-1}\mathrm{d}\xi^t - 2\rho y^{-1}\mathrm{d}y^{-1}\mathrm{d}\rho^t]. \quad (2.9c)
\end{aligned}$$

The three parameter metric on $\tilde{\mathcal{X}}_n^J$, $G_n^J(\mathbb{R})$*-invariant to the action* c) *in* Lemma 1*, is*

$$\begin{aligned}
\mathrm{d}s^2_{\tilde{\mathcal{X}}_n^J}(x,y,q,p,\kappa) &= \mathrm{d}s^2_{\mathcal{X}_n^J}(x,y,q,p) + \lambda_6^2 \\
&= \alpha\mathrm{Tr}[(y^{-1}\mathrm{d}x)^2 + (y^{-1}\mathrm{d}y)^2] \\
&+\gamma[\mathrm{d}p(xy^{-1}x + yy^{-1}y)\mathrm{d}p^t + \mathrm{d}qy^{-1}\mathrm{d}q^t + 2\mathrm{d}pxy^{-1}\mathrm{d}q^t] \\
&+\delta(\mathrm{d}\kappa - p\mathrm{d}q^t + q\mathrm{d}p^t)^2.
\end{aligned} \tag{2.10}$$

Formula (2.9) ((2.10)) is a generalisation to $\mathcal{X}_n^J$ ($\tilde{\mathcal{X}}_n^J$), $n \in \mathbb{N}$, of equation (5.25b) (respectively, (5.30)) in [25] corresponding to $n = 1$.

3. BERRY PHASE ON KÄHLER MANIFOLDS

3.1. BALANCED METRIC

Let us denote [26, page 138]:

$$\Gamma_{ijk} := g_{lj}\Gamma^l_{ik}, \quad w_{ik} := g_{lk}w^l_i.$$

Also

$$\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ijk} + \Gamma_{jik},$$

or

$$g_{ij,k} := \partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il} = 0,$$

and

$$\Gamma_{ikj} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right),\ \Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{il}}{\partial u^j} + \frac{\partial g_{jl}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^l}\right),\ g_{ij}g^{jk} = \delta^k_i. \tag{3.1}$$

Γ_{ijk} (Γ^i_{jk}) is called Christofell's symbol of first (respectively second) kind.

Let $v = x + \mathrm{i}y$, $x, y \in MS(n, \mathbb{R})$, $y > 0$, $p, q \in M(1, n, \mathbb{R})$. Then ([27, Theorem 1], [28, Proposition 3],

$$\mathrm{d}s^2_{\mathcal{X}^J_n}(x,y,q,p) = \mathrm{d}x g_{xx} \mathrm{d}x^t + \mathrm{d}y g_{yy} \mathrm{d}y^t + \mathrm{d}q g_{qq} \mathrm{d}q^t + \mathrm{d}p g_{pp} \mathrm{d}p^t + \mathrm{d}q g_{qp} \mathrm{d}p^t + \mathrm{d}p g_{pg} \mathrm{d}q^t, \tag{3.2}$$

where

$$g_{\mathcal{X}^J_n}(x,y,q,p) = \begin{pmatrix} g_{xx} & 0 & 0 & 0 \\ 0 & g_{yy} & 0 & 0 \\ 0 & 0 & g_{qq} & g_{qp} \\ 0 & 0 & g_{pq} & g_{pp} \end{pmatrix} \begin{array}{ll} g_{xx} = g_{yy} = \alpha y^{-1} \otimes y^{-1} & g_{qq} = \gamma y^{-1} \\ g_{pp} = \gamma(y + x y^{-1} x) & g_{pq} = \gamma x y^{-1} \\ g_{qp} = \gamma y^{-1} x & \end{array}. \tag{3.3}$$

3.2. PARTITIONED INVERSES

Let

$$h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}, \tag{3.4}$$

where $h_1, h_2, h_3, h_4 \in M(m,m), M(m,n), M(n,m), M(n,n)$.

We have (see *e.g.* (1) and (2) at p. 30 in [22]): if the matrices h_1, $h_4 - h_3 h_1^{-1} h_2$, h_4 and respectively $h_1 - h_2 h_4^{-1} h_3$ are nonsingular, then

$$h^{-1} = \begin{pmatrix} h^1 & h^2 \\ h^3 & h^4 \end{pmatrix} = \begin{pmatrix} (h_1 - h_2 h_4^{-1} h_3)^{-1} & -h_1^{-1} h_2 h^4 \\ -h^4 h_3 h_1^{-1} & (h_4 - h_3 h_1^{-1} h_2)^{-1} \end{pmatrix}. \tag{3.5}$$

If A (B) are non singular matrices $\in M(m,m)$, (respectively, $M(n,n)$), then we have [22, (11) (a) page 20] the relation

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

If $A,\ B \in M(m,m)$ are non-singular matrices, then [22, (5) page 29]

$$A^{-1} + B^{-1} = A^{-1}(A + B)B^{-1}.$$

Proposition 1. *The inverse of the* $MS(2n,\mathbb{R})$*-matrix* (3.3) *is*

$$g_{\mathcal{X}_n^J}(x,y,q,p)^{-1}=\begin{pmatrix} \alpha^{-1}y\otimes y & 0 & 0 & 0\\ 0 & \alpha^{-1}y\otimes y & 0 & 0\\ 0 & 0 & \gamma^{-1}(y+xy^{-1}x) & -\gamma^{-1}xy^{-1}\\ 0 & 0 & -\gamma^{-1}y^{-1}x & \gamma^{-1}y^{-1} \end{pmatrix}. \tag{3.6}$$

Proof. We apply (3.5) to the matrix

$$\begin{pmatrix} g_{qq} & g_{qp}\\ g_{pq} & g_{pp} \end{pmatrix}=\begin{pmatrix} \gamma y^{-1} & \gamma y^{-1}x\\ \gamma xy^{-1} & \gamma(y+xy^{-1}x) \end{pmatrix}. \tag{3.7}$$

If we denote

$$\begin{pmatrix} g^{qq} & g^{qp}\\ g^{pq} & g^{pp} \end{pmatrix}:=\begin{pmatrix} g_{qq} & g_{qp}\\ g_{pq} & g_{pp} \end{pmatrix}^{-1}, \tag{3.8}$$

then with formula (3.5) we get successively

$$\begin{aligned} g^{qq}&=\left\{\gamma y^{-1}-\gamma y^{-1}x[\gamma(y+xy^{-1}x)]^{-1}\gamma xy^{-1}\right\}^{-1}\\ &=\gamma^{-1}\left\{y^{-1}-(x^{-1}y)^{-1}\left(y+xy^{-1}x\right)^{-1}(yx^{-1})^{-1}\right\}^{-1}\\ &=\gamma^{-1}\left\{y^{-1}-\left[yx^{-1}(y+xy^{-1}x)x^{-1}y\right]^{-1}\right\}^{-1}\\ &=\gamma^{-1}\left\{y^{-1}-\left[yx^{-1}yx^{-1}y+yx^{-1}xy^{-1}xx^{-1}y\right]^{-1}\right\}^{-1}\\ &=\gamma^{-1}\left\{y^{-1}-\left[((yx^{-1})^2+1)y\right]^{-1}\right\}^{-1}\\ &=\gamma^{-1}\left\{y^{-1}-y^{-1}\left[1+(yx^{-1})^2\right]^{-1}\right\}^{-1}\\ &=\gamma^{-1}\left\{y^{-1}(1-[1+(yx^{-1})^2]^{-1}\right\}^{-1}. \end{aligned} \tag{3.9}$$

Let us denote

$$M:=(yx^{-1})^2.$$

We apply formula [22, (a) page 29]

$$(1+M)^{-1}=1-M+M^2-M^3+\dots,$$

and we get

$$g^{qq}=\gamma^{-1}\left\{y^{-1}M(1+M)^{-1}\right\}^{-1}=\gamma^{-1}(y+xy^{-1}x). \tag{3.10}$$

□

According with ([27, Theorem 1], [28, Proposition 3]), we consider now on the

manifold $\tilde{\mathcal{X}}_n^J$ the metric matrix

$$g'_{\tilde{\mathcal{X}}_n^J}(x,y,q,p,\kappa)=\begin{pmatrix} g_{xx} & 0 & 0 \\ 0 & g_{yy} & 0 \\ 0 & 0 & g'(q,p,\kappa) \end{pmatrix}, \tag{3.11}$$

where

$$g'(q,p,\kappa)=\begin{pmatrix} g'_{qq} & g'_{qp} & g'_{q\kappa} \\ g'_{pq} & g'_{pp} & g'_{p\kappa} \\ g'_{\kappa q} & g'_{\kappa p} & g'_{\kappa\kappa} \end{pmatrix} \begin{matrix} g'_{qq}=g_{qq}+\delta p^t\otimes p & g'_{pp}=g_{pp}+\delta q^t\otimes q & g'_{\kappa\kappa}=\delta \\ g'_{pq}=g_{pq}-\delta p^t\otimes q & g'_{q\kappa}=-\delta p^t & g'_{p\kappa}=\delta q^t. \end{matrix} . \tag{3.12}$$

Note that if $p,q\in M(1,n,\mathbb{F})$, then [22, (12) page 20]

$$p^t\otimes q=q^t\otimes p=p^tq=q^tp.$$

We write (3.12) as

$$g'(q,p,\kappa)=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{3.13}$$

where

$$A_{11}=\begin{pmatrix} g'_{qq} & g'_{qp} \\ g'_{pq} & g'_{pp} \end{pmatrix}=\begin{pmatrix} \gamma y^{-1}+\delta p\otimes p^t & \gamma y^{-1}x-\delta p\otimes q^t \\ \gamma xy^{-1}-\delta q\otimes p^t & \gamma(y+xy^{-1}x)+\delta q\otimes q^t \end{pmatrix}, \tag{3.14a}$$

$$A_{12}=\begin{pmatrix} g'_{q\kappa} \\ g'_{p\kappa} \end{pmatrix}=\begin{pmatrix} -\delta p^t \\ \delta q^t \end{pmatrix},\ A_{21}=\begin{pmatrix} g'_{\kappa q} & g'_{\kappa p} \end{pmatrix}=\begin{pmatrix} -\delta p & \delta q \end{pmatrix},\ A_{22}=g'_{\kappa\kappa}. \tag{3.14b}$$

Now we calculate the inverse matrix of (3.11)

$$g'_{\tilde{\mathcal{X}}_n^J}(x,y,q,p,\kappa)^{-1}=\begin{pmatrix} \alpha^{-1}y\otimes y & 0 & 0 \\ 0 & \alpha^{-1}y\otimes y & 0 \\ 0 & 0 & g'^{-1}(q,p,\kappa) \end{pmatrix}, \tag{3.15}$$

where

$$g'^{-1}(q,p,\kappa)=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}. \tag{3.16}$$

With the notation

$$A_{11}^{-1}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{3.17}$$

the values of the matrix (3.17) became with (3.5)

$$\begin{aligned} a_{11}=&[\gamma y^{-1}+\delta p^t\otimes p-(\gamma y^{-1}x-\delta q^t\otimes p) \\ &\times(\gamma(y+xy^{-1}x)+\delta q^t\otimes q)^{-1}(\gamma xy^{-1}-\delta p^t\otimes q)]^{-1}, \end{aligned} \tag{3.18a}$$

$$a_{12}=-(\gamma y^{-1}+\delta p^t\otimes p)^{-1}(\gamma y^{-1}x-\delta q^t\otimes p)a_{22}, \tag{3.18b}$$
$$a_{21}=-a_{22}(\gamma xy^{-1}-\delta p^t\otimes q)(\gamma y^{-1}+\delta p^t\otimes p)^{-1} \tag{3.18c}$$
$$a_{22}=[\gamma(y+xy^{-1}x)+\delta q^t\otimes q-(\gamma xy^{-1}-\delta p^t\otimes q) \times(\gamma y^{-1}+\delta p^t\otimes p)^{-1}(\gamma y^{-1}x-\delta q^t\otimes p)]^{-1}. \tag{3.18d}$$

Proposition 2. *For a_{22} appearing in* (3.17), (3.18d) *we find the simplified expression*

$$\begin{aligned} a_{22}=&[\gamma y+\delta q^t\otimes q+\delta xp^t\otimes py(1+\frac{\delta}{\gamma}p^t\otimes py)^{-1}y^{-1}x\\ &+\gamma\delta x(\gamma+\delta p^t\otimes py)^{-1}q^t\otimes p\\ &+\gamma\delta p^t\otimes q(\gamma+\delta yp^t\otimes p)^{-1}x\\ &-\delta^2p^t\otimes qy(\gamma+\delta p^t\otimes py)^{-1}q^t\otimes p]^{-1}. \end{aligned} \tag{3.19}$$

For the matrix elements in (3.16) *we find*

$$b_{11}=(A_{11}-A_{12}\delta^{-1}A_{21})^{-1}, \tag{3.20a}$$
$$b_{12}=-A_{11}^{-1}A_{12}b_{22}, \tag{3.20b}$$
$$b_{21}=-b_{22}A_{21}A_{11}^{-1}, \tag{3.20c}$$
$$b_{22}=(\delta-A_{21}A_{11}^{-1}A_{12})^{-1}, \tag{3.20d}$$

and

$$b_{11}=\begin{pmatrix} g^{qq} & g^{qp}\\ g^{pq} & g^{pp}\end{pmatrix},$$

$$b_{22}=\delta^{-1}[1-\delta(pa_{11}p^t-qa_{21}p^t-pa_{12}q^t+qa_{22}q^t)]^{-1}.$$

Proof. We write (3.18d) as

$$a_{22}=[\gamma(y+xy^{-1}x)+\delta q^t\otimes q+XX]^{-1}, \tag{3.21}$$

where

$$XX:=-(\gamma xy^{-1}-\delta p^t\otimes q)(\gamma y^{-1}+\delta p^t\otimes p)^{-1}(\gamma y^{-1}x-\delta q^t\otimes p). \tag{3.22}$$

We have successively

$$\begin{aligned}
XX &= -\gamma xy^{-1}(\gamma y^{-1}+\delta p^t\otimes p)^{-1}\gamma y^{-}x\\
&\quad +\gamma xy^{-1}(\gamma y^{-1}+\delta p^t\otimes p)^{-1}\delta q^t\otimes p\\
&\quad +\delta p^t\otimes q(\gamma y^{-1}+\delta p^t\otimes p)^{-1}\gamma y^{-1}x\\
&\quad -\delta p^t\otimes q(\gamma y^{-1}+\delta p^t\otimes p)^{-1}\delta q^t\otimes p\\
&= -\gamma^2(yx^{-1})^{-1}(\gamma y^{-1}+\delta p^t\otimes p)^{-1}(x^{-1}y)^{-1}\\
&\quad +\gamma\delta(yx^{-1})^{-1}(\gamma y^{-1}+\delta p^t\otimes p)^{-1}q^t\otimes p\\
&\quad +\gamma\delta p^t\otimes q(\gamma y^{-1}+\delta p^t\otimes p)^{-1}(x^{-1}y)^{-1}\\
&\quad -\delta^2 p^t\otimes q(\gamma y^{-1}+\delta p^t\otimes p)^{-1}q^t\otimes p\\
&= -\gamma^2[x^{-1}y(\gamma y^{-1}+\delta p^t\otimes p)yx^{-1}]^{-1}\\
&\quad +\gamma\delta[(\gamma y^{-1}+\delta p^t\otimes p)yx^{-1}]^{-1}q^t\otimes p\\
&\quad +\gamma\delta p^t\otimes q[(x^{-1}y)(\gamma y^{-1}+\delta p^t\otimes p)]^{-1}\\
&\quad -\delta^2 p^t\otimes q(\gamma y^{-1}+\delta p^t\otimes p)^{-1}q^t\otimes p\\
&= -\gamma^2[\gamma x^{-1}yx^{-1}+\delta x^{-1}yp^t\otimes pyx^{-1}]^{-1}\\
&+\quad \gamma\delta(\gamma x^{-1}+\delta p^t\otimes pyx^{-1})^{-1}q^t\otimes p\\
&+\quad \gamma\delta p^t\otimes q(\gamma x^{-1}+\delta x^{-1}yp^t\otimes p)^{-1}\\
&\quad -\delta^2 p^t\otimes q(\gamma y^{-1}+\delta p^t\otimes p)^{-1}q^t\otimes p\\
&= -\gamma^2[x^{-1}y(\gamma+\delta p^t\otimes py)x^{-1}]^{-1}\\
&\quad +\gamma\delta\left[(\gamma+\delta p^t\otimes py)x^{-1}\right]^{-1}q^t\otimes p\\
&\quad +\gamma\delta p^t\otimes q[x^{-1}(\gamma+\delta yp^t\otimes p)]^{-1}\\
&-\delta^2 p^t\otimes q(\gamma y^{-1}+\delta p^t\otimes pyy^{-1})^{-1}q^t\otimes p\\
&= -\gamma^2 x(\gamma+\delta p^t\otimes py)^{-1}y^{-1}x+XY,
\end{aligned}\tag{3.23}$$

where

$$\begin{aligned}
XY &:= \gamma\delta x(\gamma+\delta p^t\otimes py)^{-1}q^t\otimes p\\
&\quad +\gamma\delta p^t\otimes q(\gamma+\delta yp^t\otimes p)^{-1}x\\
&\quad -\delta^2 p^t\otimes qy(\gamma+\delta p^t\otimes py)^{-1}q^t\otimes p.
\end{aligned}\tag{3.24}$$

We have

$$a_{22}=[XZ+XY]^{-1},\tag{3.25}$$

where

$$\begin{aligned} XZ &:= \gamma(y+xy^{-1}x)+\delta q^t\otimes q-\gamma^2 x(\gamma+\delta p^t y)^{-1}y^{-1}x \\ &= \gamma(y+xy^{-1}x)+\delta q^t\otimes q-\gamma x(1+A)^{-1}y^{-1}x, A:=\frac{\delta}{\gamma}p^t\otimes py \\ &= \gamma y+\gamma xy^{-1}x+\delta q^t\otimes q-\gamma x(1-A+A^2-...)y^{-1}x \\ &= \gamma y+\delta q^t\otimes q+\gamma xA(1+A)^{-1}y^{-1}x, \end{aligned} \tag{3.26}$$

and we find

$$XZ=\gamma y+\delta q^t\otimes q+\delta xp^t\otimes py(1+\frac{\delta}{\gamma}p^t\otimes py)^{-1}y^{-1}x. \tag{3.27}$$

With (3.18d), (3.25), (3.27), we finally find for a_{22} the expression (3.19). □

For $\mathcal{X}_1^J$ we have

$$\begin{pmatrix} g_{qq} & g_{qp} \\ g_{pq} & g_{pp} \end{pmatrix}^{-1}=[\gamma\begin{pmatrix} \frac{1}{y} & \frac{x}{y} \\ \frac{x}{y} & \frac{x^2+y^2}{y} \end{pmatrix}]^{-1}=\frac{1}{\gamma y}\begin{pmatrix} x^2+y^2 & -x \\ -x & 1 \end{pmatrix}, \tag{3.28}$$

which is compatible with (3.6).

For $\tilde{\mathcal{X}}_1^J$ we have

$$\begin{pmatrix} g'_{qq} & g'_{qp} \\ g'_{pq} & g'_{pp} \end{pmatrix}^{-1}=\frac{1}{\gamma}\frac{1}{\gamma y+\delta[(px+q)^2+y^2p^2]}\begin{pmatrix} \gamma(x^2+y^2)+\delta q^2y & -\gamma x+\delta pqy \\ -\gamma x+\delta pqy & \gamma+\delta p^2y \end{pmatrix}. \tag{3.29}$$

If in (3.29) we take $\delta\to 0$ we get (3.28).

Let (M,g) be a n-dimensional Riemannian manifold. In a local coordinate system $x^1,\ldots,x^n$ the geodesic equations on a manifold M with components of the linear connection Γ are (see *e.g.* [29, Proposition 7.8 p 145]):

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}+\sum_{j,k=1}^{n}\Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}=0,\ i=1,\ldots,n. \tag{3.30}$$

The components Γ^i_{jk} (Christofell's symbols of second kind) of a Riemannian (Levi-Civita) connection ∇ are obtained from the Christofell's symbols of first kind $[ij,k]$ by solving the linear system of algebraic equations, see *e.g.* [29, p 160]

$$g_{lk}\Gamma^l_{ji}=[ij,k]=:\frac{1}{2}\left(\frac{\partial g_{ki}}{\partial x_j}+\frac{\partial g_{jk}}{\partial x_i}-\frac{\partial g_{ji}}{\partial x_k}\right),\ i,j,k,l=1,\ldots,n. \tag{3.31}$$

The $\frac{n^2(n+1)}{2}$ *distinct* Γ*-symbols of an* n*-dimensional manifold are given by the formulas*

$$\Gamma^m_{ij}=[ij,k]g^{km},\quad \text{where}\quad g^{mk}g_{kl}=\delta^m_l,\ i,j,k,m=1,\ldots,n. \tag{3.32}$$

Applying (3.31), (3.32) to (3.28) ((3.29)) for $\mathcal{X}_1^J$ (respectively $\tilde{\mathcal{X}}_1^J$), we get again Γ-s obtained in [6, (22)] (respectively, [6, page 29]).

We apply (3.32) to calculate Γ^x_{xy}

$$\Gamma^x_{xy} = [xy,k]g^{kx} = \frac{1}{2}(\frac{\partial g_{xx}}{\partial y} + \frac{\partial g_{yx}}{\partial x} - \frac{\partial g_{xy}}{\partial x^k})g^{kx}.$$

For example, for (3.15), (3.1) gives

$$\Gamma^x_{xy} = \frac{1}{2}(\frac{\partial g_{xx}}{\partial y} + \frac{\partial g_{yx}}{\partial x} - \frac{\partial g_{xy}}{\partial x})g^{xx} = -\frac{1}{y},$$

i.e. the first term in [6, (73)].

Similarly, with (3.14), we have on $\tilde{\mathcal{X}}_1^J$

$$g'(q,p,\kappa) = \begin{pmatrix} \gamma y^{-1} + \delta p^2 & \gamma y^{-1}x - \delta pq & -\delta p \\ \gamma x y^{-1} - \delta pq & \gamma(y + xy^{-1}x) + \delta q^2 & \delta q \\ -\delta p & \delta q & \delta \end{pmatrix}. \tag{3.33}$$

We find

$$\det g'(x,y,q,p,\kappa) = \gamma^2\delta. \tag{3.34}$$

We now calculate Γ^p_{pp}

$$\Gamma^p_{pp} = [pp,x]g^{xp} + [pp,y]g^{yp} + [pp,q]g^{qp} + [pp,p]g^{pp} + [pp,\kappa]g^{\kappa p}. \tag{3.35}$$

With formula (3.31), we find that the only non-zero Chistofell's symbol in (3.35) is

$$[pp,q] = -2\delta q$$

and

$$\Gamma^p_{pp} = -2\delta q g^{qp},$$

where

$$g^{qp} = -\frac{1}{\gamma^2\delta}\det\begin{pmatrix} \gamma x y^{-1} - \delta pq & \delta q \\ -\delta p & \delta \end{pmatrix} = -\frac{1}{\gamma}xy^{-1},$$

$$\Gamma^p_{pp} = 2\frac{\delta}{\gamma}xy^{-1}q,$$

and we regain the value calculated in [6, page 22].

The following Proposition is [25, Proposition 5.6] for $\tilde{\mathcal{X}}_1^J$

Proposition 3. *The metric on the extended Siegel–Jacobi upper half-plane $\tilde{\mathcal{X}}_1^J$, in the partial S-coordinates (x,y,p,q,κ):*

$$\begin{aligned} \mathrm{d}s^2_{\tilde{\mathcal{X}}_1^J} &= \mathrm{d}s^2_{\mathcal{X}_1^J}(x,y,p,q) + \lambda_6^2(p,q,\kappa) \\ &= \frac{\alpha}{y^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + \left[\frac{\gamma}{y}\left(x^2+y^2\right) + \delta q^2\right]\mathrm{d}p^2 + \left(\frac{\gamma}{y} + \delta p^2\right)\mathrm{d}q^2 + \delta\mathrm{d}\kappa^2 \\ &\quad + 2\left(\gamma\frac{x}{y} - \delta pq\right)\mathrm{d}p\mathrm{d}q + 2\delta(q\mathrm{d}p\mathrm{d}\kappa - p\mathrm{d}q\mathrm{d}\kappa) \end{aligned} \tag{3.36}$$

is left-invariant with respect to the action given in [25, Lemma 5.1] of the Jacobi group $G_1^J(\mathbb{R})$.

The matrix attached to metric (3.36) *is*

$$g_{\tilde{\mathcal{X}}_1^J}=\begin{pmatrix} g_{xx} & 0 & 0 & 0 & 0\\ 0 & g_{yy} & 0 & 0 & 0\\ 0 & 0 & g'_{qq} & g'_{qp} & g'_{q\kappa}\\ 0 & 0 & g'_{pq} & g'_{pp} & g'_{p\kappa}\\ 0 & 0 & g'_{\kappa q} & g'_{\kappa p} & g'_{\kappa\kappa} \end{pmatrix},$$

$$g'_{pq}=g_{pq}-\delta pq,\qquad g'_{p\kappa}=\delta q,\qquad g'_{q\kappa}=-\delta p,\qquad g'_{pp}=g_{pp}+\delta q^2,$$
$$g'_{qq}=g_{qq}+\delta p^2,\qquad g'_{\kappa\kappa}=\delta. \tag{3.37}$$

We have

$$g_{\mathcal{X}_1^J}=\begin{pmatrix} g_{xx} & 0 & 0 & 0\\ 0 & g_{yy} & 0 & 0\\ 0 & 0 & g_{qq} & g_{qp}\\ 0 & 0 & g_{pq} & g_{pp} \end{pmatrix},\quad \begin{array}{l} g_{xx}=g_{yy}=\frac{\alpha}{y^2}\\ g_{qq}=\frac{\gamma}{y},\ g_{pp}=\gamma\frac{x^2+y^2}{y},\ g_{pq}=\gamma\frac{x}{y} \end{array} \tag{3.38}$$

associated with the balanced metric

$$\mathrm{d}s^2_{\mathcal{X}_1^J}(x,y,p,q)=c_1\frac{\mathrm{d}x^2+\mathrm{d}y^2}{4y^2}+\frac{c_2}{y}\left[\left(x^2+y^2\right)\mathrm{d}p^2+\mathrm{d}q^2+2x\mathrm{d}p\mathrm{d}q\right]$$

on $\mathcal{X}_1^J$, where $\frac{c_1}{4}=\alpha, c_2=\gamma$.

The extended Siegel-Jacobi upper half-plane $\tilde{\mathcal{X}}_1^J$ does not admit an almost contact structure (Φ,ξ,η) with a contact form $\eta=\lambda_6$ and Reeb vector $\xi=\mathrm{Ker}(\eta)$.

3.3. THE INVERSE OF THE INVARIANT METRIC MATRIX ON $\tilde{\mathcal{X}}_2^J$

We particularize formula (3.14a) for the case $n=2$ and we have

Proposition 4.

$$A_{11}=\begin{pmatrix} \begin{pmatrix} \gamma\begin{pmatrix} y_{11} & y_{12}\\ y_{21} & y_{22}\end{pmatrix}^{-1}\\ +\delta\begin{pmatrix} p_1\\ p_2\end{pmatrix}\begin{pmatrix} p_1 & p_2\end{pmatrix}\end{pmatrix} & \begin{pmatrix} \gamma\begin{pmatrix} y_{11} & y_{12}\\ y_{21} & y_{22}\end{pmatrix}^{-1}\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}\\ -\delta\begin{pmatrix} q_1\\ q_2\end{pmatrix}\begin{pmatrix} p_1 & p_2\end{pmatrix}\end{pmatrix}\\ \begin{pmatrix} \gamma\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}\begin{pmatrix} y_{11} & y_{12}\\ y_{21} & y_{22}\end{pmatrix}^{-1}\\ -\delta\begin{pmatrix} p_1\\ p_2\end{pmatrix}\begin{pmatrix} q_1 & q_2\end{pmatrix}\end{pmatrix} & \begin{pmatrix} \gamma\begin{pmatrix} y_{11} & y_{12}\\ y_{21} & y_{22}\end{pmatrix}+\gamma\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}\\ \times\begin{pmatrix} y_{11} & y_{12}\\ y_{21} & y_{22}\end{pmatrix}^{-1}\begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}\\ +\delta\begin{pmatrix} q_1\\ q_2\end{pmatrix}\begin{pmatrix} q_1 & q_2\end{pmatrix}\end{pmatrix} \end{pmatrix}$$

where

$$y^{-1}=\begin{pmatrix} y_{11} & y_{12}\\ y_{21} & y_{22}\end{pmatrix}^{-1}=\frac{\begin{pmatrix} y_{22} & -y_{12}\\ -y_{21} & y_{11}\end{pmatrix}}{D},\ D:=y_{11}y_{22}-y_{12}y_{21}.$$

Particularizing (3.18) *for* $n=2$ *we get*

$$a_{11}=[\gamma y^{-1}+\delta\begin{pmatrix} p_1\\ p_2\end{pmatrix}\begin{pmatrix} p_1 & p_2\end{pmatrix}-(\gamma y^{-1}x-\delta\begin{pmatrix} q_1\\ q_2\end{pmatrix}\otimes\begin{pmatrix} p_1 & p_2\end{pmatrix})\times \tag{3.39a}$$

$$\times(\gamma(y+xy^{-1}x)+\delta\begin{pmatrix} q_1\\ q_2\end{pmatrix}\otimes\begin{pmatrix} q_1 & q_2\end{pmatrix})^{-1}(\gamma xy^{-1}-\delta\begin{pmatrix} p_1\\ p_2\end{pmatrix}\begin{pmatrix} q_1 & q_2\end{pmatrix})]^{-1} \tag{3.39b}$$

$$a_{22}=[\gamma(y+xy^{-1}x)+\delta\begin{pmatrix} q_1\\ q_2\end{pmatrix}\otimes\begin{pmatrix} q_1 & q_2\end{pmatrix}-(\gamma xy^{-1}-\delta\begin{pmatrix} p_1\\ p_2\end{pmatrix}\otimes\begin{pmatrix} q_1 & q_2\end{pmatrix})$$

$$\times(\gamma y^{-1}+\delta\begin{pmatrix} p_1\\ p_2\end{pmatrix}\otimes\begin{pmatrix} p_1 & p_2\end{pmatrix})^{-1}(\gamma y^{-1}x-\delta\begin{pmatrix} q_1\\ q_2\end{pmatrix}\otimes\begin{pmatrix} p_1 & p_2\end{pmatrix})]^{-1} \tag{3.39c}$$

$$a_{12}=-(\gamma y^{-1}+\delta\begin{pmatrix} p_1\\ p_2\end{pmatrix}\otimes\begin{pmatrix} p_1 & p_2\end{pmatrix})^{-1}(\gamma y^{-1}x-\delta\begin{pmatrix} q_1\\ q_2\end{pmatrix}\otimes\begin{pmatrix} p_1 & p_2\end{pmatrix})a_{22} \tag{3.39d}$$

$$a_{21}=-a_{22}(\gamma xy^{-1}-\delta\begin{pmatrix} p_1\\ p_2\end{pmatrix}\otimes\begin{pmatrix} q_1 & q_2\end{pmatrix})(\gamma y^{-1}+\delta\begin{pmatrix} p_1\\ p_2\end{pmatrix}\otimes\begin{pmatrix} p_1 & p_2\end{pmatrix})^{-1} \tag{3.39e}$$

With (3.16), (3.14) *and* (3.20), *we get for* $n=2$

$$b_{11}\gamma=\begin{pmatrix} y^{-1} & y^{-1}x\\ xy^{-1} & y+xy^{-1}x\end{pmatrix}^{-1}=\begin{pmatrix} (y+xy^{-1}x)^{-1} & -yx^{-1}\\ -xy^{-1} & y^{-1}\end{pmatrix} \tag{3.40a}$$

$$b_{22}\delta=\left[1-\delta\begin{pmatrix} -p & q\end{pmatrix}\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} -p^t\\ q^t\end{pmatrix}\right]^{-1}$$

$$=\left[1-\delta(pa_{11}p^t-qa_{21}p^t-pa_{12}q^t+qa_{22}q^t)\right]^{-1} \tag{3.40b}$$

$$\frac{b_{12}}{-\delta}=\begin{pmatrix} -a_{11}\begin{pmatrix} p_1\\ p_2\end{pmatrix}+a_{12}\begin{pmatrix} q_1\\ q_2\end{pmatrix}\\ -a_{21}\begin{pmatrix} p_1\\ p_2\end{pmatrix}+a_{22}\begin{pmatrix} q_1\\ q_2\end{pmatrix}\end{pmatrix}b_{22} \tag{3.40c}$$

$$\frac{b_{21}}{-\delta}=b_{22}\left(-\begin{pmatrix} p_1 & p_2\end{pmatrix}a_{11}+\begin{pmatrix} q_1 & q_2\end{pmatrix}a_{21},-\begin{pmatrix} p_1 & p_2\end{pmatrix}a_{12}+\begin{pmatrix} q_1 & q_2\end{pmatrix}a_{22}\right) \tag{3.40d}$$

Now we particularize (3.12) for $\tilde{\mathcal{X}}_2^J$

$$g'_{ij} = g'_{\tilde{\mathcal{X}}_2^J}(x,y,q,p,\kappa),\ \ i,j=1,2, \tag{3.41}$$

where the matrix elements are

$$\begin{aligned}
g'_{11} &= \gamma \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}^{-1} + \delta \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \frac{\gamma}{D} \begin{pmatrix} y_{22} & -y_{12} \\ -y_{21} & y_{11} \end{pmatrix} + \delta \begin{pmatrix} p_1p_1 & p_1p_2 \\ p_2p_1 & p_2p_2 \end{pmatrix}; \\
g'_{12} &= \gamma \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}^{-1} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} - \delta \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \begin{pmatrix} p_1 & p_2 \end{pmatrix} \\
&= \frac{\gamma}{D} \begin{pmatrix} y_{22}x_{11} - y_{12}x_{21} & y_{22}x_{12} - y_{12}x_{22} \\ -y_{21}x_{11} + y_{11}x_{21} & -y_{21}x_{12} + y_{11}x_{22} \end{pmatrix} - \delta \begin{pmatrix} q_1p_1 & q_1p_2 \\ q_2p_1 & q_2p_2 \end{pmatrix}; \\
g'_{13} &= -\delta \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}; \\
g'_{21} &= \gamma \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}^{-1} - \delta \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \begin{pmatrix} q_1, q_2 \end{pmatrix}; \\
&= \frac{\gamma}{D} \begin{pmatrix} x_{11}y_{22} - x_{12}y_{21} & -x_{11}y_{12} + x_{12}y_{11} \\ x_{21}y_{22} - x_{22}y_{21} & -x_{21}y_{12} + x_{22}y_{11} \end{pmatrix} - \delta \begin{pmatrix} p_1q_1 & p_1q_2 \\ p_2q_1 & p_2q_2 \end{pmatrix}; \\
g'_{22} &= \gamma \left[\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} + \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}^{-1} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \right] + \delta \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \begin{pmatrix} q_1 & q_2 \end{pmatrix} \\
&= \frac{\gamma}{D} \begin{pmatrix} (x_{11}^2 + y_{11}^2)y_{22} + (x_{12}^2 - y_{12}^2)y_{11} & (-x_{12}^2 - x_{11}x_{22} + y_{11}y_{22} - y_{12}^2)y_{12} \\ -2x_{11}x_{12}y_{12} & +x_{11}x_{12}y_{22} + x_{12}x_{22}y_{11} \\ (-x_{12}^2 - x_{11}x_{22} + y_{11}y_{22} - y_{12}^2)y_{12} & (x_{12}^2 - y_{12}^2)y_{22} \\ +x_{11}x_{12}y_{22} + x_{12}x_{22}y_{11} & +(x_{22}^2 + y_{22}^2)y_{11} - 2x_{12}x_{22}y_{12} \end{pmatrix} + \\
&\quad + \delta \begin{pmatrix} q_1^2 & q_1q_2 \\ q_2q_1 & q_2^2 \end{pmatrix}; \\
g'_{23} &= \delta \begin{pmatrix} q_1 \\ q_2 \end{pmatrix};\ \ g'_{31} = -\delta \begin{pmatrix} p_1 & p_2 \end{pmatrix};\ \ g'_{32} = \delta \begin{pmatrix} q_1 & q_2 \end{pmatrix};\ \ g'_{33} = \delta.
\end{aligned} \tag{3.42}$$

We calculate the determinant of the matrix $g'_{\tilde{\mathcal{X}}_2^J}$ given in (3.41) and then we find its inverse. Since $x,\ y$ are symmetric matrices, we write $g'_{\tilde{\mathcal{X}}_2^J}(x,y,q,p,\kappa) \in MS(5,\mathbb{R})$ as

$$g'_{\tilde{\mathcal{X}}_2^J}(x,y,q,p,\kappa) = (g')_{i,j=1,3} = (f)_{i,j=1,5}, \tag{3.43}$$

where

$$
\begin{aligned}
f_{1\leq j} = & \begin{array}{cccc} \frac{\gamma}{D}y_{22} & -\frac{\gamma}{D}y_{12} & \frac{\gamma}{D}(y_{22}x_{11}-y_{12}x_{12}) & \frac{\gamma}{D}(y_{22}x_{12}-y_{12}x_{22}) \\ +\delta p_1 p_1 & +\delta p_1 p_2 & -\delta q_1 p_1 & -\delta q_1 p_2 \end{array} \quad -\delta p_1 \\
f_{2\leq j} = & \begin{array}{ccc} \frac{\gamma}{D}y_{11} & \frac{\gamma}{D}(-y_{12}x_{11}+y_{11}x_{12}) & \frac{\gamma}{D}(-y_{12}x_{12}+y_{11}x_{22}) \\ \delta p_2 p_2 & -\delta q_2 p_1 & -\delta q_2 p_2 \end{array} \quad -\delta p_2 \\
f_{3\leq j} = & \begin{array}{cc} \frac{\gamma}{D}[(x_{11}^2+y_{11}^2)y_{22}+(x_{12}^2-y_{12}^2)y_{11} & \frac{\gamma}{D}[(-x_{12}^2-x_{11}x_{22}+y_{11}y_{22}-y_{12}^2)y_{12} \\ -2x_{11}x_{12}y_{12}]+\delta q_1^2 & +x_{11}x_{12}y_{22}+x_{12}x_{22}y_{11}]+\delta q_1 q_2 \end{array} \quad \delta q_1 \\
f_{4\leq j} = & \begin{array}{c} \frac{\gamma}{D}[(x_{12}^2-y_{12}^2)y_{22}+(x_{22}^2+y_{22}^2)y_{11} \\ -2x_{12}x_{22}y_{12}]+\delta q_2^2 \end{array} \quad \delta q_2 \\
f_{55} = & \ \delta
\end{aligned}
\tag{3.44}
$$

By the Laplace expansion formula [30] we have

$$
\det g'_{\tilde{\mathcal{X}}_2^J}(x,y,q,p,\kappa)=\sum_{i=1}^{5}(-1)^{1+i}f_{1i}M_{1i}=f_{11}M_{11}-f_{12}M_{12}+f_{13}M_{13}-f_{14}M_{14}+f_{15}M_{15},
$$

where the minor M_{ij} is the determinant of the 4×4 - matrix obtained from g'_{ij} removing the element ij.

We calculate M_{11}

$$
\begin{aligned}
M_{11} &= \det\begin{pmatrix} f_{22} & f_{23} & f_{24} & f_{25} \\ f_{32} & f_{33} & f_{34} & f_{35} \\ f_{42} & f_{43} & f_{44} & f_{45} \\ f_{52} & f_{53} & f_{54} & f_{55} \end{pmatrix} \\
&= -f_{25}A\begin{pmatrix} 3 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}+f_{35}A\begin{pmatrix} 2 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}-f_{45}A\begin{pmatrix} 2 & 3 & 5 \\ 2 & 3 & 4 \end{pmatrix}+f_{55}A\begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \end{pmatrix}
\end{aligned}
\tag{3.45}
$$

We obtain the determinant

$$
\det g'_{\tilde{\mathcal{X}}_2^J}(x,y,q,p,\kappa)=\delta[\gamma^4-\gamma^3\frac{\delta I_2}{D}A-\gamma^2\frac{(\delta I_2)^2}{D}B-\gamma\frac{(\delta I_2)^3}{D}A+(\delta I_2)^4], \tag{3.46}
$$

where

$$
I_2 := p_2q_1-p_1q_2, \quad A := 2[(x_{11}-x_{22})y_{12}-x_{12}(y_{11}-y_{22})], \tag{3.47a}
$$

$$
B := 4x_{12}^2+(x_{11}-x_{22})^2+y_{11}^2+y_{22}^2+2y_{12}^2. \tag{3.47b}
$$

Let

$$
N := \frac{\gamma\delta}{\det(f)\times D}. \tag{3.48}
$$

Using the *Ricci* package of *Wolfram Mathematica* system [31], we get:

Proposition 5. *The inverse of the matrix* (3.43) *is given by*

$$
\begin{aligned}
\frac{f^{11}}{N} =& \gamma^2\left[y_{11}(x_{12}^2-y_{12}^2)+y_{22}(x_{11}^2+y_{11}^2)-2x_{11}x_{12}y_{12}\right]+2\gamma\delta I_2\left[-x_{12}(x_{12}^2+y_{12}^2-x_{11}x_{22}+y_{11}y_{22})\right.\\
&\left.+y_{12}(x_{22}y_{11}+x_{11}y_{22})\right]-(\delta I_2)^2\left[y_{22}(x_{12}^2-y_{12}^2)+y_{11}(x_{22}^2+y_{22}^2)-2x_{22}x_{12}y_{12}\right]\\
\frac{f^{12}}{N} =& \gamma^2\left[x_{12}(x_{22}y_{11}+x_{11}y_{22})-y_{12}(x_{12}^2+y_{12}^2+x_{11}x_{22}-y_{11}y_{22})\right]\\
&+\gamma\delta I_2\left[x_{11}(x_{12}^2-y_{12}^2+x_{22}^2+y_{22}^2)-x_{22}(x_{12}^2-y_{12}^2+x_{11}^2+y_{11}^2)+2x_{12}y_{12}(y_{11}-y_{22})\right]\\
&+(\delta I_2)^2\left[x_{12}(x_{22}y_{11}+x_{11}y_{22})-y_{12}(x_{12}^2+y_{12}^2+x_{11}x_{22}-y_{11}y_{22})\right]\\
\frac{f^{13}}{N} =& \gamma^2(x_{12}y_{12}-x_{11}y_{22})-\gamma\delta I_2[x_{12}(x_{11}+x_{22})+y_{12}(y_{11}+y_{22})]+(\delta I_2)^2(x_{22}y_{11}-x_{12}y_{12})\\
\frac{f^{14}}{N} =& \gamma^2(-x_{12}y_{11}+x_{11}y_{12})+\gamma\delta I_2(x_{11}^2+2x_{12}^2-x_{11}x_{22}+y_{11}^2+y_{12}^2)\\
&+(\delta I_2)^2(2x_{11}y_{12}-2x_{12}y_{11}-x_{22}y_{12}+x_{12}y_{22})-\frac{D(\delta I_2)^3}{\gamma}\\
\frac{f^{15}}{N} =& \gamma^2\left[q_1(x_{11}y_{22}-x_{12}y_{12})+q_2(x_{12}y_{11}-x_{11}y_{12})+p_1\big(y_{11}(x_{12}^2-y_{12}^2)+y_{22}(x_{11}^2+y_{11}^2)\right.\\
&\left.-2x_{11}x_{12}y_{12}\big)+p_2\big(-y_{12}(x_{12}^2+y_{12}^2+x_{11}x_{22}-y_{11}y_{22})+x_{12}(x_{11}y_{22}+x_{22}y_{11})\big)\right]\\
&+\gamma\delta I_2\Big[q_1\big(x_{12}(x_{11}+x_{22})+y_{12}(y_{11}+y_{22})\big)-q_2(2x_{12}^2-x_{11}x_{22}+x_{11}^2+y_{11}^2+y_{12}^2)\\
&+2p_1\big(x_{12}(x_{11}x_{22}-y_{11}y_{22}-x_{12}^2-y_{12}^2)+y_{12}(x_{22}y_{11}+x_{11}y_{22})\big)\\
&+p_2\big(x_{11}(x_{22}^2+y_{22}^2+x_{12}^2-y_{12}^2)-x_{22}(x_{11}^2+y_{11}^2+x_{12}^2-y_{12}^2)+2x_{12}y_{12}(y_{11}-y_{22})\big)\Big]\\
&+(\delta I_2)^2\Big[q_1(x_{12}y_{12}-x_{22}y_{11})+q_2(2x_{12}y_{11}-2x_{11}y_{12}+x_{22}y_{12}-x_{12}y_{22})\\
&+p_2\big(x_{12}(x_{22}y_{11}+x_{11}y_{22})-y_{12}(y_{12}^2+x_{12}^2+x_{11}x_{22}-y_{11}y_{22})\big)\\
&-p_1\big(y_{11}(x_{22}^2+y_{22}^2)+y_{22}(x_{12}^2-y_{12}^2)-2x_{12}x_{22}y_{12}\big)\Big]+\frac{D(\delta I_2)^3}{\gamma}q_2\\
\frac{f^{22}}{N} =& \gamma^2\left[y_{11}(x_{22}^2+y_{22}^2)+y_{22}(x_{12}^2-y_{12}^2)-2x_{22}x_{12}y_{12}\right]+2\gamma\delta I_2\left[-y_{12}(x_{11}y_{22}+x_{22}y_{11})\right.\\
&\left.+x_{12}(x_{12}^2+y_{12}^2-x_{11}x_{22}+y_{11}y_{22})\right]-(\delta I_2)^2\left[y_{22}(x_{11}^2+y_{11}^2)+y_{11}(x_{12}^2-y_{12}^2)-2x_{11}x_{12}y_{12}\right]\\
\frac{f^{23}}{N} =& \gamma^2(x_{22}y_{12}-x_{12}y_{22})+\gamma\delta I_2(-2x_{12}^2+x_{11}x_{22}-x_{22}^2-y_{12}^2-y_{22}^2)\\
&+(\delta I_2)^2(x_{12}y_{11}-x_{11}y_{12}+2x_{22}y_{12}-2x_{12}y_{22})+\frac{D(\delta I_2)^3}{\gamma}\\
\frac{f^{24}}{N} =& -\gamma^2(x_{22}y_{11}-x_{12}y_{12})+\gamma\delta I_2[x_{12}(x_{11}+x_{22})+y_{12}(y_{11}+y_{22})]-(\delta I_2)^2(x_{12}y_{12}-x_{11}y_{22})
\end{aligned}
$$

$$\frac{f^{25}}{N} = \gamma^2\Big[q_1(x_{12}y_{22}-x_{22}y_{12})+q_2(x_{22}y_{11}-x_{12}y_{12})+p_1\big[x_{12}(x_{11}y_{22}+x_{22}y_{11})$$
$$-y_{12}(y_{12}^2+x_{12}^2+x_{11}x_{22}-y_{11}y_{22})\big]+p_2\big[y_{11}(x_{22}^2+y_{22}^2)+y_{22}(x_{12}^2-y_{12}^2)-2x_{12}x_{22}y_{12}\big]\Big]$$
$$-\gamma\delta I_2\Big[-q_1(2x_{12}^2-x_{11}x_{22}+x_{22}^2+y_{12}^2+y_{22}^2)+q_2\big(x_{12}(x_{11}+x_{22})+y_{12}(y_{11}+y_{22})\big)$$
$$+2p_2\big[x_{12}(x_{11}x_{22}-x_{12}^2-y_{12}^2-y_{11}y_{22})+y_{12}(x_{22}y_{11}+x_{11}y_{22})\big]$$
$$+p_1\big[-x_{11}(x_{12}^2+x_{22}^2+y_{22}^2-y_{12}^2)+x_{22}(x_{11}^2+x_{12}^2+y_{11}^2-y_{12}^2)-2x_{12}y_{12}(y_{11}-y_{22})\big]\Big]$$
$$-(\delta I_2)^2\Big[q_1(x_{12}y_{11}-x_{11}y_{12}+2x_{22}y_{12}-2x_{12}y_{22})+q_2(x_{11}y_{22}-x_{12}y_{12})$$
$$+p_1\big[-x_{12}(x_{11}y_{22}+x_{22}y_{11})+y_{12}(y_{12}^2+x_{12}^2+x_{11}x_{22}-y_{11}y_{22})\big]$$
$$+p_2\big[y_{11}(x_{12}^2-y_{12}^2)+y_{22}(x_{11}^2+y_{11}^2)-2x_{11}x_{12}y_{12}\big]\Big]-\frac{D(\delta I_2)^3}{\gamma}q_1$$

$$\frac{f^{33}}{N} = \gamma^2 y_{22}+2\gamma\delta I_2 x_{12}-(\delta I_2)^2 y_{11}$$

$$\frac{f^{34}}{N} = -\gamma^2 y_{12}-\gamma\delta I_2(x_{11}-x_{22})-(\delta I_2)^2 y_{12}$$

$$\frac{f^{35}}{N} = \gamma^2\big[-q_1y_{22}+q_2y_{12}-p_1(x_{11}y_{22}-x_{12}y_{12})+p_2(x_{22}y_{12}-x_{12}y_{22})\big]$$
$$-\gamma\delta I_2\Big[2q_1x_{12}-q_2(x_{11}-x_{22})+p_1\big(x_{12}(x_{11}+x_{22})+y_{12}(y_{11}+y_{22})$$
$$+p_2(2x_{12}^2-x_{11}x_{22}+x_{22}^2+y_{12}^2+y_{22}^2)\big)\Big]+(\delta I_2)^2\Big[q_1y_{11}+q_2y_{12}$$
$$+p_1(x_{22}y_{11}-x_{12}y_{12})+p_2(x_{12}y_{11}-x_{11}y_{12}+2x_{22}y_{12}-2x_{12}y_{22})\Big]+\frac{D(\delta I_2)^3}{\gamma}p_2$$

$$\frac{f^{44}}{N} = \gamma^2 y_{11}-2\gamma\delta I_2 x_{12}-(\delta I_2)^2 y_{22}$$

$$\frac{f^{45}}{N} = \gamma^2\Big[q_1y_{12}-q_2y_{11}+p_1(x_{11}y_{12}-x_{12}y_{11})+p_2(x_{12}y_{12}-x_{22}y_{11})\Big]$$
$$+\gamma\delta I_2\Big[q_1(x_{11}-x_{22})+2q_2x_{12}+p_1(x_{11}^2+y_{11}^2+y_{12}^2+2x_{12}^2-x_{11}x_{22})$$
$$+p_2\big(x_{12}(x_{11}+x_{22})+y_{12}(y_{11}+y_{22})\big)\Big]+(\delta I_2)^2\Big[q_1y_{12}+q_2y_{22}$$
$$+p_1(x_{12}y_{22}-x_{22}y_{12}+2x_{11}y_{12}-2x_{12}y_{11})+p_2(x_{11}y_{22}-x_{12}y_{12})\Big]-\frac{D(\delta I_2)^3}{\gamma}p_1$$

$$\frac{f^{55}}{N} = \frac{D\gamma^3}{\delta}+\gamma^2\Big[q_1^2y_{22}+q_2^2y_{11}-2q_1q_2y_{12}+p_1^2\big[(x_{12}^2-y_{12}^2)y_{11}+(x_{11}^2+y_{11}^2)y_{22}-2x_{11}x_{12}y_{12}\big]$$
$$+p_2^2\big[(x_{22}^2+y_{22}^2)y_{11}+(x_{12}^2-y_{12}^2)y_{22}-2x_{22}x_{12}y_{12}\big]+2p_1p_2\big[x_{12}(x_{11}y_{22}+x_{22}y_{11})$$
$$-(x_{12}^2+y_{12}^2+x_{11}x_{22}-y_{11}y_{22})y_{12}\big]+2p_1q_1(x_{11}y_{22}-x_{12}y_{12})+p_1q_2(x_{12}y_{22}-x_{22}y_{12})$$

$$+2p_2q_1(x_{12}y_{11}-x_{11}y_{12})+2p_2q_2(x_{22}y_{11}-x_{12}y_{12})\Big]+2(\gamma\delta I_2)\Big[(q_1^2-q_2^2)x_{12}-q_1q_2(x_{11}-x_{22})$$
$$-(p_1^2-p_2^2)\big[x_{12}(x_{12}^2+y_{12}^2-x_{11}x_{22}+y_{11}y_{22})-y_{12}(x_{22}y_{11}+x_{11}y_{22})\big]$$
$$+(p_1q_1-p_2q_2)\big[x_{12}(x_{11}+x_{22})+y_{12}(y_{11}+y_{22})\big]-\frac{1}{2}(p_1q_2+p_2q_1)(x_{11}^2-x_{22}^2+y_{11}^2-y_{22}^2)$$
$$+p_1p_2\big[x_{11}(x_{12}^2+x_{22}^2+y_{22}^2-y_{12}^2)-x_{22}(x_{12}^2+x_{11}^2+y_{11}^2-y_{12}^2)+2x_{12}y_{12}(y_{11}-y_{22})\big]\Big]$$
$$-(\delta I_2)^2\Big[q_1^2y_{11}+q_2^2y_{22}+2q_1q_2y_{12}+p_1^2\big[(x_{12}^2-y_{12}^2)y_{22}+(x_{22}^2+y_{22}^2)y_{11}-2x_{12}x_{22}y_{12}\big]$$
$$+p_2^2\big[(x_{12}^2-y_{12}^2)y_{11}+(x_{11}^2+y_{11}^2)y_{22}-2x_{12}x_{11}y_{12}\big]-2p_1q_1(x_{12}y_{12}-x_{22}y_{11})$$
$$-2p_1q_2(x_{12}y_{11}-x_{11}y_{12})-2p_2q_1(x_{12}y_{22}-x_{22}y_{12})-2p_2q_2(x_{12}y_{12}-x_{11}y_{22})$$
$$-2p_1p_2\big[x_{12}(x_{11}y_{22}+x_{22}y_{11})-y_{12}(x_{12}^2+y_{12}^2+x_{11}x_{22}-y_{11}y_{22})\big]\Big]-\frac{D(\delta I_2)^3}{\gamma}I_2$$

We systematize the results of Proposition 5:

Remark 1. *The elements of the inverse matrix of* (3.43) *can be written in the following condensed form:*

$$f^{ij}=N\Big[\frac{D\gamma^3}{\delta}P_0+\gamma^2P_1+\gamma\delta I_2P_2+(\delta I_2)^2P_3+\frac{D(\delta I_2)^3}{\gamma}P_4\Big],$$

where N *is defined in* (3.48) *and* P_0,P_1,P_2,P_3,P_4 *are polynomials in* $x=x_{ij}$ *and* $y=y_{ij}$*. We have that* $P_0=0$ *for all* f^{ij}*, excepting* f^{55}*, for which* $P_0=1$*. Similarly,* $P_4\neq 0$ *only for the elements* f^{i5}*,* $\forall i=1,...,5$*.*

Remark 2. *The inverse matrix* (3.43) *is equivalent with the inverse matrix given in* (3.16)*. To check the consistency of our calculations, we computed some of the matrix elements with the two methods: the one in Proposition 2 and the one in Proposition 5. For example, one can easily check that the element* f^{13} *is equal with the element on line 1 and column 3 of* b_{11}*.*

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