SOLVING ONE-DIMENSIONAL BRATU'S PROBLEM *VIA* KASHURI FUNDO DECOMPOSITION METHOD

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Received January 13, 2023

Abstract. The nonlinear Bratu's boundary value problem arises in a large variety of application areas such as solid fuel ignition model of thermal combustion, radiative heat transfer, thermal reaction, electrospinning process for the manufacturing of nanofibers, the Chandrasekhar model of the expansion of the universe, chemical reactor theory and nanotechnology. In this study, our aim is to solve nonlinear Bratu's problem *via* Kashuri Fundo decomposition method which is a hybrid form of the Kashuri Fundo transform method and the Adomian decomposition method.

Key words: Bratu's boundary value problem, Kashuri Fundo transform, Kashuri Fundo decomposition method.

1. INTRODUCTION

Nonlinear differential equations are of fundamental importance in science and engineering. Scientists working in these fields mostly study change. Often nonlinear differential equations also provide scientists with great benefits in observing and interpreting this change [1]. Some of these equations are difficult to solve analytically. For this reason, new methods are being researched and new studies are carried out in order to find better and newer numerical solutions to nonlinear equations.

Bratu's boundary value problem, which is an example of nonlinear differential equations arises in a large variety of application areas such as solid fuel ignition model of thermal combustion, radiative heat transfer, thermal reaction, electrospinning process for the manufacturing of nanofibers, the Chandrasekhar model of the expansion of the universe, chemical reactor theory and nanotechnology. The classical Bratu problem, which is an elliptical nonlinear differential equation with homogeneous Dirichlet boundary conditions at the boundary, is expressed as [2–5]

$$u''(x) + \lambda \mathrm{e}^{u(x)} = 0, \quad 0 \leq x \leq 1 \tag{1}$$

where $\lambda > 0$ is a constant parameter and the boundary conditions are

$$u(0) = 0, \quad u(1) = 0. \tag{2}$$

The exact solution to (1) is given by [6, 7]

$$u(x) = -2\ln\left(\frac{\cosh\left((x-\frac{1}{2})\frac{\theta}{2}\right)}{\cosh\left(\frac{\theta}{4}\right)}\right)$$

where θ is

$$\theta = \sqrt{2\lambda}\cosh\left(\frac{\theta}{4}\right).$$

Three cases can be mentioned for λ:

1. If $\lambda > \lambda_c$, then the Bratu problem has zero solution.

2. If $\lambda = \lambda_c$, then the Bratu problem has one solution.

3. If $\lambda < \lambda_c$, then the Bratu problem has two solutions,

where λ_c is the critical value and satisfies equation,

$$1 = \frac{1}{4}\sqrt{2\lambda_c}\sinh\left(\frac{\theta_c}{4}\right)$$

where

$$\lambda_c = 3.513830719.$$

Due to its simplicity, the Bratu's equation is used as a benchmarking tool for various numerical methods such as finite difference method [8], finite element approach [9], shooting method [10], variational iteration method [11–13], differential transformation method [14], homotopy analysis method [15] and various other numerical methods in the literature [16, 17].

In the literature, it is seen that methods such as Adomian decomposition method (ADM) [18, 19], one-dimensional differential transform method [20], Laplace Adomian decomposition method [21] and Laplace transform decomposition numerical method [22] are applied to find numerical solutions to the Bratu problem. In this study, we are looking for a numerical solution to the one-dimensional Bratu's problem by using the Kashuri Fundo decomposition method [23, 24], which is created by blending Kashuri Fundo transformation [25], which is one of the integral transformations, and Adomian decomposition method [26, 27].

Kashuri Fundo transform is a convenient, effective, easy-to-use and reliable method as it allows to reach the solution without going through complex calculations. For this reason, it has been the subject of many studies. There exist many studies using both Kashuri Fundo transform and its hybrid forms [28–40].

2. KASHURI FUNDO TRANSFORM

2.1. DEFINITION OF KASHURI FUNDO TRANSFORM

Definition 2.1.1 We consider functions in the set F defined as [25]

$$F = \{f(t)|\exists M, k_1, k_2 > 0 \quad \text{such that} \quad |f(t)| \leqslant M e^{\frac{|t|}{k_i^2}}, \quad \text{if} \quad t \in (-1)^i \times [0, \infty)\} \tag{3}$$

For a function belonging to the set F, the constant M must be finite number. k_1 and k_2 may be finite or infinite.
Definition 2.1.2 Kashuri Fundo transform denoted by the operator $\mathscr{K}(.)$ is defined as [25]

$$\mathscr{K}[f(t)](v) = A(v) = \frac{1}{v}\int_0^\infty e^{\frac{-t}{v^2}} f(t)\mathrm{d}t, \quad t \geqslant 0, \quad -k_1 < v < k_2. \tag{4}$$

The Kashuri Fundo transform expressed by the equation (4) can also be expressed as

$$\mathscr{K}[f(t)](v) = A(v) = v\int_0^\infty e^{-t} f(v^2 t)\,\mathrm{d}t.$$

Inverse Kashuri Fundo transform is denoted by $\mathscr{K}^{-1}[A(v)] = f(t), t \geq 0$.
Definition 2.1.3 A function $f(t)$ is said to be of exponential order $\frac{1}{k^2}$, if there exist positive constants T and M such that $|f(t)| \leq M e^{\frac{-t}{k^2}}$, for all $t \geq T$ [25].
Theorem 2.1.1 (Sufficient Conditions for Existence of Kashuri Fundo Transform) *If $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order $\frac{1}{k^2}$, then $\mathscr{K}[f(t)](v)$ exists for $|v| < k$* [25].

2.2. SOME PROPERTIES OF KASHURI FUNDO TRANSFORM

Theorem 2.2.1 (Linearity Property). *Let $f(t)$ and $g(t)$ be functions whose Kashuri Fundo integral transforms exist and c be a constant. Then [25],*

$$\mathscr{K}[(f+g)(t)](v) = \mathscr{K}[f(t)](v) + \mathscr{K}[g(t)](v)$$

$$\mathscr{K}[(cf)(t)](v) = c\mathscr{K}[f(t)](v)$$

Theorem 2.2.2 (Kashuri Fundo Transform of The Derivatives of The Function $f(t)$). *Let $A(v)$ be a Kashuri Fundo transform of $f(t)$. Then [25],*

$$\mathscr{K}[f'(t)](v) = \frac{A(v)}{v^2} - \frac{f(0)}{v} \tag{5}$$

$$\mathscr{K}[f''(t)](v) = \frac{A(v)}{v^4} - \frac{f(0)}{v^3} - \frac{f'(0)}{v} \tag{6}$$

$$\mathscr{K}[f^{(n)}(t)](v) = \frac{A(v)}{v^{2n}} - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{v^{2(n-k)-1}} \tag{7}$$

2.3. KASHURI FUNDO TRANSFORM OF SOME SPECIAL FUNCTIONS

Table 1

This table shows the Kashuri Fundo transform of some special functions [23–25]

$f(t)$	$\mathscr{K}[f(t)] = A(v)$
1	v
t	v^3
e^{ct}	$\frac{v}{1-cv^2}$
$sin(cx)$	$\frac{cv^3}{1+c^2v^4}$
$cos(cx)$	$\frac{v}{1+c^2v^4}$
$sinh(cx)$	$\frac{cv^3}{1-c^2v^4}$
$cosh(cx)$	$\frac{v}{1-c^2v^4}$
$t^n, \quad n \in \mathbb{R}^+$	$\Gamma(1+n)v^{2n+1}$
$\sum_{k=0}^{n} c_k t^k$	$\sum_{k=0}^{n} k! c_k v^{2k+1}$

3. KASHURI FUNDO DECOMPOSITION METHOD

Consider a nonlinear differential equation written in a general operator form

$$Ly(t) + Ry(t) + Ny(t) = g(t) \tag{8}$$

with initial conditions

$$y(0) = y_0, \quad y'(0) = k, \quad (k \in \mathbb{R})$$

where L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of less order than L, N is the nonlinear operator and g is the source term. y is a function dependent on the variable t.

Kashuri Fundo decomposition method is as follows [23, 24]:

Applying Kashuri Fundo transform to the equation (8) and using the equation (6), we have

$$A(v) = vy(0) + v^3y'(0) + v^4\mathscr{K}[g(t)] - v^4\mathscr{K}[Ny(t)] - v^4\mathscr{K}[Ry(t)]. \tag{9}$$

Applying the inverse of Kashuri Fundo transform to the equation (9) and using the initial conditions,

$$y(t) = y_0 + kt + \mathscr{K}^{-1}\left[v^4\mathscr{K}[g(t)]\right] - \mathscr{K}^{-1}\left[v^4\mathscr{K}[Ny(t)]\right] - \mathscr{K}^{-1}\left[v^4\mathscr{K}[Ry(t)]\right] \tag{10}$$

is obtained.

The Adomian decomposition method assumes that the function y can be expressed as an infinite series.

$$y(t)=\sum_{n=0}^{\infty}y_n(t)=y_0+y_1+y_2+y_3+\dots \tag{11}$$

where y_n can be determined iteratively. The Adomian decomposition method also supports the decomposition of nonlinear operator Ny into an infinite series of polynomial form

$$Ny=\sum_{n=0}^{\infty}A_n \tag{12}$$

where $A_n=A_n(y_0,y_1,y_2,y_3,\dots,y_n)$ is Adomian polynomials defined as

$$A_n(y_0,y_1,y_2,y_3,\dots,y_n)=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[N\left(\sum_{k=0}^{n}\lambda^k y_k\right)\right]_{\lambda=0}, \quad n=0,1,2,\dots$$

where λ is a parameter. The Adomian polynomials A_n can be defined more explicitly as

$$\begin{aligned}
A_0&=\frac{1}{0!}\frac{\mathrm{d}^0}{\mathrm{d}\lambda^0}\left[N\left(\sum_{k=0}^{0}\lambda^k y_k\right)\right]_{\lambda=0}=N(y_0)\\
A_1&=\frac{1}{1!}\frac{\mathrm{d}^1}{\mathrm{d}\lambda^1}\left[N\left(\sum_{k=0}^{1}\lambda^k y_k\right)\right]_{\lambda=0}=y_1N'(y_0)\\
A_2&=\frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left[N\left(\sum_{k=0}^{2}\lambda^k y_k\right)\right]_{\lambda=0}=y_2N'(y_0)+\frac{y_1^2}{2!}N''(y_0)\\
&\vdots
\end{aligned}$$

If the equations (11) and (12) are substituted in the equation (10), we find

$$\begin{aligned}
\sum_{n=0}^{\infty}y_n(t)=y_0+kt+\mathscr{K}^{-1}\left[v^4\mathscr{K}[g(t)]\right]-\mathscr{K}^{-1}\left[v^4\mathscr{K}\left[\sum_{n=0}^{\infty}A_n\right]\right]\\
-\mathscr{K}^{-1}\left[v^4\mathscr{K}\left[R\sum_{n=0}^{\infty}y_n(t)\right]\right]
\end{aligned} \tag{13}$$

Describing both sides of (13) will successively produce

$$\begin{aligned}
y_0(t) &= y_0 + kt + \mathscr{K}^{-1}\left[v^4\mathscr{K}[g(t)]\right] \\
y_1(t) &= -\mathscr{K}^{-1}\left[v^4\mathscr{K}[A_0]\right] - \mathscr{K}^{-1}\left[v^4\mathscr{K}[Ry_0(t)]\right] \\
y_2(t) &= -\mathscr{K}^{-1}\left[v^4\mathscr{K}[A_1]\right] - \mathscr{K}^{-1}\left[v^4\mathscr{K}[Ry_1(t)]\right] \\
y_3(t) &= -\mathscr{K}^{-1}\left[v^4\mathscr{K}[A_2]\right] - \mathscr{K}^{-1}\left[v^4\mathscr{K}[Ry_2(t)]\right] \\
&\vdots
\end{aligned}$$

Thus, the solution of the equation (8) is obtained recursively by using the Kashuri Fundo decomposition method as follows

$$\begin{aligned}
y_0(t) &= y_0 + kt + \mathscr{K}^{-1}\left[v^4\mathscr{K}[g(t)]\right] \\
y_{n+1}(t) &= -\mathscr{K}^{-1}\left[v^4\mathscr{K}[A_n]\right] - \mathscr{K}^{-1}\left[v^4\mathscr{K}[Ry_n(t)]\right].
\end{aligned}$$

As a result, the general expression of the approximate solution is expressed as

$$y(t) \approx \sum_{n=0}^{k} y_n(t) \quad \text{where} \quad \lim_{k\to\infty}\sum_{n=0}^{k} y_n(t) = y(t).$$

4. APPLICATION TO THE BRATU'S PROBLEM

Applying the Kashuri Fundo transform to the equation (1), we get

$$\mathscr{K}[u''(x)] + \lambda\mathscr{K}[\mathrm{e}^{u(x)}] = 0 \tag{14}$$

If we rearrange the equation (14) according to the equation (6) and first boundary condition, we obtain

$$\frac{\mathscr{K}[u(x)]}{v^4} - \frac{u'(0)}{v} + \lambda\mathscr{K}[\mathrm{e}^{u(x)}] = 0 \tag{15}$$

Assuming $u'(0) = k$, (k is a scalar) here and rearranging the equation (15), we get

$$\mathscr{K}[u(x)] = kv^3 - \lambda v^4\mathscr{K}[\mathrm{e}^{u(x)}]. \tag{16}$$

Our aim is to represent the solution as an infinite series using the Kashuri Fundo decomposition method, *i.e.*

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \quad \text{and} \quad f(u) = \mathrm{e}^{u(x)} = \sum_{n=0}^{\infty} A_n \tag{17}$$

where $u_n(x)$'s are to be recursively computed and A_n's are decomposed of nonlinear operator.

The first few A_n terms are given as

$$\begin{aligned}
A_0 &= f(u_0) = \mathrm{e}^{u_0} \\
A_1 &= u_1 f'(u_0) = u_1 \,\mathrm{e}^{u_0} \\
A_2 &= u_2 f'(u_0) + \frac{u_1^2}{2!} f''(u_0) = u_2 \,\mathrm{e}^{u_0} + \frac{u_1^2}{2} \,\mathrm{e}^{u_0} \\
A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f^{(3)}(u_0) = u_3 \,\mathrm{e}^{u_0} + u_1 u_2 \,\mathrm{e}^{u_0} + \frac{u_1^3}{6} \,\mathrm{e}^{u_0}.
\end{aligned} \tag{18}$$

$$\vdots$$

Substituting the equation (17) into the equation (16) yields

$$\mathscr{K}\left[\sum_{n=0}^{\infty} u_n(x)\right] = kv^3 - \lambda v^4 \mathscr{K}\left[\sum_{n=0}^{\infty} A_n\right]. \tag{19}$$

Using the linearity property of the Kashuri Fundo transform, we obtain

$$\sum_{n=0}^{\infty} \mathscr{K}[u_n(x)] = kv^3 - \lambda v^4 \sum_{n=0}^{\infty} \mathscr{K}[A_n]. \tag{20}$$

Arranging the equation (20) according to the values of $n = 0, 1, 2, \ldots$, we get the following algorithm,

$$\mathscr{K}[u_0] = kv^3 \tag{21}$$

$$\mathscr{K}[u_1] = -\lambda v^4 \mathscr{K}[A_0] \tag{22}$$

$$\mathscr{K}[u_2] = -\lambda v^4 \mathscr{K}[A_1] \tag{23}$$

$$\vdots$$

$$\mathscr{K}[u_{n+1}] = -\lambda v^4 \mathscr{K}[A_n]. \tag{24}$$

Applying the inverse Kashuri Fundo transform to the equation (21), we obtain

$$u_0 = kx. \tag{25}$$

Substituting this value of u_0 and that of A_0 given in the equation (18) into the equation (22) gives the following result

$$\mathscr{K}[u_1] = -\lambda \frac{v^5}{1 - kv^2} = \lambda \left(\frac{v^3}{k} + \frac{v}{k^2} - \frac{v}{k^2(1 - kv^2)} \right). \tag{26}$$

Applying the inverse Kashuri Fundo transform to the equation (26), we obtain

$$u_1 = \lambda \left(\frac{x}{k} + \frac{1}{k^2} - \frac{1}{k^2} \,\mathrm{e}^{kx} \right). \tag{27}$$

Substituting the values of u_0, u_1 and A_0 given in the equations (25), (27) and (18) respectively into the equation (23), we obtain

$$\mathscr{K}[u_2] = -\lambda^2\left(\frac{v^7}{k(1-kv^2)^2} + \frac{v^5}{k^2(1-kv^2)} - \frac{v^5}{k^2(1-2kv^2)}\right). \tag{28}$$

Applying the inverse Kashuri Fundo transform to the equation (28), we get

$$u_2 = -\lambda^2\left(\frac{x}{2k^3} + \frac{5}{4k^4} - \frac{\mathrm{e}^{kx}}{k^4} + \frac{x\,\mathrm{e}^{kx}}{k^3} - \frac{\mathrm{e}^{2kx}}{4k^4}\right). \tag{29}$$

Using the steps for obtaining u_1 and u_2, u_3 is also obtained as

$$u_3 = 2\lambda^3\left(-\frac{\mathrm{e}^{3kx}}{24k^6} - \frac{\mathrm{e}^{2kx}}{4k^6} + \frac{x\,\mathrm{e}^{2kx}}{4k^5} - \frac{5\,\mathrm{e}^{kx}}{8k^6} + \frac{3x\,\mathrm{e}^{kx}}{4k^5} - \frac{x^2\,\mathrm{e}^{kx}}{4k^4} + \frac{11}{12k^6} + \frac{x}{4k^5}\right). \tag{30}$$

The terms $u_4, u_5, \dots$ can be obtained in a similar way. From these founded terms, we express the approximate solution as

$$U_n = \sum_{n=0}^{\infty} u_n(x) = kx + \lambda\left(\frac{x}{k} + \frac{1}{k^2} - \frac{1}{k^2}\mathrm{e}^{kx}\right) - \lambda^2\left(\frac{x}{2k^3} + \frac{5}{4k^4} - \frac{\mathrm{e}^{kx}}{k^4} + \frac{x\,\mathrm{e}^{kx}}{k^3} - \frac{\mathrm{e}^{2kx}}{4k^4}\right) + \dots \tag{31}$$

This result is fully consistent with the results obtained by other methods in the literature [12, 19, 22].

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