DISPERSION EQUATIONS OF TRANSVERSE ELECTROMAGNETIC WAVES NARROWLY LOCALIZED NEAR THE INTERFACE BETWEEN THE MEDIA WITH DIFFERENT GRADED-INDEX PROFILES

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Received June 6, 2023

Abstract. Combinations of three pairs of contacting media with linear, exponential, and parabolic profiles of dielectric permittivity are described theoretically. Three new types of narrowly localized transverse electric waves propagating along the interfaces between the considered graded-index media are found. The exact dispersion equations for each type of the waves determining the effective refractive index in dependence on optical and geometric characteristics are obtained. The influence of the thicknesses of the graded-index layers on dispersion equation solutions is analyzed. It is found that the effective refractive index increases with an increase in the thickness of the graded-index layers in all considered combinations of the contacting media. The thickness of the parabolic graded-index profile has the least significant effect on the effective refractive index, compared to the linear and exponential ones.

Key words: electromagnetic wave, graded-index medium, graded-index profile, light localization, planar waveguide, interface, waveguide optics, surface wave, guided wave.

DOI: https://doi.org/10.59277/RomJPhys.2023.68.206

1. INTRODUCTION

Many authors use the waveguides with constant index regions (step-index thin-film waveguides) to illustrate the notion of guided waves and the basic properties [1]. However, many dielectric waveguides built on semiconductors, including gallium/aluminum arsenides, lithium niobates/fluorides and others have a graded-index profile [2–4]. A spatial modulation of the refractive index is also taken into account in photonics [5, 6]. All this makes it important to study the influence of the spatial distribution of the refractive index profile in design of optoelectronic devices [7].

One of the main characteristics here is the controlled localization of light beams and the electromagnetic field [8]. Electromagnetic surface waves in the optical range are an example of localized optical structures [9]. Therefore, surface waves in various optical media, including nonlinear [10, 11], photorefractive [12], and photonic crystals [13, 14], have been studied for quite a long time and remain relevant to this day [15–17].

Typical surface waves are characterized by localization of the field near the interface between the media and a monotonic decrease with distance from it [18, 19]. A decrease in the field in transverse direction with distance from the interface is exponential in a linear optical medium. Other profiles of the transverse field distribution are possible in various nonlinear media [20, 21] including a decrease with oscillations [22, 23].

At the same time, surface waves in graded-index media also exhibit a deviation from exponential field attenuation [24]. Graded-index media are widely used in waveguide optics when guided wave modes are also studied in addition to surface waves [25–27]. The waves localized along the planar waveguides are investigated by numerical and analytical methods [28, 29]. However, of greatest interest to theoreticians is the derivation of exact dispersion equations and analytical solutions for graded-index waveguides [30] and to the wave equations with arbitrary graded-index profiles [31]. Exact solutions are well known for waveguides with linear [32], parabolic [33], and exponential [34, 35] refractive index profiles.

Usually authors investigate the waveguides containing the graded-index layer (film) between two dielectrics (cover and substrate) [35]. Moreover, the waves propagating along a graded-index layer in contact with a nonlinear optical medium are also studied [36]. In particular, the properties of guided waves and exact dispersion equations of the waveguide with a linear graded-index film and an exponential one with nonlinear covering medium are studied in [37, 38].

Recently, we found exact solutions to the stationary wave equations that describe the transverse surface and guided waves propagating along the interfaces between linearly graded-index and intensity-dependent index layers [39], between linearly graded-index layer and the nonlinear layer formed with an increasing electric field [40], between the layer with the linearly graded-index media and photorefractive crystal [41], between linearly graded-index and nonlinear self-focusing [42] / defocusing [43] media, and parabolic graded-index and self-focusing nonlinear media [44]; see also other related works in this research area [45–49].

In our previous papers [39–44] we showed that it is possible to ensure that the field is completely localized inside the graded-index layer by selecting the values of the optical and geometric parameters of the contacting media.

In the present paper, we find the exact solutions and dispersion equations describing theoretically the new types of the electromagnetic waves narrowly localized near the interface between media with different graded-index profiles. In particular, we consider three cases: the contact of media with exponential and linear index profiles, the contact of media with exponential and parabolic index profiles, and the contact of media with linear and parabolic index profiles. The theory is formulated exclusively for TE-polarized waves, and the discussion is limited to fundamental modes only. The obtained analytical solutions for guided modes can serve as reliable benchmark solutions for numerical solvers.

This paper is organized as follows. The main equations are presented in Section 2. We analyze the main characteristics of electromagnetic waves narrowly localized near the interface between the exponential and linear graded-index media in Section 3, between the exponential and parabolic graded-index media in Section 4, and between the linear and parabolic graded-index media in Section 5. We discuss the comparison between the obtained dispersion equations in Section 6. The conclusions are presented in Section 7.

2. MAIN EQUATIONS

Let the planar interface (located at the plane $x = 0$) separates two media with different types of graded-index profiles. The dielectric permittivity ε depends on the spatial distance x, and its spatial distribution is different in media in half-spaces on different sides of the interface.

We consider the propagation of the transverse electric (TE) wave along the interface in the z-direction:

$$E_y = \psi(x)\exp(i\beta z - i\omega t)\,, \tag{1}$$

where $\beta = k_0 n$ is the propagation constant, n is the unknown effective refractive index, $k_0 = 2\pi/\lambda$ is the wave number, λ is the wavelength, ω is the frequency, and $\psi(x)$ is the electric field distribution across the interface, which can be found as the solution to the stationary wave equation

$$\psi''(x) + \{\varepsilon(x) - n^2\}k_0^2\psi(x) = 0\,, \tag{2}$$

where the spatial distribution of the dielectric function is given by

$$\varepsilon(x) = \begin{cases} \varepsilon_{(+)}(x), & x > 0, \\ \varepsilon_{(-)}(x), & x < 0. \end{cases} \tag{3}$$

The electric field distribution across the interface can be written as

$$\psi(x) = \begin{cases} \psi_{(+)}(x), & x > 0, \\ \psi_{(-)}(x), & x < 0. \end{cases} \tag{4}$$

The field distribution across the interface satisfies the boundary conditions:

$$\psi_{(+)}(+0) = \psi_{(-)}(-0) = \psi_0\,, \tag{5}$$

$$\psi'_{(+)}(+0) = \psi'_{(-)}(-0)\,, \tag{6}$$

where ψ_0 is the amplitude of the electric field at the interface.

The electric field distribution across the interface obeys the condition at infinity: $|\psi(x)| \to 0$, $|x| \to \infty$.

We focus on the study of the narrowly localized electromagnetic waves, for which the field penetration depth is smaller than the thickness of the graded-index layer. The field amplitude at the distance of the thickness of the graded-index layer is much less than the amplitude at the interface, and it can be neglected. Therefore, it is not necessary to use the boundary conditions at the distance of the thickness of the graded-index layer in the case considered. Thus, we can use the boundary conditions (5) and (6) only to describe the narrow localized electromagnetic waves.

The solution to Eq. (2) with the dielectric function defined by Eq. (3) can be presented as

$$\psi_{(\pm)}(x) = \psi_0 \frac{F_{(\pm)}(g_{(\pm)}(x))}{F_{(\pm)}(g_{(\pm)}(0))}, \tag{7}$$

where the functions $F_{(\pm)}(x)$ are the special functions that solve analytically the equations

$$F''_{(\pm)} g'_{(\pm)} + F'_{(\pm)} g''_{(\pm)} + \{\varepsilon(x) - n^2\} k_0^2 F_{(\pm)} = 0, \tag{8}$$

and $g_{(\pm)}(x)$ are auxiliary arguments. The forms of the functions $g_{(\pm)}(x)$ are associated with the change of variables that must be done in order to transform Eq. (8) to a form for which the exact solutions are known to be expressed in terms of special functions. In particular, we show below that $g_{(\pm)}(x)$ depend linearly on x as $g_{(\pm)}(x) = \alpha_{(\pm)} x + b_{(\pm)}$ in the case of linear profile, and then $g'_{(\pm)} = \alpha_{(\pm)}$ and $g''_{(\pm)} = 0$ in Eq. (8).

The functions (7) satisfy the boundary condition (5). After substitution Eq. (7) into the boundary condition (6) we obtain the general dispersion equation

$$\frac{F'_{(+)}(g_{(+)}(0))}{F_{(+)}(g_{(+)}(0))} g'_{(+)}(0) = \frac{F'_{(-)}(g_{(-)}(0))}{F_{(-)}(g_{(-)}(0))} g'_{(-)}(0), \tag{9}$$

which determines the dependence of the effective refractive index n on optical and geometric characteristics of the contacting graded-index media.

We determine the deceleration coefficient of the graded-index layer as follows

$$n_{(\pm)}(n) = \frac{F'_{(\pm)}(g_{(\pm)}(0))}{F_{(\pm)}(g_{(\pm)}(0))} \frac{g'_{(\pm)}(0)}{k_0}. \tag{10}$$

Then, the general dispersion equation (9) transforms into equation $n_{(-)}(n) = n_{(+)}(n)$, the solution of which (or Eq. (9)) determines the values of the effective refractive index n for the narrowly localized surface wave excitation near the interface between graded-index media.

Below we find explicit forms of the special functions $F_{(\pm)}(x)$ corresponding to three often used profiles:

1) the linear profile is given by

$$\varepsilon_L(x) = \varepsilon_{0L} - (\varepsilon_{0L} - \varepsilon_{fL})\frac{x}{L}, \tag{11}$$

where ε_{0L} and ε_{fL} are the dielectric constants at the interface and at the end of linearly graded-index layer of thickness L;

2) the parabolic profile is given by

$$\varepsilon_p(x) = \varepsilon_{0p} - (\varepsilon_{0p} - \varepsilon_{fp})\left(\frac{x}{h}\right)^2, \tag{12}$$

where ε_{0p} and ε_{fp} are the dielectric constants at the interface and at the end of parabolic graded-index layer of thickness h;

3) the exponential profile is given by

$$\varepsilon_e(x) = \varepsilon_{fe} + (\varepsilon_{0e} - \varepsilon_{fe})\exp(2x/a), \tag{13}$$

where ε_{0e} and ε_{fe} are the dielectric constants at the interface and at the end of exponentially graded-index medium with the characteristic thickness of a.

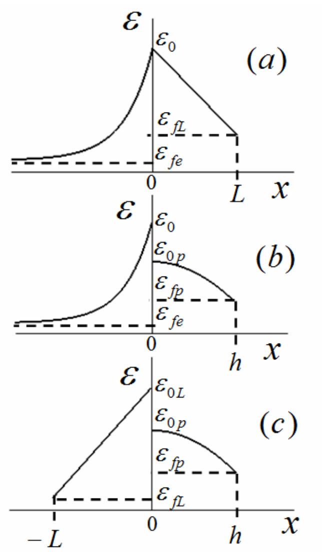

Fig. 1 – The dielectric function defined by Eq. (2).

The profiles defined by Eqs. (11–13) allow obtaining exact solutions to Eq. (8) in the terms of special functions. We find the narrowly localized surface waves propagating along the interface separating the graded-index media with three combinations of profiles defined by Eqs. (11–13), which are plotted schematically in Fig. 1. Note that the assumption of negligible fields at L and h constitutes a severe restriction of the theory.

3. THE INTERFACE BETWEEN THE EXPONENTIAL AND LINEAR GRADED-INDEX MEDIA

Here we consider the contact of the graded-index media with exponential and linear profiles. Then Eq. (3) with assumptions of $\varepsilon_{0L} = \varepsilon_{0e} = \varepsilon_0$ takes the form (see Fig. 1*a*)

$$\varepsilon(x) = \begin{cases} \varepsilon_0 - (\varepsilon_0 - \varepsilon_{fL})(x/L), \ x > 0, \\ \varepsilon_{fe} + (\varepsilon_0 - \varepsilon_{fe})\exp(2x/a), \ x < 0. \end{cases}, \tag{14}$$

In this case $F_{(+)}(x) = \mathrm{Ai}(x)$, $F_{(-)}(x) = J_\nu(x)$, where $\mathrm{Ai}(x)$ is the Airy function and $J_\nu(x)$ is the Bessel function of the first kind [50]. Therefore, the exact solution to the wave equation (2) with the dielectric function defined by Eq. (14) in the case of $\max\{\varepsilon_{fe}, \varepsilon_{fL}\} < n^2 < \varepsilon_0$ is given by

$$\psi(x) = \psi_0 \begin{cases} \dfrac{\mathrm{Ai}(x/x_L - \delta)}{\mathrm{Ai}(-\delta)}, \ x > 0, \\ \dfrac{J_{aq}(Ve^{x/a})}{J_{aq}(V)}, \ x < 0. \end{cases}, \tag{15}$$

where

$$x_L = \left(\frac{L}{k_0^2(\varepsilon_0 - \varepsilon_{fL})} \right)^{1/3}. \tag{16}$$

$$\delta = \frac{\varepsilon_0 - n^2}{\varepsilon_0 - \varepsilon_{fL}} \frac{L}{x_L}. \tag{17}$$

$$q = k_0 (n^2 - \varepsilon_{fe})^{1/2}. \tag{18}$$

$$V = ak_0(\varepsilon_0 - \varepsilon_{fe})^{1/2}. \tag{19}$$

The dispersion equation (9) of the surface waves defined by Eq. (15) takes the form

$$\frac{1}{x_L}\frac{\mathrm{Ai}'(-\delta)}{\mathrm{Ai}(-\delta)} = \frac{V}{a}\frac{J'_{aq}(V)}{J_{aq}(V)}. \tag{20}$$

The solution to the dispersion equation (20) defines the dependence of the effective refractive index n on the dielectric constants ε_{fL}, ε_{fe}, and ε_0, the wave number k_0, the thicknesses L, and a. Therefore, the solution to dispersion equation (20) $n = n(\varepsilon_{fL}, \varepsilon_{fe}, \varepsilon_0, k_0, L, a)$ must be substituted into Eqs. (17) and (18).

The solution (15) describes the narrowly localized electromagnetic waves propagating along the interface between the graded-index media with exponential and linear profiles with $\psi(L) << \psi_0$.

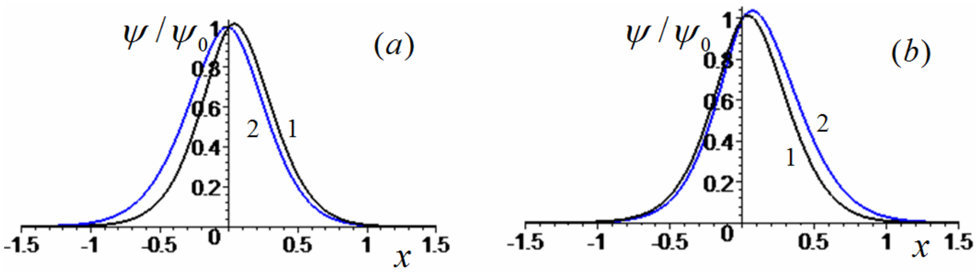

Fig. 2 – The narrowly localized electromagnetic wave determined by Eq. (15) with $k_0 = 1.5$, $\varepsilon_0 = 36$, $\varepsilon_{fL} = \varepsilon_{fe} = 0.01$ (in conventional units): *a*) $L = 1$, $a = 1$ (1), and $a = 2$ (2); *b*) $a = 1$, $L = 1$ (1), and $L = 1.5$ (2).

We plot the spatial profile of the field defined by Eq. (15) in Fig. 2. The field maximum is located in the linear graded-index layer. An increase in the value of a leads to a shift in the field distribution from the linear graded-index layer to the exponential one and to a little decrease in the field maximum intensity (Fig. 2*a*). Also, an increase in the value of L leads to an increase in the field maximum intensity and to an increase in the field penetration depth in the linear graded-index layer (Fig. 2*b*).

We focus below on studying the influence of the thickness of the graded-index layers on the value of the effective refractive index determined by the dispersion equation (20) corresponding to the narrowly localized electromagnetic wave. We obtain that the values of n^2 calculated from Eq. (20) do not approach close to the upper limit of the allowable range ε_0. We find from Eq. (20) that the effective refractive index increases with an increase in the characteristic thickness of the exponential graded-index layer *a*, with fixed values of the thickness of the linear graded-index layer *L* and dielectric constants (Fig. 3*a*).

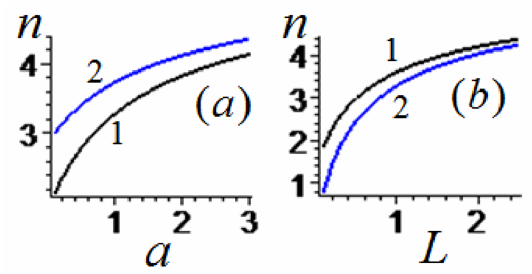

Fig. 3 – The effective refractive index *n* defined by the solution to Eq. (20) with $k_0 = 0.3$, $\varepsilon_0 = 36$, $\varepsilon_{fL} = \varepsilon_{fe} = 0.01$ (in conventional units): *a*) dependence *n* on *a* with $L = 1$ (1) and $L = 1.5$ (2); *b*) dependence *n* on *L* with $a = 1.5$ (1) and $a = 1$ (2).

There is a maximum thickness of the exponential graded-index layer, for which narrowly localized electromagnetic wave excitation is possible. After exceeding this thickness value of *a*, the wave defined by Eq. (15) ceases to be narrowly localized and is not described within the framework of the proposed theory. We derive from Eq. (20) that the effective refractive index increases with an increase in the thickness of the linear graded-index layer *L* with the fixed values of the thickness of the exponential graded-index layer *a* and dielectric constants (Fig. 3*b*). There is a maximum thickness of the linear graded-index layer, for which narrowly localized surface wave excitation is possible. After exceeding this thickness value of *L*, the wave defined by Eq. (15) ceases to be narrowly localized and is not described within the framework of the proposed theory.

It can also be noted that the influence on the value of the effective refractive index of the thickness of the linear graded-index layer *L* is more significant than that of the thickness of the exponential graded-index layer *a*.

4. THE INTERFACE BETWEEN THE EXPONENTIAL AND PARABOLIC GRADED-INDEX MEDIA

Now we consider the contact of the graded-index media with the exponential and parabolic profiles. Then Eq. (3) takes the form (see Fig. 1*b*)

$$\varepsilon(x) = \begin{cases} \varepsilon_{0p} - (\varepsilon_{0p} - \varepsilon_{fp})(x/h)^2, & x > 0, \\ \varepsilon_{fe} + (\varepsilon_{0e} - \varepsilon_{fe})\exp(2x/a), & x < 0. \end{cases} \quad (21)$$

Therefore, the exact solution to the wave equation (2) with the dielectric function defined by Eq. (21) in the case of $\varepsilon_{0p} < n^2 < \varepsilon_{0e}$ is given by

$$\psi(x) = \psi_0 \begin{cases} \dfrac{W_{\alpha,1/4}((x/x_p)^2)}{W_0\sqrt{x}}, \ x > 0, \\ \dfrac{J_{aq}(Ve^{x/a})}{J_{aq}(V)}, \ x < 0. \end{cases}, \tag{22}$$

where $W_{\alpha,l}(x)$ is the Whittaker function [51],

$$x_p^2 = \frac{h}{k_0\sqrt{\varepsilon_{0p} - \varepsilon_{fp}}}, \tag{23}$$

$$\alpha = -(n^2 - \varepsilon_{0p})(k_0 x_p / 2)^2. \tag{24}$$

Here q and V are determined by Eqs. (18) and (19), respectively, and

$$W_0 = \lim_{x \to +0} \{W_{\alpha,1/4}((x/x_p)^2)/\sqrt{x}\}.$$

The dispersion equation (9) of the electromagnetic waves defined by Eq. (22) takes the form

$$\frac{W_0'}{W_0} = \frac{V}{a}\frac{J_{aq}'(V)}{J_{aq}(V)}, \tag{25}$$

where

$$W_0' = \lim_{x \to +0} \frac{d}{dx}\{W_{\alpha,1/4}((x/x_p)^2)/\sqrt{x}\}.$$

The solution to the dispersion equation (25) defines the dependence of the effective refractive index n on the dielectric constants ε_{0p}, ε_{fp}, ε_{0e}, and ε_{fe}, the wave number k_0, and the thicknesses a and h. Therefore, the solution to the dispersion equation (25) $n = n(\varepsilon_{0p}, \varepsilon_{fp}, \varepsilon_{0e}, \varepsilon_{fe}, k_0, a, h)$ must be substituted into Eqs. (18) and (24).

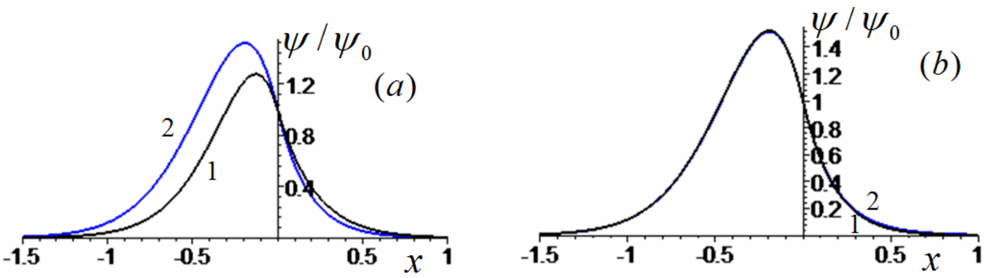

Fig. 4 – The effective refractive index *n* defined by the solution to Eq. (20) with $k_0 = 0.3$, $\varepsilon_0 = 36$, $\varepsilon_{fL} = \varepsilon_{fe} = 0.01$ (in conventional units): *a*) dependence *n* on *a* with $L = 1$ (1) and $L = 1.5$ (2); *b*) dependence *n* on *L* with $a = 1.5$ (1) and $a = 1$ (2).

The solution (22) describes the narrowly localized electromagnetic waves propagating along the interface between the graded-index media with exponential and parabolic profiles and for $\psi(h) << \psi_0$.

We plot the spatial profile of the field defined by Eq. (22) in Fig. 4. The field maximum is located in the exponential graded-index layer. An increase in the value of *a* leads to an increase in the field maximum intensity (Fig. 4*a*). A variation of the value of *h* practically does not affect the field distribution profile (Fig. 2*b*).

The effective refractive index increases with an increase in the characteristic thickness of the exponential graded-index layer *a* (Fig. 5*a*) and with an increase in the thickness of the parabolic graded-index layer *h* (Fig. 5*b*). Dependencies of the effective refractive index on *a* and on *h* are similar to the dependencies $n = n(a)$ and $n = n(L)$, which are characteristic of the electromagnetic wave near the interface between graded-index media with exponential and linear profiles described in Section 3. However, in contrast to the waves defined by Eq. (15), the influence of the characteristic thickness of the exponential graded-index layer *a* on the value of *n* is more significant. The variation of the thickness of the parabolic graded-index layer *h* has little effect on the values of *n* determined by Eq. (25).

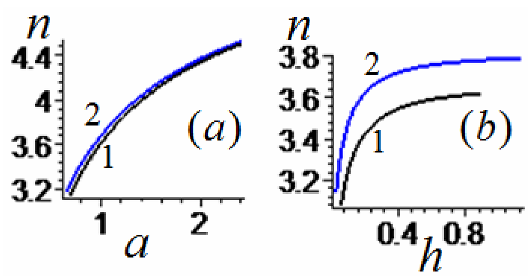

Fig. 5 – The effective refractive index *n* defined by the solution to Eq. (25) with $k_0 = 0.9$, $\varepsilon_{0e} = 36$, $\varepsilon_{0p} = 9$, $\varepsilon_{fp} = \varepsilon_{fe} = 0.01$ (in conventional units): *a*) dependence *n* on *a* with $h = 0.2$ (1) and $h = 0.3$ (2); *b*) dependence *n* on *h* with $a = 0.85$ (1) and $a = 1$ (2).

5. DISCUSSION OF THE INTERFACE BETWEEN THE LINEAR AND PARABOLIC GRADED-INDEX MEDIA

In this Section, we consider the contact of the graded-index media with linear and parabolic profiles. Then Eq. (3) takes the form (see Fig. 1*c*)

$$\varepsilon(x)=\begin{cases}\varepsilon_{0p}-(\varepsilon_{0p}-\varepsilon_{fp})(x/h)^2,\ x>0,\\ \varepsilon_{0L}+(\varepsilon_{0L}-\varepsilon_{fL})(x/L),\ x<0.\end{cases} \tag{26}$$

Therefore, the exact solution to the wave equation (2) with the dielectric function defined by Eq. (26) in the case of $\varepsilon_{0p} < n^2 < \varepsilon_{0L}$ is given by

$$\psi(x)=\psi_0\begin{cases}\dfrac{W_{\alpha,1/4}((x/x_p)^2)}{W_0\sqrt{x}},\ x>0,\\ \dfrac{\mathrm{Ai}(x/x_L-\delta)}{\mathrm{Ai}(-\delta)},\ x<0.\end{cases}, \tag{27}$$

where x_L, δ, x_p, and α are determined by Eqs. (16), (17), (23), and (24), respectively.

The dispersion equation (9) of the electromagnetic waves defined by Eq. (27) takes the form

$$\frac{W_0'}{W_0}=-\frac{1}{x_L}\frac{\mathrm{Ai}'(-\delta)}{\mathrm{Ai}(-\delta)}. \tag{28}$$

The solution to dispersion equation (28) defines the dependence of the effective refractive index n on the dielectric constants ε_{0p}, ε_{fp}, ε_{0L}, and ε_{fL}, the wave number k_0, and the thicknesses L and h. Therefore, the solution to dispersion equation (28) $n = n(\varepsilon_{0p}, \varepsilon_{fp}, \varepsilon_{0L}, \varepsilon_{fL}, k_0, L, h)$ must be substituted into Eqs. (17) and (24). The solution (27) describes the narrowly localized electromagnetic waves propagating along the interface between the graded-index media with linear and parabolic profiles and for $\psi(h) << \psi_0$.

We plot the spatial profile of the field defined by Eq. (27) in Fig. 6. The field leads to an increase in the field maximum intensity and to an increase in the field penetration depth in the linear graded-index layer (Fig. 6*a*), but the field penetration depth in the parabolic graded-index layer remains almost the same. Also, an increase in the value of h leads to an increase in the field maximum intensity and to an increase in the field penetration depth in the parabolic graded-index layer (Fig. 6*b*), but the field penetration depth in the linear graded-index layer remains almost the same.

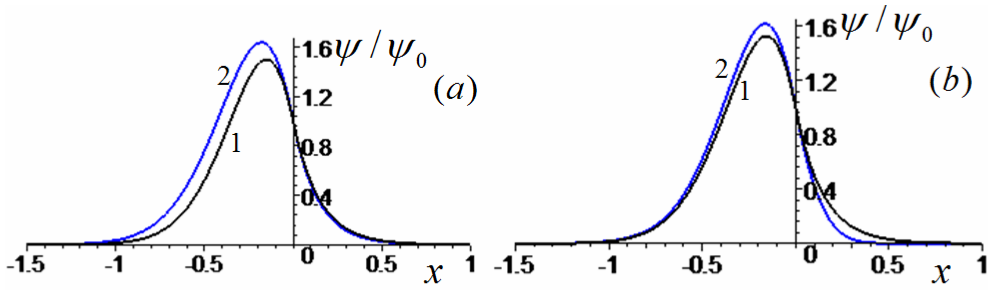

Fig. 6 – The narrowly localized electromagnetic wave determined by Eq. (27) with $k_0 = 1.5$, $\varepsilon_{0L} = 49$, $\varepsilon_{0p} = 9$, $\varepsilon_{fp} = \varepsilon_{fL} = 0.01$ (in conventional units): *a*) $h = 2$, $L = 1$ (1), and $L = 1.2$ (2); *b*) $L = 1$, $h = 0.3$ (1), and $h = 5$ (2).

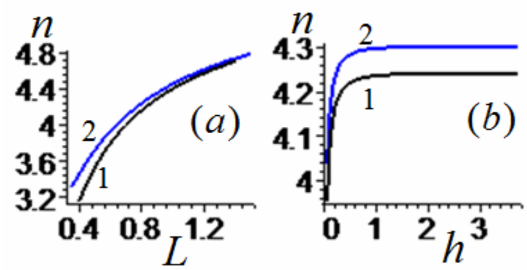

Fig. 7 – The effective refractive index *n* defined by the solution to Eq. (28) with $k_0 = 0.9$, $\varepsilon_{0L} = 36$, $\varepsilon_{0p} = 9$, $\varepsilon_{fp} = \varepsilon_{fL} = 0.01$ (in conventional units): *a*) dependence *n* on *a* with $h = 0.2$ (1) and $h = 0.3$ (2); *b*) dependence *n* on *h* with $a = 0.85$ (1) and $a = 1$ (2).

The effective refractive index increases with an increase in the thickness of the linear graded-index layer *L* (Fig. 7*a*) and with an increase in the thickness of the parabolic graded-index layer *h* (Fig. 7*b*). Dependencies of the effective refractive index on *L* and on *h* are similar to the dependencies $n = n(L)$ and $n = n(h)$, which are characteristic of the electromagnetic waves described in Sections 3 and 4, respectively. It follows from the dispersion equation (28) that the dependence $n = n(L)$ is more smooth than the dependence $n = n(h)$. The effective refractive index changes at small values of the thickness of the parabolic graded-index layer *h*, and then its value practically does not change with a further increase in *h* (Fig. 7*b*).

6. DISCUSSION

Here, we discuss the comparison between dispersion equations of the obtained above types of narrowly localized electromagnetic waves.

First, we compare the influence of the characteristic thickness of the exponential graded-index layer *a* on the effective refractive index in the cases of contacts an exponential graded-index medium with a linear one (dielectric function (14), Fig. 1*a*) and with a parabolic one (dielectric function (21), Fig. 1*b*). We plot the dependencies $n = n(a)$ calculated from the dispersion equation (20) (line 1 in Fig. 8 *a*) and from the dispersion equation (25) (line 2 in Fig. 8*a*).

Both dispersion equations demonstrate an increase in the effective refractive index with the increase of the characteristic thickness of the exponential graded-index layer. However, the values of n calculated from the dispersion equation (20) are greater than those calculated from the dispersion equation (25) at the same values of a and optical parameters. Therefore, the choice of the contact of an exponential graded-index medium with a linear one makes it possible to obtain higher values of the refractive index than for the contact of an exponential graded-index medium with a parabolic one at the same values of optical characteristics.

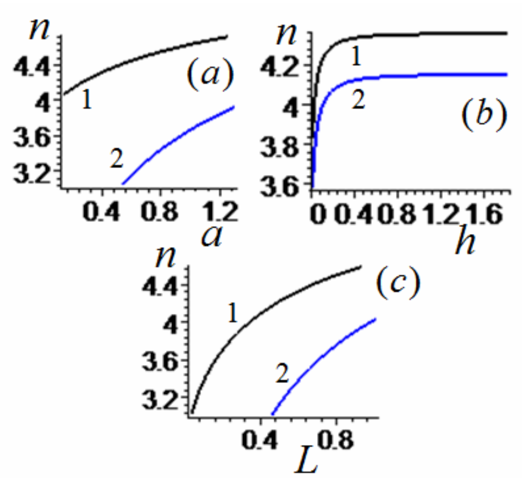

Fig. 8 – The dependencies of effective refractive index n on the thicknesses of the graded-index layers with $\varepsilon_{0L} = \varepsilon_{0e} = 36$, $\varepsilon_{0p} = 9$, $\varepsilon_{fp} = \varepsilon_{fL} = \varepsilon_{fe} = 0.01$: *a*) dependence n on a with $k_0 = 0.8$ calculated from Eq. (20) with $L = 1$ (1) and calculated from Eq. (25) with $h = 1$ (2); *b*) dependence n on h with $k_0 = 1.2$ calculated from Eq. (28) with $L = 0.9$ (1) and calculated from Eq. (25) with $a = 1.1$ (2); *c*) dependence n on L with $k_0 = 0.8$ calculated from Eq. (20) with $a = 1$ (1) and calculated from Eq. (28) with $h = 1$ (2).

Next, we compare the influence of the thickness of the parabolic graded-index layer h on the effective refractive index in the cases of contacts of a parabolic graded-index medium with an exponential one (dielectric function (21), Fig. 1*b*) and with a linear one (dielectric function (26), Fig. 1*c*). We plot the dependencies $n = n(h)$ calculated from the dispersion equation (28) (line 1 in Fig. 8*b*) and from the dispersion equation (25) (line 2 in Fig. 8*b*). In both cases an increase in the thickness of the parabolic graded-index layer leads to an increase in the effective refractive index. However, the values of n calculated from the dispersion equation

(28) are greater than those calculated from the dispersion equation (25) at the same values of *h* and optical parameters. Therefore, the choice of the contact of a parabolic graded-index medium with a linear one makes it possible to obtain higher values of the refractive index than for the contact of a parabolic graded-index medium with an exponential one at the same values of optical characteristics.

Finally, we compare the influence of the thickness of the linear graded-index layer *L* on the effective refractive index in the cases of contacts of a linear graded-index medium with an exponential one (dielectric function (14), Fig. 1*a*) and with a parabolic one (dielectric function (26), Fig. 1*c*). We plot the dependencies $n = n(L)$ calculated from the dispersion equation (20) (line 1 in Fig. 8*c*) and from the dispersion equation (28) (line 2 in Fig. 8*c*). Solutions to the both dispersion equations (20) and (28) demonstrate that an increase in the thickness of the linear graded-index layer leads to an increase in the effective refractive index. However, the values of *n* calculated from the dispersion equation (20) are greater than those calculated from the dispersion equation (28) at the same values of *L* and optical parameters. Therefore, the choice of the contact of a linear graded-index medium with a parabolic one makes it possible to obtain higher values of the refractive index than for the contact of a linear graded-index medium with an exponential one at the same values of optical characteristics.

Thus, it is expedient to use a parabolic profile to describe the optical properties in the case when the thickness of the graded-index layer has little effect on the effective refractive index (or propagation constant). An exponential and a linear profile are characterized by a more significant effect on the growth of the effective refractive index. Therefore, such profiles can be used to describe the optical properties of crystals in which the thickness of the graded-index layer significantly changes the effective refractive index.

Note that the propagation constant and the wave frequency are given by $\beta = k_0 n$, and $\omega = k_0 c$, respectively. Therefore, we find $n = \beta/k_0$ and $k_0 = \omega/c$. The substitution of these equations into Eqs. (20), (25), and (28) allows obtaining the dependence $\omega = \omega(\beta)$. Therefore, Eqs. (20), (25), and (28) can be considered as the dispersion equations of the waveguides. This allows a similar analysis of the influence of the thickness of the graded-index layer on the wave frequency.

7. CONCLUSIONS

We considered the combinations of three pairs of contacting media with linear, exponential, and parabolic profiles of dielectric permittivity. We found three new types of narrowly localized transverse electric waves propagating along the interface between the considered graded-index media. We obtained the exact dispersion equation for each type of the waves coupling the effective refractive index with optical and geometric characteristics of the contacting media. The waves of considered types are characterized by non-exponential attenuation with increasing distance from the interface and the field penetration depth is less than the thickness of the graded-index layer.

We analyzed the influence of the thicknesses of the graded-index layers on dispersion equation solutions determining the effective refractive index. We found that the effective refractive index increases with an increase in the thickness of the graded-index layers in all considered combinations of contacting media. It was shown that the thickness of the parabolic graded-index profile has the least significant effect on the effective refractive index, compared to the linear and exponential ones.

The results of our studies can be useful in choosing a profile shape for describing the optical properties of semiconductor heterostructures characterized by the refractive index depending on a spatial distance. The application of the obtained results is also possible when choosing the refractive index profile in weakly dissipative graded-index planar optical waveguides.

Acknowledgments. The study was carried out using equipment of the Center of High Technologies of the Belgorod V. G. Shukhov State Technological University.

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