CONSTRUCTING FAMILIES OF SOLUTIONS TO AN INTEGRABLE TIME-DEPENDENT SHALLOW WATER WAVE EQUATION IN (1+1)-DIMENSIONS

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Abstract. In this paper, an integrable shallow water wave equation with time-dependent coefficients in (1+1)-dimensions is taken into account. Through employing the generalized three-wave methods, a series of (double) solitary wave solutions and periodic (solitary) wave solutions to the considered equation are presented with the aid of symbolic calculation. Furthermore, by specifying relevant functions and parameters, the localized structures of some resulting solutions are displayed *via* some figures. These results enrich the diversity of nonlinear waves in physics.

Key words: Shallow water wave equation, generalized three-wave methods, solitary wave solution, periodic wave solution.

1. INTRODUCTION

Waves occur in most scientific and engineering fields, such as fluid mechanics, optics, electromagnetism, solid mechanics, structural mechanics, and quantum mechanics. The waves for all these applications are described by solutions to linear or nonlinear partial differential equations (NLPDEs). Therefore, studies of abundant solutions to these equations to characterize related physical phenomena are of great significance [1]. Until now, various advanced methods and their generalizations have been proposed and applied to investigate solutions to NLPDEs, such as the Hirota's bilinear method [2–7], the Darboux transformation method [8–11], the Lie symmetry analysis method [12–15], the Riemann-Hilbert method [16–19], and others [20–22].

The research work on NLPDEs with time-dependent coefficients is booming due to the discovery that the research results on NLPDEs with time-dependent coefficients are closer to scientific applications than those of NLPDEs with constant coefficients. In 2020, a shallow water wave model with time-dependent coefficients in (1+1)-dimensions [23]

$$f_1(t)u_t + f_2(t)u_x u_t + f_3(t)u_{xxt} + f_4(t)u_x + f_5(t)u_x^2 + f_6(t)u_{xxx} = 0 \qquad (1)$$

was established, where $f_i(t)(i = 1, 2, ..., 6)$ are non-zero differentiable functions of the temporal variable t. To ensure integrability, the compatibility condition requires that $f_1(t) = f_2(t) = f_3(t) = f(t), f_4(t) = g(t)$, and $f_5(t) = f_6(t) = h(t)$. Equation (1) is derived from the shallow water wave equation with constant coefficients [2]

$$u_t - 3u_x u_t - u_{xxt} + u_x + 3u_x^2 + u_{xxx} = 0.$$

For equation (1), the Painlevé analysis was conducted to show its complete integrability for any analytic time-dependent coefficients defined by compatibility condition. Moreover, a series of novel exact solutions were explored by adopting the simplified Hirota's direct method of real and complex criteria, as well as two ansätz methods [23]. The specific objective of this paper is to work out some exact solutions to the above developed equation (1) under the requirement for compatibility condition using the generalized three-wave methods [24–28].

2. APPLICATION OF THE GENERALIZED THREE-WAVE METHODS

This section is devoted to determination of some exact solutions to equation (1). To this end, a variable transformation

$$u(x,t) = 6[\ln(v(x,t))]_x \tag{2}$$

is first employed to cast equation (1) into the following equation in v

$$\begin{aligned} f(t)(vv_{xt} - v_x v_t + 3v_{xt}v_{xx} + v_{xxxt}v - 3v_{xxt}v_x - v_{xxx}v_t) \\ + g(t)(vv_{xx} - v_x^2) + h(t)(3v_{xx}^2 + vv_{xxxx} - 4v_{xxx}v_x) = 0. \end{aligned} \tag{3}$$

It is manifest that if we present a solution to equation (3), then a solution to equation (1) can be generated *via* the transformation (2).

2.1. Ansatz I

In this subsection, with regard to equation (3), the expression of solution is assumed to be of the form

$$v = k_1 \cos\xi_1 + k_2 \cosh\xi_2 + k_3 e^{-\xi_3} + k_4 e^{\xi_3}, \tag{4}$$

where $\xi_1 = a_1(x + a_2(t)), \xi_2 = a_3(x + a_4(t)), \xi_3 = a_5(x + a_6(t))$, and $a_j, k_l (j = 1,3,5; l = 1,2,3,4)$ are several unknown parameters, and $a_\iota(t) (\iota = 2,4,6)$ are functions of the temporal variable t. Then directly on substituting (4) into equation (3) and setting the coefficients of $\cos\xi_1 \cosh\xi_2, \sin\xi_1 \sinh\xi_2, \cos\xi_1 e^{\xi_3}$ and others to zeros, we obtain a complicated system of determining equations with respect to the undetermined functions and parameters. Here we omit to show this system for brevity. After performing some operations on the resulting system by symbolic software Maple, we derive the following cases:

Case 1.

$$a_6(t) = -\int \frac{4a_5^2 h(t) + g(t)}{(4a_5^2 + 1) f(t)} dt, k_1 = 0, k_2 = 0, k_3 = k_3, k_4 = k_4.$$

Inserting the results in this case into (4), we have

$$v = k_3 e^{-\xi_3} + k_4 e^{\xi_3},$$

where $\xi_3 = a_5\left(x - \int \frac{4a_5^2 h(t)+g(t)}{(4a_5^2+1)f(t)}dt\right)$. Thus, the solitary wave solution to equation (1) is written as

$$u = \frac{6\left(-k_3 a_5 e^{-a_5\left(x-\int \frac{4a_5^2 h(t)+g(t)}{(4a_5^2+1)f(t)}dt\right)} + k_4 a_5 e^{a_5\left(x-\int \frac{4a_5^2 h(t)+g(t)}{(4a_5^2+1)f(t)}dt\right)}\right)}{\left(k_3 e^{-a_5\left(x-\int \frac{4a_5^2 h(t)+g(t)}{(4a_5^2+1)f(t)}dt\right)} + k_4 e^{a_5\left(x-\int \frac{4a_5^2 h(t)+g(t)}{(4a_5^2+1)f(t)}dt\right)}\right)}. \tag{5}$$

By assumption of $k_3 k_4 > 0$, then the solution (5) becomes

$$u = 6a_5 \tanh\left(a_5 x - a_5 \int \frac{4a_5^2 h(t)+g(t)}{(4a_5^2+1)f(t)}dt + \frac{1}{2}\ln\left(\frac{k_4}{k_3}\right)\right). \tag{6}$$

Now, if we set $a_5 = i\tilde{a}_5$ and $k_3 = k_4$ in solution (6), where $\tilde{a}_5$ is a real constant, then we acquire the periodic wave solution

$$u = -6\tilde{a}_5 \tan\left(\tilde{a}_5 x - \tilde{a}_5 \int \frac{-4\tilde{a}_5^2 h(t)+g(t)}{(1-4\tilde{a}_5^2)f(t)}dt\right).$$

Case 2.

$$a_4(t) = -\int \frac{(a_5^4 - 2a_3^2 a_5^2 + a_3^2 + 3a_5^2 + a_3^4)h(t) + (a_3^2 - a_5^2 + 1)g(t)}{(a_3^2 + 1 - 2a_3 a_5 + a_5^2)(a_3^2 + 1 + 2a_3 a_5 + a_5^2)f(t)}dt,$$
$$a_6(t) = -\int \frac{(a_5^4 - 2a_3^2 a_5^2 + a_5^2 + 3a_3^2 + a_3^4)h(t) + (a_5^2 - a_3^2 + 1)g(t)}{(a_3^2 + 1 - 2a_3 a_5 + a_5^2)(a_3^2 + 1 + 2a_3 a_5 + a_5^2)f(t)}dt,$$
$$k_1 = 0, k_2 = k_2, k_3 = k_3, k_4 = \frac{a_3^2 k_2^2 (a_5^2 + 3a_3^2 + 3)}{4a_5^2 k_3 (3a_5^2 + a_3^2 + 3)}.$$

Regarding this case, the results are substituted into (4) to yield

$$v = k_2 \cosh\xi_2 + k_3 e^{-\xi_3} + \frac{a_3^2 k_2^2 (a_5^2 + 3a_3^2 + 3)}{4a_5^2 k_3 (3a_5^2 + a_3^2 + 3)} e^{\xi_3},$$

where

$$\begin{aligned} \xi_2 &= a_3\left(x - \int \frac{(a_5^4 - 2a_3^2 a_5^2 + a_3^2 + 3a_5^2 + a_3^4)h(t) + (a_3^2 - a_5^2 + 1)g(t)}{(a_3^2 + 1 - 2a_3 a_5 + a_5^2)(a_3^2 + 1 + 2a_3 a_5 + a_5^2)f(t)}dt\right), \\ \xi_3 &= a_5\left(x - \int \frac{(a_5^4 - 2a_3^2 a_5^2 + a_5^2 + 3a_3^2 + a_3^4)h(t) + (a_5^2 - a_3^2 + 1)g(t)}{(a_3^2 + 1 - 2a_3 a_5 + a_5^2)(a_3^2 + 1 + 2a_3 a_5 + a_5^2)f(t)}dt\right). \end{aligned} \tag{7}$$

Therefore, the solitary wave solution to equation (1) reads

$$u=\frac{6\left(a_3k_2\sinh\xi_2-a_5k_3e^{-\xi_3}+\frac{a_3^2k_2^2(a_5^2+3a_3^2+3)}{4a_5k_3(3a_5^2+a_3^2+3)}e^{\xi_3}\right)}{k_2\cosh\xi_2+k_3e^{-\xi_3}+\frac{a_3^2k_2^2(a_5^2+3a_3^2+3)}{4a_5^2k_3(3a_5^2+a_3^2+3)}e^{\xi_3}},\tag{8}$$

where ξ_2 and ξ_3 are determined by (7). It is obvious that $k_3k_4=\frac{a_3^2k_2^2(a_5^2+3a_3^2+3)}{4a_5^2(3a_5^2+a_3^2+3)}>0$. Thereby, the solution (8) is further converted into

$$u=\frac{6\left(a_3k_2\sinh\xi_2+\sqrt{\frac{a_3^2k_2^2(a_5^2+3a_3^2+3)}{3a_5^2+a_3^2+3}}\sinh\left(\xi_3+\frac{1}{2}\ln\left(\frac{a_3^2k_2^2(a_5^2+3a_3^2+3)}{4a_5^2k_3^2(3a_5^2+a_3^2+3)}\right)\right)\right)}{k_2\cosh\xi_2+\sqrt{\frac{a_3^2k_2^2(a_5^2+3a_3^2+3)}{a_5^2(3a_5^2+a_3^2+3)}}\cosh\left(\xi_3+\frac{1}{2}\ln\left(\frac{a_3^2k_2^2(a_5^2+3a_3^2+3)}{4a_5^2k_3^2(3a_5^2+a_3^2+3)}\right)\right)}.\tag{9}$$

Case 3.

$$\begin{aligned}
a_2(t)&=-\int\frac{(a_1^4+2a_1^2a_5^2-a_1^2+a_5^4+3a_5^2)h(t)+(1-a_1^2-a_5^2)g(t)}{(a_1^2+2a_1+1+a_5^2)(a_1^2-2a_1+1+a_5^2)f(t)}dt,\\
a_6(t)&=-\int\frac{(a_5^4+2a_1^2a_5^2+a_5^2+a_1^4-3a_1^2)h(t)+(a_5^2+a_1^2+1)g(t)}{(a_1^2+2a_1+1+a_5^2)(a_1^2-2a_1+1+a_5^2)f(t)}dt,\\
k_1&=k_1,k_2=0,k_3=k_3,k_4=-\frac{a_1^2k_1^2(a_5^2-3a_1^2+3)}{4a_5^2k_3(3a_5^2-a_1^2+3)}.
\end{aligned}$$

For this case, insertion of the results into (4) directly leads to

$$v=k_1\cos\xi_1+k_3e^{-\xi_3}-\frac{a_1^2k_1^2(a_5^2-3a_1^2+3)}{4a_5^2k_3(3a_5^2-a_1^2+3)}e^{\xi_3},$$

where

$$\begin{aligned}
\xi_1&=a_1\left(x-\int\frac{(a_1^4+2a_1^2a_5^2-a_1^2+a_5^4+3a_5^2)h(t)+(1-a_1^2-a_5^2)g(t)}{(a_1^2+2a_1+1+a_5^2)(a_1^2-2a_1+1+a_5^2)f(t)}dt\right),\\
\xi_3&=a_5\left(x-\int\frac{(a_5^4+2a_1^2a_5^2+a_5^2+a_1^4-3a_1^2)h(t)+(a_5^2+a_1^2+1)g(t)}{(a_1^2+2a_1+1+a_5^2)(a_1^2-2a_1+1+a_5^2)f(t)}dt\right).
\end{aligned}\tag{10}$$

As a consequence, we attain the periodic solitary wave solution

$$u=\frac{6\left(-a_1k_1\sin\xi_1-a_5k_3e^{-\xi_3}-\frac{a_1^2k_1^2(a_5^2-3a_1^2+3)}{4a_5k_3(3a_5^2-a_1^2+3)}e^{\xi_3}\right)}{k_1\cos\xi_1+k_3e^{-\xi_3}-\frac{a_1^2k_1^2(a_5^2-3a_1^2+3)}{4a_5^2k_3(3a_5^2-a_1^2+3)}e^{\xi_3}}\tag{11}$$

with ξ_1 and ξ_3 being given by (10). Under the constraint of $(3a_1^2-a_5^2-3)(3a_5^2-$

$a_1^2+3)>0$, the solution (11) is further turned into

$$u=\frac{6\left(-a_1k_1\sin\xi_1+\sqrt{\frac{a_1^2k_1^2(3a_1^2-a_5^2-3)}{3a_5^2-a_1^2+3}}\sinh\left(\xi_3+\frac{1}{2}\ln\left(\frac{a_1^2k_1^2(3a_1^2-a_5^2-3)}{4a_5^2k_3^2(3a_5^2-a_1^2+3)}\right)\right)\right)}{k_1\cos\xi_1+\sqrt{\frac{a_1^2k_1^2(3a_1^2-a_5^2-3)}{a_5^2(3a_5^2-a_1^2+3)}}\cosh\left(\xi_3+\frac{1}{2}\ln\left(\frac{a_1^2k_1^2(3a_1^2-a_5^2-3)}{4a_5^2k_3^2(3a_5^2-a_1^2+3)}\right)\right)}. \tag{12}$$

2.2. Ansatz II

In what follows, in order to present more solutions, we suppose that the expression of solution for equation (3) is

$$v=m_1\cos^2\eta_1+m_2\sinh^2\eta_2+m_3e^{-\eta_3}+m_4e^{\eta_3}, \tag{13}$$

where $\eta_1=b_1(x+b_2(t)),\eta_2=b_3(x+b_4(t)),\eta_3=b_5(x+b_6(t))$, and $b_j,m_l(j=1,3,5;l=1,2,3,4)$ are some parameters to be determined, and $b_\iota(t)(\iota=2,4,6)$ are functions of t. By substitution of (13) into equation (3) and setting the coefficients of $\cos^2\eta_1,\cosh^2\eta_2,\cos^2\eta_1e^{\eta_3}$ and others to zeros, a system of determining equations about the unknowns can be obtained at once. Computing this system with the aid of symbolic software reveals the following cases:

Case 1.

$$b_1=\epsilon ib_3,b_2(t)=Mt+C_1,b_4(t)=-\int\frac{(16Mb_3^2+M)f(t)+32b_3^2h(t)+2g(t)}{(16b_3^2+1)f(t)}dt,$$
$$m_1=m_2,m_2=m_2,m_3=0,m_4=0,\epsilon=\pm1.$$

Carrying the results in this case into (13), we find

$$v=m_2\cos^2\eta_1+m_2\sinh^2\eta_2,$$

in which

$$\begin{aligned}&\eta_1=\epsilon ib_3(x+Mt+C_1),\\&\eta_2=b_3\left(x-\int\frac{(16Mb_3^2+M)f(t)+32b_3^2h(t)+2g(t)}{(16b_3^2+1)f(t)}dt\right).\end{aligned} \tag{14}$$

Hence, equation (1) possesses the solitary wave solution

$$u=\frac{12\epsilon b_3m_2\cosh\tilde{\eta}_1\sinh\tilde{\eta}_1+12b_3m_2\cosh\eta_2\sinh\eta_2}{m_2\cosh^2\tilde{\eta}_1+m_2\sinh^2\eta_2},$$

where $\tilde{\eta}_1=\epsilon b_3(x+Mt+C_1)$, and η_2 is expressed by (14).

Case 2.

$$b_1=\frac{\epsilon}{2}\sqrt{4b_3^2+1},b_2(t)=-\int\frac{(8b_3^2+3)h(t)-g(t)}{2(4b_3^2+1)f(t)}dt,$$
$$b_4(t)=-\int\frac{(8b_3^2-1)h(t)+g(t)}{8b_3^2f(t)}dt,m_1=m_2,m_2=m_2,$$
$$m_3=0,m_4=0,\epsilon=\pm1.$$

Plugging the results in this case into (13), we attain

$$v=m_2\cos^2\eta_1+m_2\sinh^2\eta_2,$$

where

$$\begin{aligned}\eta_1&=\frac{\epsilon}{2}\sqrt{4b_3^2+1}\left(x-\int\frac{(8b_3^2+3)h(t)-g(t)}{2(4b_3^2+1)f(t)}dt\right),\\ \eta_2&=b_3\left(x-\int\frac{(8b_3^2-1)h(t)+g(t)}{8b_3^2f(t)}dt\right).\end{aligned}\tag{15}$$

Accordingly, we arrive at the periodic solitary wave solution

$$u=\frac{-6\epsilon m_2\sqrt{4b_3^2+1}\cos\eta_1\sin\eta_1+12b_3m_2\cosh\eta_2\sinh\eta_2}{m_2\cos^2\eta_1+m_2\sinh^2\eta_2}\tag{16}$$

with η_1 and η_2 being given by (15).

As a matter of fact, taking $b_3=i\tilde{b}_3$ in solution (16), where $\tilde{b}_3$ is a real constant, can generate the periodic wave solution

$$u=\frac{-6\epsilon m_2\sqrt{1-4\tilde{b}_3^2}\cos\eta_1\sin\eta_1-12\tilde{b}_3m_2\cos\tilde{\eta}_2\sin\tilde{\eta}_2}{m_2\cos^2\eta_1-m_2\sin^2\tilde{\eta}_2},$$

and

$$\eta_1=\frac{\epsilon}{2}\sqrt{1-4\tilde{b}_3^2}\left(x-\int\frac{(3-8\tilde{b}_3^2)h(t)-g(t)}{2(1-4\tilde{b}_3^2)f(t)}dt\right),$$
$$\tilde{\eta}_2=\tilde{b}_3\left(x+\int\frac{(-8\tilde{b}_3^2-1)h(t)+g(t)}{8\tilde{b}_3^2f(t)}dt\right).$$

Case 3.

$$b_1=\epsilon_1ib_3,b_2(t)=-\int\frac{16b_3^2h(t)+g(t)}{(1+16b_3^2)f(t)}dt,b_4(t)=-\int\frac{16b_3^2h(t)+g(t)}{(1+16b_3^2)f(t)}dt,$$
$$b_5=\epsilon_2\sqrt{-12b_3^2-3},b_6(t)=-\int\frac{(128b_3^4+48b_3^2+3)h(t)-(8b_3^2+1)g(t)}{2(1+16b_3^2)(1+4b_3^2)f(t)}dt,$$
$$m_1=m_2,m_2=m_2,m_3=m_3,m_4=0,\epsilon_1=\epsilon_2=\pm1.$$

For this case, carrying the results into (13) gives

$$v = m_2\cos^2\eta_1 + m_2\sinh^2\eta_2 + m_3e^{-\eta_3},$$

where

$$\begin{aligned}
\eta_1 &= \epsilon_1 i b_3\left(x - \int\frac{16b_3^2h(t)+g(t)}{(1+16b_3^2)f(t)}dt\right),\\
\eta_2 &= b_3\left(x - \int\frac{16b_3^2h(t)+g(t)}{(1+16b_3^2)f(t)}dt\right),\\
\eta_3 &= \epsilon_2\sqrt{-12b_3^2-3}\left(x - \int\frac{(128b_3^4+48b_3^2+3)h(t)-(8b_3^2+1)g(t)}{2(1+16b_3^2)(1+4b_3^2)f(t)}dt\right).
\end{aligned} \tag{17}$$

Thus, the expression of solution to equation (1) is written as

$$u = \frac{12\epsilon_1 b_3 m_2\cosh\tilde{\eta}_1\sinh\tilde{\eta}_1 + 12b_3m_2\cosh\eta_2\sinh\eta_2 - 6\epsilon_2 m_3\sqrt{-12b_3^2-3}e^{-\eta_3}}{m_2\cosh^2\tilde{\eta}_1 + m_2\sinh^2\eta_2 + m_3e^{-\eta_3}},$$

where $\tilde{\eta}_1 = \epsilon_1 b_3\left(x - \int\frac{16b_3^2h(t)+g(t)}{(1+16b_3^2)f(t)}dt\right)$, and η_2, η_3 are expressed by (17).

Case 4.

$$b_1 = \epsilon i b_3, b_2(t) = -\int\frac{16b_3^2h(t)+g(t)}{(16b_3^2+1)f(t)}dt, b_4(t) = -\int\frac{16b_3^2h(t)+g(t)}{(16b_3^2+1)f(t)}dt, b_5 = 2b_3,$$
$$b_6(t) = -\int\frac{16b_3^2h(t)+g(t)}{(16b_3^2+1)f(t)}dt, m_1 = m_2, m_2 = m_2, m_3 = m_3, m_4 = m_4, \epsilon = \pm 1.$$

Now, substituting the results in this case into (13) can obtain

$$v = m_2\cos^2\eta_1 + m_2\sinh^2\eta_2 + m_3e^{-\eta_3} + m_4e^{\eta_3},$$

where

$$\begin{aligned}
\eta_1 &= \epsilon i b_3\left(x - \int\frac{16b_3^2h(t)+g(t)}{(16b_3^2+1)f(t)}dt\right),\\
\eta_2 &= b_3\left(x - \int\frac{16b_3^2h(t)+g(t)}{(16b_3^2+1)f(t)}dt\right),\\
\eta_3 &= 2b_3\left(x - \int\frac{16b_3^2h(t)+g(t)}{(16b_3^2+1)f(t)}dt\right).
\end{aligned} \tag{18}$$

As a result, the solitary wave solution to equation (1) is represented as

$$u = \frac{12\epsilon b_3 m_2\cosh\tilde{\eta}_1\sinh\tilde{\eta}_1 + 12b_3m_2\cosh\eta_2\sinh\eta_2 - 12b_3m_3e^{-\eta_3} + 12b_3m_4e^{\eta_3}}{m_2\cosh^2\tilde{\eta}_1 + m_2\sinh^2\eta_2 + m_3e^{-\eta_3} + m_4e^{\eta_3}}$$

with $\tilde{\eta}_1 = \epsilon b_3\left(x - \int\frac{16b_3^2h(t)+g(t)}{(16b_3^2+1)f(t)}dt\right)$, and η_2, η_3 being determined by (18).

3. CONCLUDING REMARKS

Some figures are made first in this section to exhibit the localized structures of some solutions using Maple plot tool. We begin with solution (6), where the function and parameters involved are set as $h(t) = 1, a_5 = \frac{1}{5}, k_3 = 1, k_4 = 1$. Figure 1 presents two types of kink solitary waves, namely, periodic and kink types, due to the choices of $f(t) = 1, g(t) = \sin\frac{t}{2}$ and $f(t) = \cosh\frac{t}{2}, g(t) = 1$. Concerning solution (9), if we take $a_3 = 1, a_5 = \frac{1}{2}, k_2 = 1, k_3 = 1, f(t) = 1, h(t) = 1$, then different choices of $g(t)$ can generate different types of double solitary waves. Figure 2 displays parabolic two kink solitary waves of periodic and kink types, respectively. From the expression of solution (12), we can see that it includes both trigonometric and hyperbolic functions. Therefore, in the three-dimensional figure, the waves take on structural characteristics of both periodicity and kink, as shown in Fig. 3. Compared to the right panel, the periodicity of the wave in the left panel is along the t-axis because of $g(t) = 0$. If we make other selections for the functions and parameters in the solution, then spatial structure of the wave will be changed accordingly.

To sum up, an integrable time-dependent shallow water wave equation in (1+1)-dimensions was considered by application of the constructive methods. Consequently, abundant solutions including the (double) solitary wave, periodic wave and periodic solitary wave solutions with some free functions and parameters were written out. And taking the functions and parameters involved properly, a few three-dimensional figures were drawn to illustrate that these solutions possess rich physical structures.

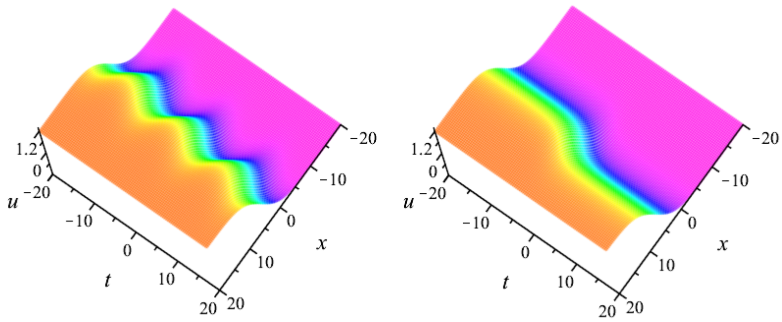

Fig. 1 – Profile of solution (6) with $a_5 = \frac{1}{5}, k_3 = 1, k_4 = 1, h(t) = 1$; left panel: $f(t) = 1, g(t) = \sin\frac{t}{2}$; right panel: $f(t) = \cosh\frac{t}{2}, g(t) = 1$.

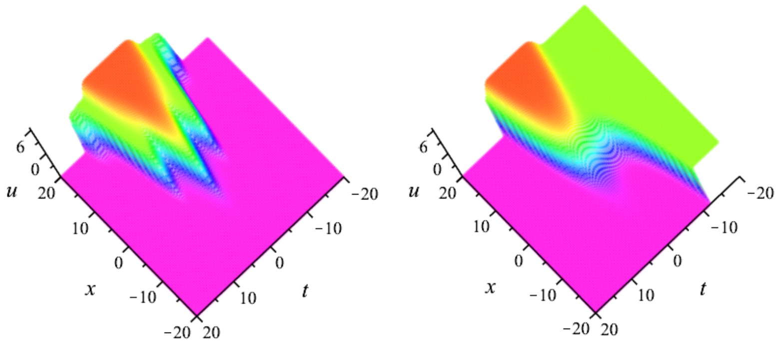

Fig. 2 – Profile of solution (9) with $a_3 = 1, a_5 = \frac{1}{2}, k_2 = 1, k_3 = 1, f(t) = 1, g(t) = t$; left panel: $h(t) = 5\sin t$; right panel: $h(t) = \frac{t^2}{10}$.

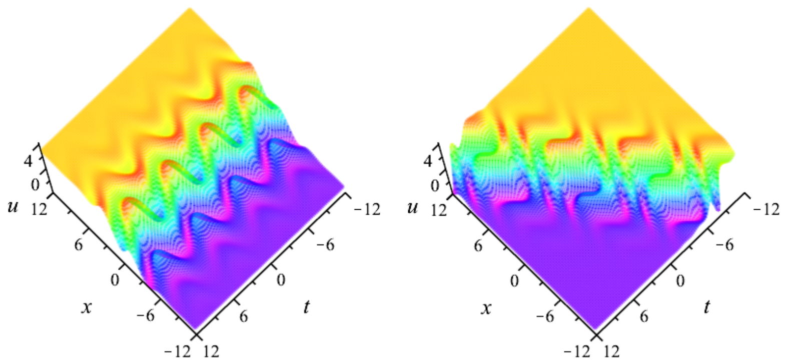

Fig. 3 – Profile of solution (12) with $a_1 = \sqrt{2}, a_5 = \frac{1}{2}, k_1 = 1, k_3 = 1, f(t) = 1, h(t) = \sin t$; left panel: $g(t) = 0$; right panel: $g(t) = 1$.

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