DERIVATION OF LUMP SOLUTIONS TO INTEGRABLE (2+1)- AND (3+1)-DIMENSIONAL EXTENDED KDV AND KP EQUATIONS

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Abstract. In this study, we investigate the lump solutions for both integrable (2+1)- and (3+1)-dimensional extended KdV and KP equations using symbolic computation with Maple and the Hirota bilinear (HB) form. For each integrable model, we create positive quadratic function solutions to the HB equation. Graphs of the derived lump solutions are displayed with the proper parameter values.

Key words: Extended KdV equation; extended KP equation; lump solutions; multiple soliton solutions; Painlevé analysis.

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1. INTRODUCTION

Integrable equations disclose real characteristics as well as being useful for the investigation of the mysterious nature of nonlinear phenomena [1–16]. These equations provide both qualitative and quantitative characteristics of various natural phenomena, including but not limited to fluid mediums, plasma physics, excitations of ultra-cold atoms in Bose-Einstein condensates, and wave propagation in shallow water areas. Consequently, the investigation of integrable models has emerged as a prominent area of research in the realm of nonlinear scientific disciplines. Research on fully integrable equations is thriving both in the theoretical and experimental domains of the scientific literature [17–41].

The (2+1)-dimensional Korteweg-de Vries (KdV) equation reads

$$v_t + v_{xxx} + 3(v\partial_y^{-1} v_x)_x = 0. \tag{1}$$

For $x = y$, Eq. (1) reduces to the standard KdV equation. Note that for $v(x,y,t) = u_y(x,y,t)$, Eq. (1) becomes

$$u_{ty} + u_{xxxy} + 3(u_y u_x)_x = 0. \tag{2}$$

By utilizing the concept of the weak Lax pair, this equation was derived for the first time by Boiti *et al.* [1]. Also, Eq. (1) is commonly referred to as the Boiti-Leon-Manna-Pempinelli equation. It has been demonstrated to exhibit various characteristics such as the presence of Lax pair, an infinite number of conservation laws, integrability properties, and multiple soliton solutions. Various approaches have been extensively examined to explore alternative solutions for this model [1–15]. Moreover, this equation has been shown to possess families of localized solutions. Furthermore, at $x = y$, this equation can be simplified to the conventional KdV equation.

In Ref. [2], we extended Eq. (1) to the following two (2+1)- and (3+1)-dimensional KdV equations, respectively,

$$v_t + v_{xxx} + (v\partial_y^{-1} v_x)_x + (\partial_y^{-1} v_{xx}) + (\partial_y^{-1} v_{yy}) = 0, \tag{3}$$

and

$$v_t + v_{xxx} + (v\partial_y^{-1} v_x)_x + (\partial_y^{-1} v_{xx}) + (\partial_y^{-1} v_{yy}) + (\partial_y^{-1} v_{yz}) = 0. \tag{4}$$

The Painlevé test was used to confirm that the two extended Eqs. (3) and (4) are fully integrable. In Ref. [2], multiple soliton solutions (SSs) were obtained for both models (3) and (4).

Moreover, in our work [3], we introduced two integrable (2+1)- and (3+1)-dimensional extensions of the Kadomtsev–Petviashvili (KP) equation given by

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} + u_{xx} = 0, \tag{5}$$

and

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} + u_{xz} = 0, \tag{6}$$

respectively. The KP extensions (5) and (6) were confirmed to be Painlevé integrable with multiple soliton solutions for each model.

As previously mentioned, both KdV and KP extensions are fully integrable, allowing for the investigation of multiple SSs in each model. As is well-known, solitons (solitary waves) are identified as exponentially localized solutions in all space-directions. Nonetheless, an alternative category of exact solutions, known as lump solutions (LSs), has been extensively examined. In comparison to SSs, the LSs are distinct types of rational function solutions that are confined in all space directions. These solutions experience rational decay towards an asymptotic value and exhibit uniform velocity during the motion [6–24]. Nonlinear patterns appearing in a shallow water surface with high surface tension are described by the solutions called *lumps*. When surface tension dominates over a shallow water surface, nonlinear patterns can be described as lumps [10–16]. Additionally, lumps interact with each other without scattering, only experiencing a parallel change in the asymptotic motion [16–29]. There has been an increasing interest in the utilization of lump or lump-type solutions to address various types of nonlinear equations.

The investigation of lumps, breathers, and freak or rogue waves [(FWs) or (RWs)] has experienced significant expansion and garnered increased interest within the nonlinear scientific disciplines and engineering domains. Solitons, breathers, lumps, and RWs are categorized as discrete forms of nonlinear localized waves. Solitons are a type of wave that is both localized and stable. In contrast, RWs and breathers are localized structures that possess unstable characteristics. Concerning solutions in the form of rational functions, the solutions of lump-type demonstrate localization in nearly all directions within space. RWs, which are characterized by their sudden appearance and disappearance without a trace [42, 43], are spatially and temporally localized waveforms that have been implicated in a multitude of maritime catastrophes. In recent years, numerous investigations have been carried out on the distinct features of RWs in a variety of physical systems, particularly in plasma physics and optical fibers. This type of wave has been investigated based on the study of modulational instability (MI) of a nonlinear Schrödinger (NLS) equation [44–48]. Moreover, there are many studies about this type of wave in the framework of KdV-type equation [49–51].

The objective of this study is to supplement our previous research endeavors in Refs. [2, 3], wherein we conducted the Painlevé test and formally obtained multiple SSs for each equation. In order to advance the study of the extended KdV and KP Eqs. (3)–(6), the proposed approach involves the utilization of symbolic computation *via* Maple software, as well as the application of Hirota's bilinear (HB) form for analysis. This methodology aims to formally derive LSs for the (2+1)- and (3+1)-dimensional extended KdV and KP Eqs. (3)–(6). Moreover, some obtained LSs will be presented graphically.

2. LUMPS FOR THE (2+1)-DIMENSIONAL KdV EQUATION

In Eq. (3), for $v(x,y,t) = u_y(x,y,t)$, the following extended (2+1)-dimensional KdV equation is obtained

$$u_{ty} + u_{xxxy} + (u_y u_x)_x + u_{xx} + u_{yy} = 0. \tag{7}$$

To derive LSs for Eq. (7), we first transform Eq. (7) into a bilinear equation in operators form

$$\left(D_t D_y + D_x^3 D_y + D_x^2 + D_y^2\right) f \cdot f = 0. \tag{8}$$

Here, D_t and D_t indicate the HB derivative operators. According to Eq. (8) and by using $u(x,y,t) = 6(\ln f(x,y,t))_x$, then Eq. (7) is transformed to

$$\begin{aligned}&(ff_{ty} - f_t f_y) + (ff_{xxxy} - 3f_x f_{xxy} + 3f_{xx} f_{xy} - f_{xxx} f_y)\\ &+(ff_{xx} - f_x f_x) + (ff_{yy} - f_y f_y) = 0.\end{aligned} \tag{9}$$

To obtain the LSs for Eq. (7), the following assumptions are considered

$$\begin{aligned} g &= a_1 x + a_2 y + a_3 t + a_4, \\ h &= a_5 x + a_6 y + a_7 t + a_8, \\ f &= g^2 + h^2 + a_9, \end{aligned} \tag{10}$$

where $a_j, 1 \leq j \leq 9$ are real variables that will be identified later. Now, by inserting Eq. (10) into Eq. (9), we finally get a polynomial in (x, y, t) variables. By following the same procedures as in Refs. [1–10], we finally, get the values of parameters a_j. Thus we get the following set of equations restricting the various parameters:

$$\begin{aligned} a_i = a_i \& i &= 1, 2, 4, 5, 6, 8, \\ a_3 &= -\frac{a_2(a_1^2 + a_2^2 - a_5^2 + a_6^2) + 2a_1 a_5 a_6}{a_2^2 + a_6^2}, \\ a_7 &= \frac{a_6(a_1^2 + a_2^2 - a_5^2 + a_6^2) - 2a_1 a_2 a_5}{a_2^2 + a_6^2}, \\ a_9 &= -\frac{(a_1^2 + a_5^2)(a_2^2 + a_6^2)(a_1 a_2 + a_5 a_6)}{(a_1 a_6 - a_2 a_5)^2}, a_1 a_2 + a_5 a_6 < 0, \end{aligned} \tag{11}$$

which must fulfill the following determinant condition

$$\Delta = \begin{vmatrix} a_1 & a_5 \\ a_2 & a_6 \end{vmatrix} \neq 0. \tag{12}$$

Also the condition $a_9 > 0$ must be satisfied to ensure a well-defined function $f(x, y, t)$ that is positive and localizes $u(x, y, t)$ in all space-directions. By inserting the obtained parameters given in Eq. (11) into Eq. (10), we get the class of positive quadratic function solutions (PQFSs). This, in turn, will give a class of LSs to Eq. (7) by using $u = (6 \ln f(x,t))_x$ where f, g, and h are defined in Eq. (10). Furthermore, the obtained LSs $u(x, y, t) \to 0$ if and only if $g^2 + h^2 \to \infty$.

For example, selecting

$$a_1 = 1, a_2 = -2, a_4 = 2, a_5 = 1, a_6 = -1, a_8 = 2, \tag{13}$$

gives

$$a_3 = \frac{12}{5}, a_7 = \frac{9}{5}, a_9 = 90. \tag{14}$$

Recall that $v = u_y$ and by using the selected values for the parameters we obtain the following LS:

$$v = \left(\frac{6\,\left(\frac{42t}{5} + 4x - 6y + 8\right)}{\left(\frac{12t}{5} + x - 2y + 2\right)^2 + \left(\frac{9}{5}t + x - y + 2\right)^2 + 90} \right)_y. \tag{15}$$

3. LUMPS FOR THE (3+1)-DIMENSIONAL KdV EQUATION

Using $v(x,y,z,t) = u_y(x,y,z,t)$ in Eq. (4), the following extended (3+1)-dimensional KdV equation is obtained

$$u_{ty} + u_{xxxy} + (u_y u_x)_x + u_{xx} + u_{yy} + u_{yz} = 0. \tag{16}$$

Equation (16) is Painlevé integrable as proved in Ref. [2]. Moreover, multiple real and multiple complex SSs were furnished in Ref. [2]. By following the same methodology as we did above and transform Eq. (16) into a bilinear equation in the following operators form, we can derive LSs:

$$\left(D_t D_y + D_x^3 D_y + D_x^2 + D_y^2 + D_y D_z\right) f \cdot f = 0. \tag{17}$$

Here, D_R, are as defined earlier, where $R = x,y,z,t$. Accordingly, Eq. (16) is transformed to the following form using $u(x,y,z,t) = 6(\ln f(x,y,z,t))_x$:

$$\begin{array}{l}(ff_{ty} - f_t f_y) + (ff_{xxxy} - 3f_x f_{xxy} + 3f_{xx} f_{xy} - f_{xxx} f_y) \\ +(ff_{xx} - f_x f_x) + (ff_{yy} - f_y f_y) + (ff_{yz} - f_y f_z) = 0.\end{array} \tag{18}$$

To obtain the LSs for Eq. (16), the following assumptions are introduced

$$\begin{aligned} g &= a_1 x + a_2 y + a_3 z + a_4 t + a_5, \\ h &= a_6 x + a_7 y + a_8 z + a_9 t + a_{10}, \\ f &= g^2 + h^2 + a_{11}, \end{aligned} \tag{19}$$

where $a_j, 1 \le j \le 11$ are real parameters to be determined.

Now, by inserting Eq. (19) into Eq. (18), we finally obtain a polynomial of (x,y,z,t) variables. By following the same methodology used above, the values of the parameters $a_j, 1 \le j \le 11$ can be obtained. Following the same process as before, we arrive at the following set of equations that constrain the various parameters:

$$\begin{aligned} a_i = a_i \& i &= 1,2,4,5,6,7,9,10 \\ a_3 &= -\frac{a_2(a_1^2 - a_6^2 + a_7^2) + a_4(a_2^2 + a_7^2) + (2a_1 a_6 a_7 + a_2^3)}{a_2^2 + a_7^2}, \\ a_8 &= \frac{a_7(a_1^2 - a_2^2 - a_6^2 + a_7^2) - a_9(a_2^2 + a_7^2) - 2a_1 a_2 a_6}{a_2^2 + a_7^2}, \\ a_{11} &= -\frac{3(a_1^2 + a_6^2)(a_2^2 + a_7^2)(a_1 a_2 + a_5 a_6)}{(a_1 a_7 - a_2 a_6)^2}, \end{aligned} \tag{20}$$

which must fulfill the following determinant condition

$$\Delta = \begin{vmatrix} a_1 & a_6 \\ a_2 & a_7 \end{vmatrix} \neq 0 \tag{21}$$

and

$$a_1 a_2 + a_5 a_6 < 0, \tag{22}$$

to force $a_{11} > 0$ to ensure a well-defined function $f(x,y,z,t)$ that is positive and localizes $u(x,y,z,t)$ in all space-directions. The obtained parameters given in Eq.

(20) can generate the class of PQFSs by substituting Eq. (20) into Eq. (19). This, in turn will give a class of LSs to Eq. (16) by using $u = (6\ln f(x,y,z,t))_x$, where f, g, and h are given Eq. (19). In addition, the obtained LSs $u(x,y,z,t) \to 0$ if and only if $g^2 + h^2 \to \infty$.

For example, by considering $a_1 = 1$, $a_2 = 2$, $a_4 = 2$, $a_5 = 1$, $a_6 = 2$, $a_7 = -2$, $a_9 = 1$, $a_{10} = 2$, with $v(x,y,z,t) = u_y(x,y,z,t)$, we get

$$a_3 = -\frac{9}{4}, a_8 = \frac{3}{4}, a_{11} = \frac{20}{3}, \tag{23}$$

which lead to the following LS:

$$v = \left(\frac{6\left(10x - 4y - \frac{3}{2}z + 8t + 10\right)}{\left(x + 2y - \frac{9}{4}z + 2t + 1\right)^2 + \left(2x - 2y + \frac{3}{4}z + t + 2\right)^2 + \frac{20}{3}} \right)_y. \tag{24}$$

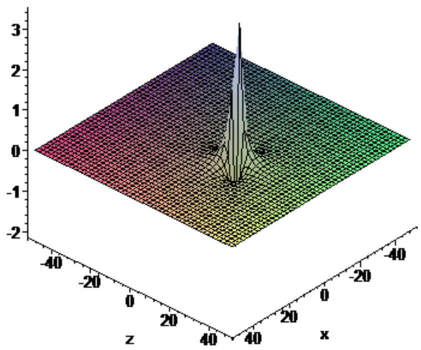


Fig. 1 – Profile of the LS for $-50 \le x, z \le 50, y = 10, t = 0$.

4. LUMPS FOR THE (2+1)-DIMENSIONAL KP EQUATION

The extended integrable (2+1)-dimensional KP equation reads

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} + u_{xx} = 0, \tag{25}$$

where this extension is obtained by including the linear term u_{xx} to the standard KP equation. This equation was introduced in Ref. [3], where its integrability was emphasized and multiple SSs were derived. In a similar manner to the previously presented analysis, LSs can be derived. Consequently, Eq. (25) can be transformed into a bilinear equation in the following operators form

$$\left(D_t D_x + D_x^4 + D_x^2 - D_y^2\right) f \cdot f = 0, \tag{26}$$

where D_t, D_x and D_y are the HB derivative operators.

By using $u(x,y,t) = 6(\ln f(x,y,t))_{xx}$, then Eq. (25) can be transformed to

$$(ff_{tx} - f_t f_x) + (ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}f_{xx}) + (ff_{xx} - f_x f_x) - (ff_{yy} - f_y f_y) = 0. \tag{27}$$

To obtain the LSs for Eq. (25), we will use the same assumptions given in Eq. (10) and by following the same procedures, we can get the values of the parameters $a_j, 1 \leq j \leq 9$. Thus, we arrive at the following set of equations that constrain the various parameters:

$$\begin{aligned} a_i = a_i \& i &= 1,2,4,5,6,8, \\ a_3 &= -\frac{a_1(a_1^2 - a_2^2 + a_5^2 + a_6^2) - 2a_2 a_5 a_6}{a_1^2 + a_5^2}, \\ a_7 &= -\frac{a_5(a_1^2 + a_2^2 + a_5^2 - a_6^2) - 2a_1 a_2 a_6}{a_1^2 + a_5^2}, \\ a_9 &= \frac{(a_1^2 + a_5^2)^3}{(a_1 a_6 - a_2 a_5)^2}, \end{aligned} \tag{28}$$

which must fulfill the following determinant condition

$$\Delta = (a_1 a_6 - a_2 a_5) = \begin{vmatrix} a_1 & a_5 \\ a_2 & a_6 \end{vmatrix} \neq 0. \tag{29}$$

Also the condition $a_9 > 0$ must be satisfied to ensure a well-defined function $f(x,y,t)$ that is positive and localizes $u(x,y,t)$ in all space-directions. By inserting the obtained parameters given in Eq. (28) into Eq. (10), we get the class of PQFSs. This, in turn, will give a class of LSs to Eq. (25) by using $u = (6\ln f(x,t))_x$. For example, by using $a_1 = 1, a_2 = 2, a_4 = 2, a_5 = 1, a_6 = 4, a_8 = 2$, we get

$$a_3 = 1, a_7 = 13, a_9 = 6. \tag{30}$$

Accordingly, the following LS is obtained

$$\begin{aligned} u(x,y,t) &= \frac{8}{(x+2y+t+2)^2 + (x+4y+13t+2)^2 + 6} \\ &- \frac{2(24x+12y+28t+8)^2}{((x+2y+t+2)^2 + (x+4y+13t+2)^2 + 6)^2}. \end{aligned} \tag{31}$$

5. LUMPS FOR THE (3+1)-DIMENSIONAL KP EQUATION

The extended (3+1)-dimensional KP equation reads

$$(u_t + 6uu_x + u_{xxx})_x - u_{yy} + u_{xz} = 0, \tag{32}$$

where we added the linear term u_{xz} to the standard KP equation.

Equation (32) is Painlevé integrable as proved in Ref. [3]. Moreover, multiple real and multiple complex SSs were furnished in Ref. [3] by using the simplified Hirota's method for real and complex structures.

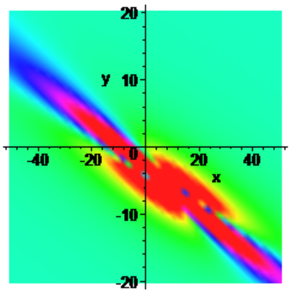
! 

Fig. 2 – Profile of the density curve for $-50 \leq x \leq 50, -30 \leq y \leq 30, t = 1$.

To find the LSs for Eq. (32), we follow the previous analysis and we transform Eq. (32) into a bilinear equation in operators form

$$\left(D_t D_x + D_x^4 - D_y^2 + D_x D_z\right) f \cdot f = 0. \tag{33}$$

Now, using Eq. (33) with $u(x,y,z,t) = 2(\ln f(x,y,z,t))_{xx}$, thus Eq. (32) can be transformed to

$$\begin{gathered}(f f_{tx} - f_t f_x) + (f f_{xxxx} - 4 f_x f_{xxx} + 3 f_{xx} f_{xx}) \\ +(f f_{xx} - f_x f_x)(f f_{yy} - f_y f_y) + (f f_{xz} - f_x f_z) = 0.\end{gathered} \tag{34}$$

To obtain the LSs for Eq. (32), we will use the assumptions given in Eq. (19) and in a similar manner to the previously presented analysis, we can get the values of the parameters $a_j, 1 \leq j \leq 11$. Following the same procedures as before, we arrive at the following set of equations that constrain the various parameters:

$$\begin{aligned} a_i = a_i\&i &= 1,2,4,5,6,7,8,10, \\ a_3 &= -\tfrac{a_4(a_1^2+a_6^2)-a_1(a_2^2-a_7^2)-2a_2a_6a_7}{a_1^2+a_6^2}, \\ a_9 &= \tfrac{a_8(a_1^2+a_6^2)+a_6(a_2^2-a_7^2)-2a_1a_2a_7}{a_1^2+a_6^2}, \\ a_{11} &= \tfrac{3(a_1^2+a_6^2)^3}{(a_1a_7-a_2a_6)^2}, \end{aligned} \tag{35}$$

which must fulfill the following determinant condition

$$\Delta = (a_1 a_7 - a_2 a_6) = \begin{vmatrix} a_1 & a_6 \\ a_2 & a_7 \end{vmatrix} \neq 0 \tag{36}$$

and

$$a_1 \neq 0 \,\text{and}\, a_6 \neq 0, \tag{37}$$

to guarantee $a_{11} > 0$ and to ensure a well-defined function $f(x,y,z,t)$ that is positive and localizes $u(x,y,z,t)$ in all space-directions. Now, by inserting the obtained parameters given in Eq. (35) into Eq. (19), we get the class of PQFSs. This, in turn, will give a class of LSs to Eq. (32) by using $u = (6 \ln f(x,y,z,t))_{xx}$. For example, by using the values $a_1 = 1, a_2 = 2, a_4 = 2, a_5 = 1, a_6 = 2, a_7 = 1, a_8 = 2, a_{10} = 2$, we get

$$a_3 = \frac{1}{5}, a_8 = -\frac{12}{5}, a_{11} = \frac{125}{3}. \tag{38}$$

Accordingly, the following LS is obtained (see Figs. 1 and 2)

$$\begin{aligned} u(x,y,z,t) &= \frac{20}{(x+2y+\frac{1}{5}z+2t+1)^2+(2x+y+2z-\frac{12}{5}t+2)^2+\frac{125}{3}} \\ &- \frac{2(10x+8y+\frac{42z}{5}-\frac{28}{5}t+10)^2}{((x+2y+\frac{1}{5}z+2t+1)^2+(2x+y+2z-\frac{12}{5}t+2)^2+\frac{125}{3})^2}. \end{aligned} \tag{39}$$

6. DISCUSSION

In the paper, we studied the integrable (2+1)- and (3+1)-dimensional extended KdV and KP equations. Using symbolic computation and Hirota's bilinear form, we derived positive quadratic function solutions to the corresponding bilinear equation for each distinct model. The obtained positive quadratic function solutions generate families of lump solutions to these equations by using the related dependent variable transformation for each equation. The lump solutions contain free distinct parameters, with sufficient restricted conditions for each examined equation, to guarantee the existence of these solutions. Using suitable values of the parameters involved, graphs of the obtained solutions are also reported.

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