PHONON MEDIATED COLLECTIVE DYNAMICS OF COHERENTLY PUMPED TWO-LEVEL EMITTERS

VICTOR CEBAN[1,a], MIHAI A. MACOVEI[1,b], AURELIAN ISAR[2,c]

[1]Institute of Applied Physics, Moldova State University,
Academiei str. 5, MD-2028 Chişinău, Moldova
[2]Horia Hulubei National Institute of Physics and Nuclear Engineering,
Reactorului 30, RO-077125, Magurele-Bucharest, Romania
E-mail [a]: victor.ceban@ifa.md
E-mail [b]: mihai.macovei@ifa.md
Corresponding author[c]: isar@theory.nipne.ro

Received July 23, 2023

Abstract. We investigate the collective quantum dynamics of an ensemble of two-level emitters, embedded in a crystal, and coherently pumped by a moderately intense, and externally applied coherent electromagnetic field. The ensemble is damped preponderantly *via* the surrounding phonon reservoir which mediates the inter-particle collective interactions. We have found that generally phonon transitions among the corresponding dressed states are taking place involving simultaneously many single emitters or pairs of two-level emitters, respectively. In both cases the phonon intensity can be proportional to the squared number of involved two-level emitters.

Key words: Collective phenomena, two-level emitters, phonons, external coherent pumping.

DOI: https://doi.org/10.59277/RomJPhys.2023.68.114

1. INTRODUCTION

Collective phenomena among many few-level emitters [1], embedded in solid-state media, attracted a lot of attention nowadays. These effects, *i.e.* superradiance or superfluorescence, were/are abundantly investigated and observed in multiple samples like molecular aggregates and crystals, molecular centres in solids or nuclear ensembles embedded in thin films, or quantum dots and quantum wells, respectively [2–7]. The main features of the superradiance phenomenon are that a sample of N initially excited atoms forms a collective dipole moment, which leads to effects such as a superradiant fluorescence intensity proportional to N^2 and a quantum dynamics N times faster than for single atoms. This occurs because of vacuum mediated inter-particle interactions among the excited emitters. Then a big number of possible applications were suggested or implemented.

Particularly, a steady-state superradiant laser operation was demonstrated in [8], which may improve the stability of passive atomic clocks. Synchronization of two ensembles of atoms *via* superradiance was theoretically predicted in Ref. [9], that may have implications towards ultrastable lasers and precision measurements. The performance of a cooperative quantum heat engine is significantly enhanced by the collective interactions among its constituents [10], whereas in certain cases its absorption power may be proportional to squared number of atoms, *i.e.* $P \propto N^2$ [11], or being dependent on the nature of the environmental reservoir [12]. Cooperative effects among artificial qubits were suggested to cool a quantum LC circuit whom they couple with [13], while superradiance based laser cooling of crystals and glasses doped with rare-earth ions was theoretically predicted too, in Refs. [14, 15]. Other phenomena referring to quantum effects in pumped two-level few-qubit samples were addressed as well [16–20]. In this broad research area, we also mention some recent works on quantum affinity of qubits subjected to noisy quantum channels [21], on quantum entanglement and quantum steering of two bosonic modes in noisy environments [22], on quantum steering of two bosonic modes in the two-reservoir model [23], and on open quantum dynamics of Gaussian Hilbert-Schmidt geometric discord [24].

Actually, phonon superradiance where the excited two-level emitters are synchronized *via* the vibrational phonons of the environmental solid-state host was predicted as well [25, 26]. It was already demonstrated that crystals of molecular nanomagnets can exhibit giant magnetic relaxation due to the Dicke superradiance of electromagnetic waves [27]. Phonon laser effects and superradiance in these materials were additionally demonstrated [28]. Furthermore, optomechanical superradiance was demonstrated as well, see *e.g.* Refs. [29, 30]. Notice that earlier works focused also on solving exactly the Dicke model in an external coherent laser field and in the steady-state, with the corresponding description of the established collective quantum dynamics [31, 32].

We investigate here an ensemble of optical impurities, which are modelled as two-level atoms or molecules embedded in a solid crystal. Each emitter interacts resonantly with an externally applied and moderately intense coherent electromagnetic wave, whereas the inter-particle correlations are mainly induced because of lattice thermal vibrations of the crystal. We have obtained the corresponding master equation describing the externally driven multiqubit ensemble, which is concomitantly damped *via* the environmental phonon bath, and obtained its steady-state solution for some cases of interest. Particularly, we have found that phonon transitions among the corresponding semi-classical dressed states are taking place involving many single emitters or pairs of two-level emitters simultaneously. Furthermore, for larger ensembles, pairs of emitters contribute to phonon transitions mainly at resonant driving, whereas bunches of individual emitters at off-resonance external coherent pumping, respectively. The collective interaction nature among the two-level emitters is characterized *via* squared emitters number dependence of the phonons intensity.

This paper is organized as follows. In Sec. 2 we describe the analytical approach and the system of interest, while in Sec. 3 we present and analyse the obtained results. The article concludes with a summary in Sec. 4.

2. THEORETICAL FRAMEWORK

The system of interest consists of an ensemble of externally pumped N two-level emitters embedded in a crystal. These can be impurity atoms or molecules whose collective inter-particle interactions are mediated by the environmental thermal phonon reservoir inside the solid crystal. The Hamiltonian describing the pumped two-level sample in the corresponding interaction picture, is:

$$H = \sum_{\chi} \hbar\omega_{\chi} b^{\dagger}_{\chi} b_{\chi} + \hbar\Delta S_z + \hbar\Omega\big(S^{+} + S^{-}\big) + \frac{\hbar}{N}\sum_{\chi} g_{\chi} S^{+}S^{-}\big(b^{\dagger}_{\chi} + b_{\chi}\big). \quad (1)$$

Here, the first two terms describe the free energies of the phonon bath and two-level emitters, while the last two components account for the interaction of the emitter's subsystem with an external applied coherent electromagnetic field and the environmental phonon reservoir [25], respectively. The detuning between the emitter's frequency ω_0 and the applied coherent electromagnetic field source frequency ω_L is $\Delta = \omega_0 - \omega_L$. The corresponding Rabi frequency is given by Ω, whereas the emitter-phonon coupling strength is denoted by g_{χ}. In the Hamiltonian (1), the collective two-level emitters operators $S^{+} = \sum_{j=1}^{N} |2\rangle_{jj}\langle 1|$ and $S^{-} = [S^{+}]^{\dagger}$ obey the usual commutation relations for su(2) algebra, namely, $[S^{+}, S^{-}] = 2S_z$ and $[S_z, S^{\pm}] = \pm S^{\pm}$, where $S_z = \sum_{j=1}^{N}(|2\rangle_{jj}\langle 2| - |1\rangle_{jj}\langle 1|)/2$ is the bare-state inversion operator. Further, $|2\rangle_j$ and $|1\rangle_j$ are the excited and ground state of the emitter j, respectively, while $b^{\dagger}_{\chi}$ and b_{χ} are the creation and the annihilation operator of the environmental phonon field reservoir, and satisfy the standard bosonic commutation relations, *i.e.*, $[b_{\chi}, b^{\dagger}_{\chi'}] = \delta_{\chi\chi'}$, and $[b_{\chi}, b_{\chi}] = [b^{\dagger}_{\chi}, b^{\dagger}_{\chi}] = 0$ [33].

In the following, we shall describe our system using the laser-qubit semiclassical dressed-state formalism defined as:

$$\begin{aligned} |+\rangle_j &= \sin\theta|1\rangle_j + \cos\theta|2\rangle_j, \\ |-\rangle_j &= \cos\theta|1\rangle - \sin\theta|2\rangle_j, \end{aligned} \quad (2)$$

with $\tan 2\theta = 2\Omega/\Delta$. Applying this transformation to (1), one arrives then at the following dressed-state Hamiltonian:

$$\begin{aligned} H &= \sum_{\chi} \hbar\omega_{\chi} b^{\dagger}_{\chi} b_{\chi} + \hbar\bar{\Omega} R_z + \frac{1}{2N}\sin 2\theta \sum_{\chi} \hbar g_{\chi} \\ &\times \big(\cos^2\theta(R_z R^{-} + R^{+}R_z) - \sin^2\theta(R^{-}R_z + R_z R^{+})\big)\big(b^{\dagger}_{\chi} + b_{\chi}\big) \\ &- \frac{1}{4N}\sin^2 2\theta \sum_{\chi} \hbar g_{\chi}\big(R^{+2} + R^{-2}\big)\big(b^{\dagger}_{\chi} + b_{\chi}\big), \end{aligned} \quad (3)$$

with $\bar{\Omega} = \sqrt{\Omega^2 + (\Delta/2)^2}$ being the corresponding generalized Rabi frequency. The new quasi-spin operators, *i.e.*, $R^+ = \sum_{j=1}^{N} |+\rangle_{jj}\langle -|$, $R^- = [R^+]^\dagger$ and $R_z = \sum_{j=1}^{N} (|+\rangle_{jj}\langle +| - |-\rangle_{jj}\langle -|)$ are operating in the dressed-state picture. They obey the following commutation relations: $[R^+, R^-] = R_z$ and $[R_z, R^\pm] = \pm 2R^\pm$. Notice that the terms proportional to R_z^2, R^+R^- and R^-R^+ were omitted from the Hamiltonian (3) because they do not contribute to transitions among the established collective dressed-states. While inspecting the dressed Hamiltonian (3), one can observe that transitions among the dressed-states within the whole ensemble are taken place *via* phonon absorption-emission processes involving individual emitters (*i.e.*, the second line) or pairs of emitters (the last line), respectively. These processes occur at different generalized Rabi frequencies, namely, $2\bar{\Omega}$ and $4\bar{\Omega}$, respectively.

In the Heisenberg picture, the time evolution of the mean value of an arbitrary collective operator $Q(t)$, belonging to the emitter subsystem only, can be written as follows [4]:

$$\frac{d}{dt}\langle Q(t)\rangle = \frac{i}{\hbar}\langle [H, Q(t)]\rangle, \tag{4}$$

where the notation $\langle \cdots \rangle$ indicates averaging over the initial state of both the two-level emitters and the environment phonon reservoir, respectively. Substituting the expression for the Hamiltonian (3) in the Eq. (4), one obtains:

$$\begin{aligned} &\frac{d}{dt}\langle Q(t)\rangle - i\bar{\Omega}\big\langle [R_z, Q(t)]\big\rangle = \frac{i}{2N}\sin 2\theta \sum_{\chi} g_{\chi} \\ \times\ &\big\langle b_{\chi}^{\dagger}\big[\cos^2\theta(R_zR^- + R^+R_z) - \sin^2\theta(R^-R_z + R_zR^+), Q(t)\big]\big\rangle \\ -\ &\frac{i}{4N}\sin^2 2\theta \sum_{\chi} g_{\chi}\big\langle b_{\chi}^{\dagger}\big[R^{+2} + R^{-2}, Q(t)\big]\big\rangle + H.c., \end{aligned} \tag{5}$$

where, in general, for the non-Hermitian collective atomic operators Q, the $H.c.$ terms should be evaluated without conjugating Q, *i.e.*, by replacing Q^+ with Q in the Hermitian conjugate part. Now, assuming that the two-level emitter subsystem couples weakly to the surrounding phonon reservoir, one can eliminate from the above equation of motion the phonon degrees of freedom. On solving formally the Heisenberg equations for phonon field operators one can represent the solutions in the form:

$$b_{\chi}^{\dagger}(t) = b_{v}^{\dagger}(t) + b_{s}^{\dagger}(t), \tag{6}$$

with $b_{\chi}(t) = [b_{\chi}^{\dagger}(t)]^{\dagger}$. The vacuum and source parts from Eq. (6) are represented as follows:

$$b_{v}^{\dagger}(t) = b_{\chi}^{\dagger}(0)e^{i\omega_{\chi}t},$$

and

$$b_s^{\dagger}(t) = \frac{ig_\chi}{2N}\sin 2\theta\big(\cos^2\theta R^+R_z - \sin^2\theta R_zR^+\big)\xi(\omega_\chi - 2\bar{\Omega}) - \frac{ig_\chi}{4N}\sin^2 2\theta R^{+2}\xi(\omega_\chi - 4\bar{\Omega}).$$

Here,

$$\xi(\omega_\chi - x) = \pi\delta(\omega_\chi - x) + iP_c\frac{1}{\omega_\chi - x},$$

where P_c is the Cauchy principal part. Introducing Eq. (6) in the master equation (5) and using Bogoliubov lemma [34] to represent the free emitter-phonon-field correlators $\langle b_v^{\dagger}(t)B(t)\rangle$ and $\langle B(t)b_v(t)\rangle$ *via* the emitter operators and the phonon bath characteristics only, we obtain the following master equation but in the Schrödinger picture, describing the quantum dynamics of any collective emitters operator, namely,

$$\begin{aligned}\frac{d}{dt}\rho(t) + i\bar{\Omega}[R_z,\rho] &= -\frac{\Gamma_1}{4}(1+\bar{n}_1)\sin^2 2\theta\big[(\cos^2\theta R^+R_z \\ &- \sin^2\theta R_zR^+),(\cos^2\theta R_zR^- - \sin^2\theta R^-R_z)\rho\big] \\ &- \frac{\Gamma_1}{4}\bar{n}_1\sin^2 2\theta\big[(\cos^2\theta R_zR^- - \sin^2\theta R^-R_z),(\cos^2\theta R^+R_z - \sin^2\theta R_zR^+)\rho\big] \\ &- \frac{\Gamma_2}{16}(1+\bar{n}_2)\sin^4 2\theta\big[R^{+2},R^{-2}\rho\big] - \frac{\Gamma_2}{16}\bar{n}_2\sin^4 2\theta\big[R^{-2},R^{+2}\rho\big] + H.c. \qquad (7)\end{aligned}$$

Here $\Gamma_1 = \gamma_1/N^2$, with

$$\gamma_1 \equiv \sum_\chi g_\chi^2\xi(\omega_\chi - 2\bar{\Omega}),$$

and $\Gamma_2 = \gamma_2/N^2$, with

$$\gamma_2 \equiv \sum_\chi g_\chi^2\xi(\omega_\chi - 4\bar{\Omega}),$$

while

$$\bar{n}_1 = \left[\exp\left(\frac{2\bar{\Omega}\hbar}{k_BT}\right) - 1\right]^{-1},$$

and

$$\bar{n}_2 = \left[\exp\left(\frac{4\bar{\Omega}\hbar}{k_BT}\right) - 1\right]^{-1},$$

respectively. Here k_B is the corresponding Boltzmann constant, whereas T is the environmental temperature. Notice here that the equation for the density matrix $\rho(t)$

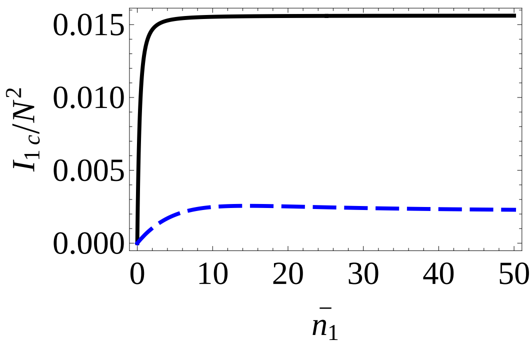

Fig. 1 – The steady-state scaled collective phonon intensity I_{1c}/N^2 [in units of γ_1] as a function of the mean thermal phonon number $\bar{n}_1$. Here $\Delta/(2\Omega)$=1 and the solid line corresponds to $N = 2$, while the dashed one to $N = 100$, respectively.

was obtained using the following identity:

$$\mathrm{Tr}\left(\frac{d}{dt}Q(t)\rho(0)\right) = \mathrm{Tr}\left(\frac{d}{dt}\rho(t)Q(0)\right),$$

and after performing the secular approximations, *i.e.*, neglecting faster oscillating terms at generalized Rabi frequency and higher ones.

In the following section, we shall investigate the established phonon steady-state quantum dynamics when either (i) $\Gamma_2 \to 0$ and $\Gamma_1 \neq 0$ or (ii) $\Gamma_1 \to 0$ and $\Gamma_2 \neq 0$, respectively.

3. RESULTS AND DISCUSSION

Using the master equation (7), it is straightforward to obtain a simple analytical solution for the steady-state density operator of the system. The solution can be written in the form

$$\rho_s = Z^{-1} e^{-\alpha R_z}, \tag{8}$$

where

$$\alpha = \ln\big[(1+\bar{n}_1)/\bar{n}_1\big]/2, \tag{9}$$

when $\Gamma_1 \neq 0$ and $\Gamma_2 \to 0$, while

$$\alpha = \ln\big[(1+\bar{n}_2)/\bar{n}_2\big]/4, \tag{10}$$

if $\Gamma_2 \neq 0$ and $\Gamma_1 \to 0$, respectively. The parameter Z is the normalization constant such that $\mathrm{Tr}\{\rho_s\} = 1$.

The phonon intensity is proportional to the mean phonon number from all phonon modes, namely, $I \propto \sum_\chi g_\chi \langle b^\dagger_\chi b_\chi\rangle$. Taking into account the expressions for

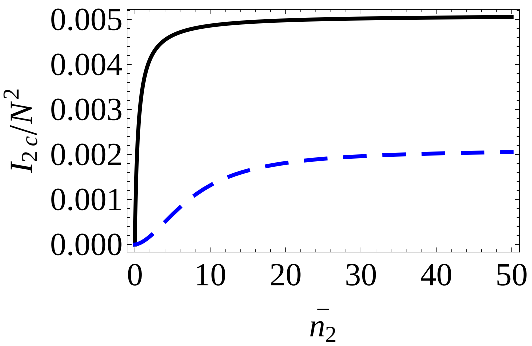

Fig. 2 – The steady-state scaled collective phonon intensity I_{2c}/N^2 [in units of γ_2] as a function of the mean thermal phonon number $\bar{n}_2$. Here $\Delta/(2\Omega)$=0.1 and the solid line corresponds to $N = 2$, while the dashed one to $N = 100$, respectively.

the phonon field operators, *i.e.* (6) and its Hermitian conjugate, one arrives at the following expressions for the phonon intensity expressed through the atomic correlators and thermal phonon bath characteristics only, that is,

$$I_1 = \bar{n}_1 + I_{1c}, \tag{11}$$

with

$$\begin{aligned} I_{1c} &= \frac{\gamma_1}{4N^2}\sin^2 2\theta\Big(\cos^2 2\theta\langle R_z^2 R^+ R^-\rangle \\ &- 4\cos 2\theta\cos^2\theta\langle R_z R^+ R^-\rangle + 4\cos^4\theta\langle R^+ R^-\rangle\Big), \end{aligned} \tag{12}$$

when $\Gamma_1 \neq 0$ and $\Gamma_2 = 0$, whereas

$$I_2 = \bar{n}_2 + I_{2c}, \tag{13}$$

with

$$I_{2c} = \frac{\gamma_2}{16N^2}\sin^4 2\theta\langle R^{+2} R^{-2}\rangle, \tag{14}$$

if $\Gamma_2 \neq 0$ and $\Gamma_1 = 0$, respectively. Considering an atomic coherent state $|n\rangle$, denoting a symmetrized N-atom state in which $N-n$ particles are in the lower dressed state $|-\rangle$ and n atoms are excited to the upper dressed state $|+\rangle$, and that $R^-|n\rangle = \sqrt{n(N-n+1)}|n-1\rangle$, $R^+|n\rangle = \sqrt{(N-n)(n+1)}|n+1\rangle$, and $R_z|n\rangle = (2n-N)|n\rangle$ [31, 35, 36], one can calculate the expectation values of any atomic correlators entering in expressions (12,14). A first observation here is that for off-resonant driving, the phonon intensity, for larger atomic ensembles, is proportional to squared number of emitters, *i.e.* $I_1 \propto N^2$, while the same occurs for the second situation too, that is $I_2 \propto N^2$, but regardless of the resonance condition.

Figure 1 depicts the steady-state phonon intensity coming from the externally pumped two-level ensemble, and based on expression (12). Larger thermal phonon bath temperatures lead to saturation of phonon intensity. Moreover, for larger atomic ensembles, the phonon intensity enhances for off-resonant pumping compared to the resonance driving. This is because the first two terms from expression (12) vanish for $\Delta = 0$, respectively. On the other side, Fig. 2 shows the phonon intensity when pairs of two-level qubits contribute to phonon transitions among the involved semi-classical dressed states. Again here the intensity saturates for larger environmental temperatures, while it is maximal for near resonance external pumping, see expression (14). Thus, summarising here, near the resonance atomic pairs are contributing to the generated phonon field, while for off-resonant pumping – bunches of individual emitters.

4. CONCLUSIONS

We have investigated the collective quantum dynamics of an externally coherently pumped ensemble of two-level emitters placed inside a crystal. The emitters are close to each others on a length-scale commensurable to the emission phonon wavelengths and, as a consequence, the lattice thermal vibrational phonons mediate the inter-particle correlations. We have found that the generated phonon field at the generalized Rabi frequencies, characterizing the coherently pumped two-level emitters, are proportional to the squared number of radiators from the sample. Furthermore, we have demonstrated that bunches of individual single emitters or pairs of emitters contribute to the generated field simultaneously. We have calculated the corresponding phonon intensities when either multiple individual emitters or pairs of two-level emitters are involved in the phonon generation. This is possible because these processes occur at different frequencies, *i.e.* $2\bar{\Omega}$ and $4\bar{\Omega}$. *Via* engineering of the host solid state sample, one can arrange that one of the above described phonon emission process is enhanced while the other one is suppressed, respectively. Also, for larger ensembles, pairs of emitters contribute to phonon transitions mainly at resonant driving, whereas bunches of individual emitters, correspondingly, at off-resonance external coherent pumping.

Acknowledgements. MM is grateful for the hospitality of the Department of Theoretical Physics at the Horia Hulubei National Institute of Physics and Nuclear Engineering, Magurele-Bucharest, Romania. As well, VC and MM acknowledge the financial support from the Moldavian National Agency for Research and Development, *via* grant No. 20.80009.5007.07. A.I. acknowledges the financial support received from the Romanian Ministry of Research, Innovation and Digitization, through the Project PN 23 21 01 01 / 2023.

REFERENCES

1. R. H. Dicke, Coherence in spontaneous radiation processes, Phys. Rev. **93**, 99 (1954).
2. K. Cong, Q. Zhang, Y. Wang, G. T. Noe, A. Belyanin, and J. Kono, Dicke superradiance in solids, J. Opt. Soc. Am. B **33**, C80 (2016).
3. B. W. Adams, C. Buth, S. M. Cavaletto, J. Evers, Z. Harman, C. H. Keitel, A. Palffy, A. Picon, R. Rohlsberger, Y. Rostovtsev, and K. Tamasaku, X-ray quantum optics, J. Mod. Opt. **60**, 2 (2013).
4. G. S. Agarwal, "*Quantum Statistical Theories of Spontaneous Emission and their Relation to Other Approaches*" (Springer, Berlin, 1974).
5. L. Allen and J. H. Eberly, "*Optical Resonance and Two-Level Atoms*" (Wiley, New York, 1975).
6. M. Gross and S. Haroche, Superradiance: an essay on the theory of collective spontaneous emission, Phys. Rep. **93**, 301 (1982).
7. A. V. Andreev, V. I. Emelyanov, and Yu. A. Ilinskii, "*Cooperative Effects in Optics: Superfluorescence and Phase Transitions*" (IOP Publishing, London, 1993).
8. J. G. Bohnet, Z. Chen, J. M. Weiner, D. Meiser, M. J. Holland and J. K. Thompson, A steady-state superradiant laser with less than one intracavity photon, Nature **484**, 78 (2012).
9. M. Xu, D. A. Tieri, E. C. Fine, J. K. Thompson, and M. J. Holland, Synchronization of Two Ensembles of Atoms, Phys. Rev. Lett. **113**, 154101 (2014).
10. M. A. Macovei, Performance of the collective three-level quantum thermal engine, Phys. Rev. A **105**, 043708 (2022).
11. Sh. Kamimura, H. Hakoshima, Y. Matsuzaki, K. Yoshida, and Y. Tokura, Quantum-Enhanced Heat Engine Based on Superabsorption, Phys. Rev. Lett. **128**, 180602 (2022).
12. M. Cuzminschi, A. Zubarev, and A. Isar, Extractable quantum work from a two-mode Gaussian state in a noisy channel, Sci. Rep. **11**, 24286 (2021).
13. M. A. Macovei, Cooling a quantum circuit *via* coupling to a multiqubit system, Phys. Rev. A **81**, 043411 (2010).
14. S. V. Petrushkin and V. V. Samartsev, Superradiance regime of laser cooling of crystals and glasses doped with rare-earth ions, Laser Phys. **11**, 948 (2001).
15. G. Nemova and R. Kashyap, Alternative technique for laser cooling with superradiance, Phys. Rev. A **83**, 013404 (2011).
16. E. Cecoi, V. Ciornea, A. Isar, and M. A. Macovei, Entanglement of a laser-driven pair of two-level qubits *via* its phonon environment, J. Opt. Soc. Am. B **35**, 1127 (2018).
17. R. Grimaudo, A. Isar, T. Mihaescu, I. Ghiu, and A. Messina, Dynamics of quantum discord of two coupled spin-1/2's subjected to time-dependent magnetic fields, Results Phys. **13**, 102147 (2019).
18. I. Ghiu, R. Grimaudo, T. Mihaescu, A. Isar, and A. Messina, Quantum correlation dynamics in controlled two-coupled-qubit systems, Entropy **22**, 785 (2020).
19. E. Cecoi, V. Ciornea, A. Isar, and M. A. Macovei, Entanglement *versus* cooling in the system of a driven pair of two-level qubits longitudinally coupled with a boson-mode field, J. Phys. B: At. Mol. Opt. Phys. **53**, 065501 (2020).
20. T. Mihaescu, E. Cecoi, M. Macovei, and A. Isar, Geometric discord for a driven two-qubit system, Rom. Rep. Phys. **73**, 101 (2021).
21. I. Ghiu, The quantum affinity of qubits subjected to noisy quantum channels, Rom. Rep. Phys. **73**, 119 (2021).
22. M. Cuzminschi and A. Isar, Quantum entanglement and quantum steering of two bosonic modes in noisy environments, Rom. J. Phys. **66**, 112 (2021).
23. M. Cuzminschi and A. Isar, Quantum steering of two bosonic modes in the two-reservoir model, Rom. Rep. Phys. **73**, 110 (2021).

24. A. Dobre and A. Isar, Open quantum dynamics of Gaussian Hilbert-Schmidt geometric discord, Rom. Rep. Phys. **74**, 101 (2022).
25. I. K. Kudryavtsev and A. S. Shumovsky, On a modification of the Dicke model, Optica Acta **26**, 827 (1979).
26. W. Bialek, Phonon super-radiance, Phys. Lett. A **103**, 349 (1984).
27. E. M. Chudnovsky and D. A. Garanin, Superradiance from Crystals of Molecular Nanomagnets, Phys. Rev. Lett. **89**, 157201 (2002).
28. E. M. Chudnovsky and D. A. Garanin, Phonon Superradiance and Phonon Laser Effect in Nanomagnets, Phys. Rev. Lett. **93**, 257205 (2004).
29. V. Ceban, P. Longo, and M. A. Macovei, Fast phonon dynamics of a nanomechanical oscillator due to cooperative effects, Phys. Rev. A **95**, 023806 (2017).
30. S. B. Jager, J. Cooper, M. J. Holland, and G. Morigi, Dynamical Phase Transitions to Optomechanical Superradiance, Phys. Rev. Lett. **123**, 053601 (2019).
31. R. R. Puri, "*Mathematical Methods of Quantum Optics*" (Springer, Berlin, 2001).
32. S. Ya. Kilin, Critical phenomena in cooperative resonant fluorescence of a system of interacting atoms, Sov. Phys. JETP **55**, 38 (1982).
33. D. F. Walls, and G. J. Milburn, "*Quantum Optics*" (Springer, Berlin, Heidelberg, 2008).
34. N. N. Bogolyubov and N. N. Bogolyubov, Jr., Fiz. Elem. Chastits At. Yadra **11**, 245 (1980) [Sov. J. Part. Nucl. **11**, 93 (1980)].
35. S. S. Hassan, G. P. Hildred, R. R. Puri, and R. K. Bullough, Incoherently driven Dicke model, J. Phys. B: At. Mol. Phys. **15**, 2635 (1982).
36. N. N. Bogolubov Jr., A. S. Shumovsky, and Tran Quang, Squeezing in collective resonance fluorescence, Phys. Lett. A **118**, 315 (1986).