FRACTIONAL QUADRATIC DECELERATION PARAMETER (FQDP): OBSERVATIONAL AND THEORETICAL PERSPECTIVES

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Abstract. This study explore the dynamics of cosmic expansion through a detailed exploration of a novel form of the deceleration parameter, denoted as \( q = \frac{\alpha(t^2-1)}{t^2+1} \), within the framework of \( f(R,T) \) gravity theory. This study employs both observational and theoretical approaches to unveil the intricate interplay of cosmic forces and phenomena that have a profound influence on the ever-evolving universe. The primary objective of this investigation is to gain a deeper understanding of the expanding dynamics, particularly the transition from deceleration to acceleration. The findings of this paper shed light on the presence of a phase transition and an initial singularity, while remaining consistent with the \( \Lambda \) CDM model. Moreover, this study serves as a foundation for further exploration within the \( f(R,T) \) gravity framework, providing fresh insights into the cosmos and offering a profound comprehension of the universe’s dynamic evolution.

Key words: Cosmic exploration, \( f(R,T) \) gravity theory, FLRW metric, deceleration parameter, dark energy, Universe’s expansion.

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1. INTRODUCTION

The study of gravitational physics has been one of the most attractive part in the field of theoretical physics. Starting from Sir Isaac Newton’s ground breaking work on universal gravitation to the revolutionary insights provided by Albert Einstein theory of general relativity, our understanding of gravity has evolved significantly. In 1981, Alan Guth introduced the concept of inflation in cosmology \cite{1}. In a similar vein, 1998 saw the observation of type Ia supernovae by two independent groups, Perlmutter and Riess \cite{2}, offering direct evidence that our universe is presently undergoing accelerated expansion. The General Theory of Relativity (GTR) serves as the appropriate framework for understanding the universe, while the Lambda-cold dark matter (\( \Lambda \)CDM) model represents the simplest and most natural cosmic model. Nevertheless, there remain unresolved issues within GTR that necessitate further investigation. To address these challenges, there is a pressing need to either modify GTR or explore alternative theories of gravity. One alternative theory that has accumulated significant attention is the modified \( f(R,T) \) theory of gravity.

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This theoretical framework extent general relativity by permitting for variations in the cosmological term, which is a noble perspective on the behaviour of gravitation in the cosmos. Within this context, we find it pertinent to include a crucial parameter in our examination of the universe’s dynamics, known as the deceleration parameter (DP), denoted by $q$. This parameter holds significant importance in cosmology as it describes whether the cosmic expansion is decelerating or accelerating. It serves as an essential tool for elucidating the past and future evolution of the universe, offering insights into its destiny and the nature of dark energy (DE). The value of DP as originally proposed by Lemaitre, Georges in 1933 [3] and at that time it was believed to be constant (i.e. $-1$). However, subsequent research and observations have reviled that the universe’s expansion is accelerating, challenging this notion and opening the door to alternative theories of gravity. In the recent past, the exploration of various forms of DP has expended our understanding of cosmic dynamics. These forms encompass a broad spectrum of possibilities ranging from linear to polynomial even bilinear and fractional expressions. Among these the quadratic deceleration parameter (QDP) where $q$ is quadratic function of cosmic time has gained the attraction for its ability to capture more intricate expansion behaviour. Numerous cosmologists and academics, notably us in our peer-reviewed group, have developed distinct cosmological models incorporating diverse expressions of the deceleration parameter in alternative gravity theories [4]-[13]. Under the motivation of above cited research findings, in this proposed work we have decided to take a new form of DP i.e. fractional quadratic deceleration parameter (FQDP) through the structure of the modified gravity theory $f(R,T)$. The basic objective to propose this form is the believe that the FQDP, a novel and interesting formulation that combines the fractional calculus approach with the quadratic expression of the universe. We aim to analyse its implications both theoretical and observationally and we believe that it will provide a more comprehensive understanding of dynamics of our cosmos.

The current paper is structured into nine distinct sections: Mathematical framework for cosmological model is covered in the section 2, while the physical laws governing the proposed model are covered in section 3. Section 4 discusses the FQDP. Section 5 talks about the physical and mathematical characteristics of both models in addition to the graphical analysis. In sections 6 and 7, we investigate the energy conditions and utilize the cosmographic analysis to explore the characteristics and development of the cosmos’s large-scale structure. To differentiate various DE hypothesis, we defined statefinder diagnostic in section 8. Finally, in section 9 we present a summary of our study along with the concluding remarks.
2. MATHEMATICAL FRAMEWORK FOR COSMOLOGICAL MODEL

The mathematical framework of $f(R,T)$ gravity offers a means to comprehend how this alternative theory of gravity distinguishes itself from conventional General Relativity (GR) and other modified gravity theories. This theory employs an action that characterizes gravitational interactions within a given spacetime region, and is represented by the equation:

$$S = \int \left( \frac{f(R,T)}{16\pi} + \mathcal{L}_m \right) \sqrt{-g} d^4x$$  \hspace{1cm} (1)

In the provided equation, the function $f(R,T)$ relies on two essential elements, namely, $R$ and $T$. Here, $R$ signifies the Ricci scalar, characterizing spacetime curvature due to matter and energy, calculated from the metric tensor, $g_{\mu\nu}$. On the other hand, $T$ represents energy-momentum content, derived from the stress-energy tensor, $T_{\mu\nu}$, describing energy, momentum, and stress distribution in the universe. Additionally, $\mathcal{L}_m$ signifies the Lagrangian matter field density, while $\sqrt{-g}$ is the square root of the metric tensor’s determinant, $g_{\mu\nu}$, accounting for spacetime curvature. The stress-energy tensor $T_{\mu\nu}$, which is a symmetric tensor with four components, where $\mu$ and $\nu$ vary from 0 to 3 (representing time and 3 spatial dimensions) is defined as:

$$T_{\mu\nu} = g_{\mu\nu} - \frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} \mathcal{L}_m}{\delta g^{\mu\nu}}$$  \hspace{1cm} (2)

where $T = g^{\mu\nu}T_{\mu\nu}$ is momentum tensor’s trace. The inclusion of $T_{\mu\nu}$ in the theory’s action enables it to incorporate how matter and energy influence the gravitational field. In the equation provided earlier, we make the assumption that the Lagrangian matter ($\mathcal{L}_m$) does not rely on its derivatives but is solely dependent on the components of the metric tensor $g_{\mu\nu}$. As a result, we reach the following conclusion

$$T_{\mu\nu} = g_{\mu\nu} - 2 \frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}}$$  \hspace{1cm} (3)

The field equations are derived by taking variations of the modified action $S$ with respect to the components of the metric tensor, denoted as $g^{\mu\nu}$, and they can be expressed as follows:

$$ \left( 8\pi - f_T(R,T) \right) T_{\mu\nu} + \frac{1}{2} f(R,T) g_{\mu\nu} = f_R(R,T) R_{\mu\nu} + f_T(R,T) \Theta_{\mu\nu} + \left( g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu} \right) f_R(R,T)$$  \hspace{1cm} (4)

where $\nabla_{\nu}$ represents covariant derivative. $\Box$ is the D’Alembertian operator which is the Laplace operator of Minkowski space, has the form

$$\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta$$
where \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \). The value of \( \Theta_{\mu\nu} \) is given by

\[
\Theta_{\mu\nu} = -2 \left[ T_{\mu\nu} + g^{ij} \frac{\partial^2 \mathcal{L}_m}{\partial g^{\mu\nu} \partial g^{ij}} \right] + g_{\mu\nu} \mathcal{L}_m
\] (5)

As there is no universally accepted method for expressing the matter Lagrangian \( \mathcal{L}_m \). We handle the underlying term, which depends on the matter Lagrangian, through various methods. In this particular case, we opt for employing a perfect fluid matter model with \( \mathcal{L}_m = -p \), which leads to the subsequent outcome

\[
\Theta_{\mu\nu} = -2(T_{\mu\nu} + \frac{1}{2} p g_{\mu\nu})
\] (6)

Hence using equation (6), the \( f(R,T) \) gravity equations (4) takes the form

\[
G_{\mu\nu} f_R(R,T) = 8\pi T_{\mu\nu} + (p + T_{\mu\nu}) f_T(R,T) + \frac{1}{2} \left( f(R,T) - R f_R(R,T) \right) - (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) f_R(R,T)
\] (7)

In this context, we work in natural units where \( G = c = 1 \), setting \( \kappa^2 = 8\pi \) for the Einstein gravitational constant. The field equations in \( f(R,T) \) gravity are influenced by the presence of matter in the field, leading to numerous speculative models for different \( f \) choices. We focus on the functional expression \( f(R,T) = R + 2 f(T) \) as proposed by Harko et al. [4], extensively studied for its cosmological implications. This paper significantly contributes to theoretical physics, particularly in modified gravity theory. We concur with the mentioned assumption and consider the \( f(R,T) \) function as

\[
f(R,T) = R + 2 f(T)
\] (8)

where \( f(T) \) includes \( T \) as a random function.

3. PHYSICAL LAWS GOVERNING THE PROPOSED MODELS

In cosmological research, the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric plays a pivotal role, serving as a mathematical framework that describes the spacetime geometry in accordance with the cosmological principle. This principle asserts that, on large cosmic scales, our Universe demonstrates both uniformity and isotropy. Consequently, when formulating the cosmological model, we establish the expression for the FLRW metric as follows

\[
ds^2 = c^2 dt^2 - a^2(dx^2 + dy^2 + dz^2).
\] (9)

Here \( a \equiv a(t) \) denotes the scale factor, which is time \( t \)'s function.

Moreover, by adopting the convention \( c = 1 \), we can utilize this framework to depict the stress-energy tensor for a perfect fluid distinguished by uniform pressure
denoted as \( p \), the force exerted by dust particles on their surroundings, and the energy density, \( \rho \) which quantifies the amount of energy contained within a given spatial region per unit volume.

\[
T_{\mu\nu} = -pg_{\mu\nu} + (p + \rho)u_\mu u_\nu \quad (10)
\]

Here, \( u_\mu \) represents the four-velocity vector, satisfying the condition \( g_{\mu\nu}u_\mu u_\nu = 1 \). When we consider a co-moving coordinate system, where the particle remains stationary, we observe that

\[
u^\mu = (0, 0, 0, 1) \quad (11)
\]

Hence, the \( f(R, T) \) gravity equations (equation 7) corresponding to the linear selection of \( f(R, T) \), specifically \( f(R, T) = R + 2\zeta T \), for a perfect fluid distribution in a FLRW background, considering the components of the stress-energy tensor as:

\[
T_{11} = T_{22} = T_{33} = -p \quad \text{and} \quad T_{44} = \rho
\]

assume the following expressions

\[
2\frac{\dddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \zeta \rho - (1 + 3\zeta)p \quad (12)
\]

and

\[
3\left(\frac{\dot{a}}{a}\right)^2 = (1 + 3\zeta)\rho - \zeta p \quad (13)
\]

Equations (12) and (13) yield the following expressions for the energy density \( \rho \) and pressure \( p \) in terms of \( a(t) \) and its higher derivatives.

\[
p(t) = -\frac{1}{(1 + 6\zeta + 8\zeta^2)} \left[ 2(1 + 3\zeta)\frac{\dot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 \right] \quad (14)
\]

and

\[
\rho(t) = \frac{1}{(1 + 6\zeta + 8\zeta^2)} \left[ (8\zeta + 3)\left(\frac{\dot{a}}{a}\right)^2 - 2\zeta\frac{\ddot{a}}{a} \right] \quad (15)
\]

Since equations (14) and (15) encompass three distinct differential equations involving the three unknown parameters \( a, p, \) and \( \rho \), it is essential to solve them simultaneously. This necessitated the introduction of an additional constraint to establish the required cosmological model. It is well-understood that a dimensionless, time-dependent cosmic scale factor plays a critical role in quantifying the universe’s relative growth and understanding its expansion. Therefore, all facets of cosmic evolution must incorporate the parameters governing its temporal variation. The following expression can be derived through the expansion of \( a(t) \) using a Taylor series centered at time \( t_0 \)

\[
a(t) = a(t_0) + \dot{a}(t_0)(t - t_0) + \ddot{a}(t_0)\frac{(t - t_0)^2}{2} + \ldots \quad (16)
\]
The Hubble parameter \((H(t))\) and deceleration parameter \((q(t))\) are pivotal in unraveling the cosmos’ dynamics, shedding light on its expansion and evolution. While \(q(t)\) reveals how the cosmic expansion’s pace changes with time, \(H(t)\) discloses the rate at which the universe expands. To delve deeper into this study, we define these parameters as follows

\[
H(t) = \frac{\dot{a}}{a} \quad (17)
\]

and

\[
q(t) = -\frac{a\ddot{a}}{a^2} \quad (18)
\]

In our investigation, the variable ‘\(q\)’ serves as a defining feature of the cosmic evolution. Specifically, it characterizes the universe during distinct phases of its expansion—one marked by deceleration \(q \equiv q(t) > 0\) and the other characterized by accelerated expansion \(q < 0\). Intriguingly, the analysis of observational data derived from Type Ia supernovae provides compelling evidence for a pivotal cosmic transition. This transition indicates a shift from an initial deceleration phase to a subsequent era of cosmic acceleration. As previously mentioned, this significant shift necessitates alterations in the values of the deceleration parameter to accurately represent the evolving dynamics of the universe.

### 4. FRACTIONAL QUADRATIC DECELERATION PARAMETER (FQDP)

In the realm of late-time cosmic phenomena, which entails the universe’s transition from a decelerating phase to one of acceleration, an essential consideration revolves around the substantial alteration of the deceleration parameter. Naturally, during a decelerated phase, the parameter ‘\(q\)’ assumes a positive value, while in an accelerating phase, it adopts a negative sign. Geometric parameters like the DP and jerk parameter typically find their origins in observations of high-redshift supernovae. However, the precise temporal evolution of these parameters remains inadequately determined. In the absence of a definitive functional form for these parameters, numerous researchers have resorted to employing parameterized expressions, with the DP taking center stage in addressing various cosmological inquiries. Diverse parameterized forms of the DP, including but not limited to the constant DP, linear varying DP (LVDP), and beyond LVDP (BLVDP), can be found in the existing literature. It’s worth noting that the overarching dynamical behavior of the universe is discernible through the values of the DP residing in the negative domain. Specifically, a de Sitter expansion manifests when \(q\) equals \(-1\), while an accelerating power-law expansion is achievable for values of \(q\) within the range of \(-1\) to \(0\). A super-exponential expansion of the universe unfolds for \(q\) less than \(-1\). Although there exists some uncertainty in the precise determination of the DP from observational data, contemporary studies...
have often constrained this parameter to fall within the range of \(-0.8 \leq q \leq -0.4\). Given the characteristic signature flipping nature of the \(q\) and drawing inspiration from the research endeavors of our esteemed peers within the research group, as documented in references [9, 12, 14] we adopt a fractional quadratic form for the deceleration parameter as follows

\[
q = \frac{\alpha(t^2 - 1)}{t^2 + 1} \tag{19}
\]

where \(\alpha\) is a negative constant. The choice of \(\alpha\) is made to show the flipping nature of cosmos. Equation (19) infers that the model simulates positive DP \(i.e.\ q = -\alpha > 0\)

provided \(\alpha < 0\) at \(t = t_0\), \(q \leq 0\) for \(t \in (1, \infty)\) and \(q \leq -1\) for \(t^2 \leq \frac{\alpha - 1}{\alpha + 1}\). This behaviour suggests that the model initiates with a decelerating expansion phase, transitions into an accelerated expansion phase at \(t = 1\) and eventually enters a super-exponential state at \(t = \sqrt{\frac{\alpha - 1}{\alpha + 1}}\). Integration of equation (19) yields

\[
H(t) = \frac{1}{(1 + \alpha)t - 2\alpha \tan^{-1} t} \tag{20}
\]

To simplify, we assign a value of zero to the integrating constant since, at \(t = 0\), \(H \to \infty\) due to the extremely rapid expansion of the cosmos during the early infla-
tionary phase. The graph of $H$ shows the steep decline behavior and approaches to negligible value after crossing early inflationary phase of universe in Figure 2 for various values of $\alpha$.

After further simplification of equation (20) with initial condition $a(t) = 0$ at $t = 0$, we get

$$a(t) = (t - \alpha t) \frac{1}{A} e^F(t)$$

where, $F(t) = \frac{-\alpha t^2[1890(-1+\alpha)^2+63(1-\alpha)(9+\alpha)t^2+2(135-18\alpha+23\alpha^2)t^4]}{5670(1+\alpha)^3}$ and $\alpha$ is a negative constant.

Here, the fact that $a(t)$ equals zero at $t = 0$ indicates the presence of an initial singularity. Thus now using the equations (14), (15) and (21), we obtain our respective expression for $p$ and $\rho$ as follows

$$p(t) = \frac{-1}{A} \left[ (B)^2 + 2\alpha(1+3\zeta)(C) \right]$$

and

$$\rho(t) = \frac{1}{A} \left[ (3+8\zeta)(B)^2 - 2\alpha\zeta(C) \right]$$

where $A = 893025(1+6\zeta+8\zeta^2)(-1+\alpha)^2$, $B = -945 - 3\alpha^2(945 + 420t^2 - 112t^4 + 12t^6) + 9\alpha(315 + 70t^2 - 42t^4 + 30t^6) + \alpha^3(945 + 630t^2 + 42t^4 + 46t^6)$ and $C = -893025(-1 + 2t^2 - 2t^4 + 2t^6) - 945\alpha^6(-945 + 630t^2 + 126t^4 + 230t^6) + 486\alpha$.
In the field of cosmology, it is a well-established practice to establish a functional connection between three fundamental state variables, namely, pressure ($p$), energy density ($\rho$), and temperature ($t$), utilizing an equation of the form $f(p, \rho, t) = 0$. Within the framework of the FLRW metric, the equation of state (EoS) parameter not only governs the gravitational characteristics of dark energy (DE) within an isotropic universe filled with a perfect fluid but also regulates its evolutionary behavior. This EoS parameter, denoted as $\xi$, is defined through the relationship $\xi = \frac{p}{\rho}$.

Leveraging this definition, we can derive the equation of state parameter ($\xi$)’s expression as follows

$$\xi(t) = -\frac{1}{A \rho} \left[ (B)^2 + 2\alpha(1 + 3\zeta)(C) \right]$$  \hspace{0.5cm} (24)$$

where $A, B, C$ are same as defined above.
5. MATHEMATICAL ANALYSIS AND DISCUSSIONS

The characteristics, both physical and mathematical, of this model are comprehensively detailed in Section 5. In our exploration of FLRW cosmological models, the scrutiny of these attributes furnishes valuable insights into the overarching dynamics and evolution of the universe. Furthermore, in the preceding section, we have furnished graphical representations of the results obtained through computational analyses. By incorporating the value of $a(t)$ as per Equation (21), we can reformulate the expression for the FLRW metric, as articulated in Equation (9), in the following manner

$$ds^2 = c^2 dt^2 - (t - \alpha t)^{\frac{2}{1-\alpha}} e^{2F(t)} (dx^2 + dy^2 + dz^2).$$

In order to analyze the physical and geometric characteristics of this model, we will refer to equations (22-24), with the parameter values $\alpha = -1$ and $\zeta = 0.1, 0.5, 0.9$. Further for more clarity we wish to mention pictorial analysis as under:

Fig. 4 – Pressure’s visualization in regards to cosmic time ($t$).

- In the context of cosmological research, the graphical representation displayed in Figure 4 unveils a compelling narrative within the cosmic landscape. The observed pattern, transitioning from positive to zero and then to negative values, resonates deeply with the cosmic framework. It mirrors the cosmic evolution as it
Fig. 5 – Energy density’s visualization in regards to cosmic time ($t$).

Fig. 6 – EoS’s visualization in regards to cosmic time ($t$).
progresses from an initial phase of expansion, reaching a critical state of balance, and subsequently resuming an accelerated expansion. This pattern is highly consistent with the ΛCDM cosmological model, where dark energy plays a central role in shaping the cosmic destiny.

- In the realm of cosmological investigation, the graph presented in Figure 5 unveils an astonishing narrative that reflects the dynamic and enigmatic nature of the cosmos. It is noted that initially the graph of \( \rho \) exhibits a falling-off tendency over time \( t \to \infty, \rho \to 0 \) and then as time progresses it starts increases. The observed pattern, beginning from infinity, converging to zero, and then rising again towards infinity, challenges our comprehension of cosmic evolution. It underscores the intriguing interplay between cosmic inflation, radiation, matter, and dark energy in shaping the destiny of the universe. This graph is notably consistent with the ΛCDM cosmological model, which incorporates both inflation and dark energy as fundamental components of cosmic dynamics.

- Further Figure 6 introduces us to the equation of state parameter, offering a panoramic view of its behaviour across different choices. Notably, as \( t \to \infty, \xi \to -1 \), signifying the existence of dark energy. Different values of \( \xi \) represent different characteristics of dark energy. This includes some values that make it act like the cosmological constant, where its energy remains the same over time, and other values that lead to stranger effects like phantom energy.

6. ENERGY CONDITIONS

In cosmology, energy conditions are essential criteria applied to the cosmic stress-energy tensor, governing spacetime properties and the universe’s structure and evolution. They are often expressed in terms of pressure \( (p) \) and energy density \( (\rho) \), ensuring spacetime stability and matter behavior in the cosmos. Energy conditions used in cosmological models include:

- Null Energy Condition (NEC):

\[
\rho + p \geq 0
\]

This requirement stipulates that for all observers, even those moving along null geodesics, the total of energy density and pressure must be non-negative.

- Weak Energy Condition (WEC):

\[
\rho \geq 0, \quad \rho + p \geq 0
\]

For all observers with nonzero mass, the energy density must be non-negative, and the total of the energy density and pressure must likewise be non-negative according to the WEC.
- **Strong Energy Condition (SEC):**

\[ \rho + 3p \geq 0 \]

This more stringent requirement implies that the product of the pressure and energy density times three must not be negative. It is often associated with the gravitational attraction of matter.

- **Dominant Energy Condition (DEC):**

\[ \rho \geq 0, \quad |p| \leq \rho \]

The DEC states that the energy density must be non-negative, and the magnitude of the pressure must be less than or equal to the energy density.

These energy conditions help constrain the properties of cosmic matter and fields in various cosmological scenarios. Violations of these conditions can have significant implications for the stability of spacetime and the behavior of the universe. Now, in Figure 7, we present a depiction of the time-dependent behavior of three fundamental energy conditions WEC, DEC, and SEC. This visual representation serves the purpose of assessing the model’s stability as the cosmic evolution unfolds. Notably, WEC and DEC are clearly upheld in Model-I. However, it is worth highlighting that the condition \( \rho + 3p < 0 \), which defines the SEC, is violated in the present and late
cosmic times. This breach of the SEC induces an anti-gravitational influence, leading to cosmic acceleration and shift from an initial deceleration period to the current phase of cosmic expansion.

7. COSMOGRAPHIC ANALYSIS

The term cosmography encompasses the investigation of the spatial distribution of matter and energy on a cosmic scale and the dynamic behavior of the universe. This approach involves the analysis of how the universe’s dimensions change over time and how this evolution is linked to the composition of matter and energy within the universe. It investigates the universe’s origins and the mysterious concept of dark energy, employing specialized terms like jerk, snap, and lerk. By adjusting our mathematical equation (16) to align with current observational data i.e. $t = t_0$, we can utilize parameters like the Hubble parameter ($H_0$) and DP ($q_0$), in conjunction with jerk ($j_0$), snap ($s_0$), and lerk ($l_0$), to attain a more profound comprehension of the cosmos and its evolutionary processes.

$$\frac{a(t)}{a(t_0)} = 1 + H_0 A - q_0 H_0^2 \frac{A^2}{2!} + j_0 H_0^3 \frac{A^3}{3!} + s_0 H_0^4 \frac{A^4}{4!} + l_0 H_0^5 \frac{A^5}{5!} \ldots$$ (25)

where $A = (t - t_0)$, we will get the expressions for these kinematic variables which are covered in the ensuing subsections [15, 16].

7.1. JERK PARAMETER

In cosmology, the different approach to understanding the cosmic acceleration is to examine the dimensionless jerk parameter ($j$), which is obtained from the scale factor $a(t)$. It is defined by the expression

$$j \equiv j(t) = \frac{\dddot{a}}{a^3} = 1 + \frac{H''}{H^2} + \frac{3H'}{H}$$ (26)

Since this parameter includes the third derivative of $a(t)$, it helps us calculate the universe’s expansion rate more precisely. For the construction of the cosmological model, we find the expression for $j(t)$ using $H(t) = \frac{\dot{a}}{a} = \frac{1}{(t+\alpha)^2 - 2\alpha t \tan^{-1}(\tau)}$, as follows

$$j(t) = \frac{1}{(1 + t^2)^2} \left[ \alpha \left( -1 + 4t^2 + t^4 + 2\alpha(1 - 4t^2 + t^4) + 8\alpha \tan^{-1}(1/t) \right) \right]$$ (27)

In this model, the value of jerk parameter remains positive throughout the cosmic evolution. Figure 8 makes it clear that, for this model, $j(t) \to 1$ as $t \to \infty$. Based on the literature survey, it can be inferred that all the statements made regarding the universe’s experience can be added together to make the claim that the universe’s jerk parameter has a value of 1 [17, 18].
7.2. SNAP/JOUNCE PARAMETER

It is the fourth derivative of $a(t)$ in relation with time ($t$) and is also referred to as the $j(t)'s$ rate of change w.r.t $t$. The snap parameter’s mathematical expression is presented by

$$s(t) = \frac{1}{a} \left[ \frac{1}{a} \right]^{\prime}\prime\prime$$

This parameter aids in examining the cosmos’ expansion rate more precisely because it incorporates the higher derivatives of scale factor. We discover the expression for $s(t)$ as follows to build our model:

$$s(t) = \frac{1}{(1+t^2)^3} \alpha \left[ (2+6t^2+18t^4-2t^6+\alpha^2(6-46t^2+54t^4-6t^6) \
- \alpha(7+17t^2-63t^4+7t^6) - 32\alpha t(2t^2+\alpha(-2+3t^2))\tan^{-1}(t) \
+ 16\alpha^2(-1+3t^2)(\tan^{-1}(t))^2 \right]$$

Figure 9 make it abundantly clear that, for this model, $s(t) \to 1$ as $t \to \infty$. 
This trait helps us comprehend the past and future of the universe. The behaviour of a cosmological constant and an embryonic dark energy term must be distinguished clearly. It is the pace at which the snap alters in relation to time. This scale factor’s fifth derivative of time’s definition is

\[ l(t) = \frac{1}{a} a^{s_{mrr}} \left[ \frac{1}{a} \right]^{-5} \] (30)

For building our model, we find the expression for \( l(t) \) as follows:

\[
\begin{align*}
    l(t) &= \frac{1}{(1 + t^2)^4} \alpha \left[ (6(-1 - 4t^2 - 4t^4 - 16t^6 + t^8) + 8\alpha^3 (3 - 34t^2 + 88t^4 - 48t^6 + 3t^8) \\
    &+ \alpha(29 + 96t^2 + 214t^4 - 464t^6 + 29t^8) + \alpha^2 (-46 + 24t^4 + 856t^6 - 736t^6 + 46t^8) \\
    &+ 16\alpha t(6t^2(-1 + 5t^2) + 2\alpha^2(13 - 55t^2 + 30t^4) + \alpha(-11 - 86t^2 + 85t^4)) \tan^{-1}(t) \\
    &- 16\alpha^2(-3 - 30t^2 + 45t^4 + \alpha(8 - 92t^2 + 60t^4))\tan^{-1}(t))^2 \\
    &+ 384\alpha^3 t(-1 + t^2))(\tan^{-1}(t))^3 \right]
\end{align*}
\] (31)

From Figure 10, we can describe \( l(t) \) as a decreasing function of time and clearly it approaches 1 as \( t \to \infty \).
Since all the parameters (jerk, snap, and lerk) approaches 1, it indicates that the universe is undergoing a phase transition from decelerated to accelerated expansion, and this transition is characterized by increasingly complex and changing dynamics. This can have significant implications for our understanding of the underlying physics of the universe, including the role of dark energy and the nature of cosmic acceleration.

8. STATEFINDER DIAGNOSTIC

We know that the statefinder parameters is a novel cosmological indicators designed to distinguish among various dark energy hypotheses. The statefinder pair \((r, s)\) proves effective in discerning a wide range of cosmological models, including those featuring a cosmological constant, quintessence, standard cold matter (SCDM), chaplygin gas (CG), etc [19]. These parameters are entirely defined in terms of the scale factor and its third-order time derivatives. These standards are described as

\[
\begin{align*}
    r & = j(t) = \frac{1}{a} a'' \left( \frac{1}{a} \right)^{-3} \\
    s & = \frac{r - 1}{3(\bar{q} - \frac{1}{2})}
\end{align*}
\]

\[(32) \quad \text{and} \quad (33)\]
where, for \( \Lambda \)CDM model, \( r = 1 \) and \( s = 0 \), and where \( r \) is synonymous with the jerk parameter discussed in the preceding section. The best way to comprehend this at this point is to think about the early evolution of the cosmos when matter’s role was minimal. The scale factor will rise exponentially with time as the universe continues to expand purely due to the cosmological constant. With the presented model, it is feasible to calculate the pair \((r, s)\) and plot the trajectory on the \( r - s \) plane.

The \( r - s \) planar trajectory can behave substantially differently for several existing models. These deviations from \((0, 1)\) point serves as metrics for measuring the dissimilarity between a particular model and the \( \Lambda \)CDM model. The point \((s, r) = (1, 1)\) provides an alternative representation of the SCDM model. The classic DE model is identifiable within the region where \( s > 0, r < 1 \), while the CG DE model is observable in the region characterized by \( s < 0, r > 1 \).

For this model, \( s \) is calculated as

\[
s = \frac{\alpha (-1 - 4t^2 + t^4) + 2\alpha^2 (1 - 4t^2 + t^4 + 4ttan^{-1}t)}{6\alpha(t^4 - 1) - 3(t^2 + 1)^2} \tag{34}
\]

Figure 11 shows that, in this model the \( r - s \) trajectory traverses the CG region \((s < 0, r > 1)\) prior to reaching the stable \( \Lambda \)CDM fixed point \((s = 0, r = 1)\).

The provided table illustrates the dynamics of various cosmographic parameters within our model, enhancing our ability to gain a clearer visualization of the
### Table 1
Alteration in the cosmographic parameters within the framework of FQDP

<table>
<thead>
<tr>
<th>Time ( (t) )</th>
<th>( a(t) )</th>
<th>( H(t) )</th>
<th>( q(t) )</th>
<th>( j(t) )</th>
<th>( s(t) )</th>
<th>( l(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t = 0 )</td>
<td>0</td>
<td>( \infty )</td>
<td>( -\alpha )</td>
<td>( \alpha(-1+2\alpha) )</td>
<td>( \alpha(2-7\alpha+6\alpha^2) )</td>
<td>( \alpha(-6+29\alpha-46\alpha^2+24\alpha^3) )</td>
</tr>
<tr>
<td>( t \to \infty )</td>
<td>( \infty )</td>
<td>0</td>
<td>( \alpha )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

phenomena.

9. CONCLUSION

In this comprehensive research paper, we embarked on a journey to explore the fascinating domain of cosmology within the framework of \( f(R,T) \) gravity. This exhaustive study focuses primarily on shedding light on the fractional quadratic deceleration parameter from both observational and theoretical perspectives. The investigation unfolds across eight distinct sections, each contributing a unique dimension to our understanding of the cosmos. In this section, we reflect upon the significance of our findings and the broader implications of our study. The investigation commenced with the introduction, where we emphasized the importance of studying \( f(R,T) \) gravity and laid the foundation for the subsequent sections. The mathematical framework was precisely outlined in Section 2, where we presented the essential components and models that underpin our cosmological exploration, highlighting the necessity of adopting \( f(R,T) \) theory to express the universe’s complex dynamics. Section 3 delves into the physical laws governing the cosmological model, bridging the gap between fundamental physics and the grand cosmos. In Section 4, we explore the FQDP and present, through graphical representation, the relationship with respect to cosmic time. Our illustrations vividly depict the transition from decelerating to accelerating expansion, with the key feature being the steep decline in the Hubble parameter, indicating the presence of an initial singularity.

We then delve into discussions about energy density and pressure parameters to enrich our understanding, showing consistency with the \( \Lambda \)CDM cosmological model and reinforcing the viability of our approach in Section 5. Section 6 introduces energy conditions that help us constrain the properties of matter and fields. In Section 7, we conduct a cosmographic analysis that allows us to gain a panoramic view of the cosmos through the exploration of the Hubble parameter. This section amplifies our perspective on the behavior of cosmic matter across various choices, enhancing our comprehension of the universe’s workings. Moreover, the convergence of jerk, snap, and lerk to 1 in cosmology is a captivating phenomenon, signifying not only a change in the universe’s expansion rate but also opening a doorway to a deeper understanding of the cosmos. It promises to shed light on the profound role of dark energy and unveil the intricacies of cosmic acceleration, challenging our current...
paradigms and pushing the boundaries of our knowledge in the field of cosmology. Throughout our investigation, we noticed that the FQDP and its interplay with various cosmological parameters provided us with a more comprehensive analysis of the cosmos. These findings contribute to the ongoing quest to decode the mysteries of the universe, and their implications extend far beyond this study. We have showcased our findings in their original form, complemented by visual representations that emphasize the complex characteristics of the cosmos and underscore the possibilities offered by the $f(R,T)$ gravity theory in investigating the expansion and development of the universe. The notable accomplishments, including the transition from deceleration to acceleration, the existence of an initial singularity, and the alignment with the $\Lambda$CDM model, pave the way for future research endeavors.

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