NEW ANALYTICAL SOLUTIONS FOR THE REACTION-CONVECTION-DIFFUSION EQUATION AGAINST ITS NUMERICAL SOLUTIONS

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Abstract. We will achieve new configuration of the analytical solutions to the reaction-convection-diffusion equation (RCDE) which possess a significant role in mathematical physics applications and biological science. These new configurations of the analytical solution have been documented *via* the Paul-Painlevé approach method (PPAM) and the (G'/G)-expansion method. These two techniques are used for the first time to achieve these new configurations of the analytical solution of this equation. In addition, the numerical solutions analogous to all achieved analytical solutions have been detected *via* the variational iteration method (VIM) that usually gives good numerical solutions.

Key words: reaction-convection-diffusion equation, Paul-Painlevé approach method, traveling wave solutions, numerical solutions.

1. INTRODUCTION

In our current article we will apply two distinct techniques to construct new configurations of the exact solution behavior to the Reaction-convection-diffusion equation. The first one is the Paul-Painlevé approach PPAM [1–5] which is one of the important approaches that discovered latterly and achieved surprise results in various branches of nonlinear science because it possesses high quality results. While the second one is the (G'/G)-expansion method [6–8] that continuously achieves good results for the exact solutions to any nonlinear partial differential equation problem. The main target of this article focused on how we can performing comparison presentation between the exact solutions achieved by this approach and the numerical solutions realized using the variational iteration method VIM [9–11] to the Reaction-convection-diffusion equations system [12–15]. This equation has many significant applications in various branches of nonlinear sciences subtended to the influence of four principal bases namely:

(1) The local reaction that develop the focuses population
(2) The diffusion term implies samples distribution in space
(3) The convection drifting under the influence of external forces.

The suggested system can be reduced to the Reaction-convection-diffusion equation RCDE which is second order NLPDE widely involved natural phenomena include the change of focused population of one or more substances species distributed in space. In literature the suggested model can be written as

$$v_t = (\lambda + \lambda_0 v)v_{xx} + \lambda_1 v v_x - \lambda_2 v - \lambda_3 v^2, \tag{1}$$

where v denotes to the studied species concentration, while λ, λ_i, = 1, 2, 3 are random parameters. This equation is the main original source of a lot of equations particularly if λ = 1, λ_0 = 0 equation (1) will reduced to the well-known Murray equation which used impressively in many branches of physics and biology in addition to it is a general form to the well-known Fisher equation, furthermore when $\lambda_1 = \lambda_2 = 0$ it reduced to the classical Burgers equation. Many authors proposed the solutions to this equation with its different models through different methods [16–26]. When Eq. (1) is surrenders to the transformation $v(x,t) = \varphi(\zeta)$, $\zeta = C_1(x + C_2 t)$ it will be converted to

$$C_1^2 \varphi'' + \lambda_1 C_1 \varphi \varphi' - \lambda_2 \varphi - \lambda_3 \varphi^2 - C_1 C_2 \varphi' = 0, \tag{2}$$

where C_1, C_2 are arbitrary constants.

2. THE PAUL-PAINLEVÉ APPROACH SCHEMA

To discuss this technique we firstly investigate the formalism of any NLPDE

$$\Upsilon(\varphi, \varphi_x, \varphi_t, \varphi_{xx}, \varphi_{tt}, ...) = 0, \tag{3}$$

where Υ includes $\varphi(x,t)$, its consecutive partial derivatives and the nonlinear terms. When Eq. (3) is surenders to $\varphi(x,t) = \varphi(\zeta), \zeta = x - C_0 t$ it will be reduced to the following ODE:

$$\mathrm{T}(\varphi', \varphi'', \varphi''', ...) = 0, \tag{4}$$

where T in terms of $\varphi(\zeta)$ and its total derivatives.

The exact solution of Eq. (4) in the framework of the PPAM [1] is

$$\varphi(\zeta) = A_0 + \sum_{i=1}^{m} A_i \, e^{-iN\zeta} R^i(X), X = W(\zeta), \tag{5}$$

where $X = W(\zeta) = d_1 - \frac{e^{-N\zeta}}{N}$, $R(X)$ in Eq. (5) achieves the Riccati-equation in the form $R_X - AR^2 = 0$ which has solution this solution

$$R(X) = \frac{1}{AX + X_0}. \tag{6}$$

Via applying the PPAM to extracting the traveling wave solution of the RCDE equation (2) mention above

$$C_1^2\varphi'' - \lambda_1 C_1 \varphi\varphi' + \lambda_2\varphi - \lambda_3\varphi^2 - C_2 C_1 \varphi' = 0 .$$

When the homogeneous balance implemented between $\varphi'', \varphi\varphi'$ it implies $m = 1$ consequently, the solution in the framework of PPAM is

$$\varphi(\zeta) = A_0 + A_1 R(X) e^{-N\zeta} \tag{7}$$

$$\varphi' = -NA_1 e^{-N\zeta} R(X) - AA_1 e^{-2N\zeta} R^2(X) \tag{8}$$

$$\varphi'' = N^2 A_1 e^{-N\zeta} R(X) + 3AA_1 N\, e^{-2N\zeta} R^2(X) + 2A_1 A^2\, e^{-3N\zeta} R^3(X) \tag{9}$$

$$\varphi\varphi' = -A_0 ANe^{-N\zeta} R(X) - (A_0 A_1 A + A_1^2 N) e^{-2N\zeta} R^2(X) - A_1^2 A e^{-3N\zeta} R^3(X) \tag{10}$$

$$\varphi^2 = A_0^2 + 2A_0 A_1\, e^{-N\zeta} R(X) + A_1^2 e^{-2N\zeta} R^2(X). \tag{11}$$

Via inserting $\varphi, \varphi^2, \varphi_\zeta, \varphi\varphi_\zeta, \varphi_{\zeta\zeta}$ into Eq. (2), implementing the equivalence between for the coefficients of various powers of $R^{iN}(\zeta) e^{-N\zeta}$ lead to the following system

$$2\, AC_1 + \lambda_1 A_1 = 0, \tag{12}$$

$$3ANC_1^2 + A_0 A\lambda_1 C_1 + NA_1\lambda_1 C_1 - \lambda_3 A_1 + C_1 C_2 A = 0, \tag{13}$$

$$A_1 C_1^2 N^2 + \lambda_1 C_1 A_0 AN + \lambda_2 A_1 - 2\lambda_3 A_0 A_1 + C_1 C_2 AA_1 = 0, \tag{14}$$

$$\lambda_2 A_0 - \lambda_3 A_0^2 = 0. \tag{15}$$

When we solve the above system we obtain only one acceptable solution and the remaining are refused, hence we have unique solution which is,

$$A_0 = \frac{-\lambda_2}{\lambda_3},\ A_1 = \frac{2C_1C_2}{C_2\lambda_1 - 2\lambda_3},\ A = \frac{C_2\lambda_1}{C_2\lambda_1 - 2\lambda_3},\ N = \frac{-C_2\lambda_1 + 2\lambda_3}{C_1\lambda_1}. \tag{16}$$

This result can be simplified to be $A_0 = -1,\ A_1 = -2,\ A = -1,\ N = -1,$ $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and the solution in the framework of the suggested manner is, $\varphi(\zeta) = A_0 + A_1 e^{-N\zeta} R(X)$. *Via* choosing $X_0 = d_1 = 1$ and inserting $R(X)$ this solution become,

$$\varphi(\zeta) = A_0 + A_1 \frac{e^{-N\zeta}}{A(1 - \frac{e^{-N\zeta}}{N}) + 1}. \tag{17}$$

Substituting about the obtained result we get

$$\varphi(\zeta) = -1 + \frac{e^{\zeta}}{2 + e^{\zeta}} \tag{18}$$

$$\varphi(\zeta) = \frac{-2}{2 + e^{\zeta}} \tag{19}$$

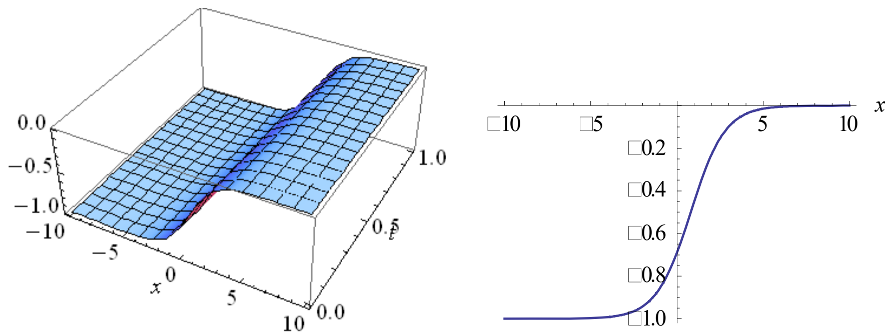

Fig. 1 – The draw of Eq.(19) in 2D and 3D with values:
$A_0 = 1$, $C_0 = C_1 = C_2 = \lambda_1 = \lambda_2 = \lambda_3 = X_0 = d_1 = 1$, $A_1 = 1$, $A = 1$, $N = 1$, for $2Dt = 0.1$.

3. THE (G'/G)-EXPANSION SCHEMA

The exact solution of Eq. (4) in the framework of this technique is

$$\varphi(\zeta) = A_0 + \sum_{k=1}^{m} A_k \left[\frac{G'}{G}\right]^k, A_m \neq 0, \tag{20}$$

where $G(\zeta)$ that appears in Eq. (6) achieves the differential equation $G'' + \mu G' + \lambda G = 0$ that generates the following forms of solutions, while the integer $m = 1$ that is determined before.

Case 1. If $\mu^2 - 4\lambda \succ 0$, the solution is

$$\frac{G'}{G} = \frac{\sqrt{\mu^2 - 4\lambda}}{2}\left(\frac{l_1 \sinh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta + l_2 \cosh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}{l_1 \cosh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta + l_2 \sinh(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}\right) - \frac{\mu}{2}. \tag{21}$$

Case 2. If $\mu^2 - 4\lambda \prec 0$, the solution is

$$\frac{G'}{G} = \frac{\sqrt{\mu^2 - 4\lambda}}{2}\left(\frac{-l_1 \sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta + l_2 \cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}{l_1 \cos(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta + l_2 \sin(\frac{\sqrt{\mu^2-4\lambda}}{2})\zeta}\right) - \frac{\mu}{2}. \tag{22}$$

Case 3. If $\mu^2 - 4\lambda = 0$, the solution is

$$\frac{G'}{G} = (\frac{l_2}{l_1 + l_2\zeta}) - \frac{\mu}{2}, \tag{23}$$

where l_1, l_2 are constants.

By applying this schema for Eq. (2) the solution is

$$\varphi(\zeta) = A_0 + A_1\left(\frac{G'}{G}\right) \tag{24}$$

Hence

$$\varphi' = -A_1\left(\lambda\left(\frac{G'}{G}\right) + \mu\left(\frac{G'}{G}\right)^2\right), \tag{25}$$

$$\varphi'' = -A_1(\lambda^2 + 2\mu)\left(\frac{G'}{G}\right) - 3A_1\lambda\left(\frac{G'}{G}\right)^2 - 2A_1\left(\frac{G'}{G}\right)^3 - A_1\lambda\mu. \tag{26}$$

By inserting the relations (24–26) into equation (2), equating the coefficients of different powers of $\left(\frac{G'}{G}\right)^i$ to zero we get system of equation whose solution is

$$\begin{aligned}
(1)\quad & A_1 = \frac{-2C_1}{\lambda_1},\ A_0 = \frac{-2\lambda\lambda_1 C_1 + C_2\lambda_2 - \lambda_1\sqrt{C_2^2 - 4\lambda_2}}{2\lambda_1^2}, \\
& \lambda_3 = \frac{1}{4}\left(-C_2\lambda_1 - \lambda_1\sqrt{C_2^2 - 4\lambda_2}\right), \\
& \mu = \frac{2\lambda^2 C_1^2 - C_2^2 + 2\lambda_2 + C_2\sqrt{C_2^2 - 4\lambda_2}}{8C_1^2}.
\end{aligned} \tag{27}$$

$$\begin{aligned}
(2)\quad & A_1 = \frac{-2C_1}{\lambda_1},\ A_0 = \frac{-2\lambda\lambda_1 C_1 + C_2\lambda_2 + \lambda_1\sqrt{C_2^2 - 4\lambda_2}}{2\lambda_1^2}, \\
& \lambda_3 = \frac{1}{4}\left(-C_2\lambda_1 + \lambda_1\sqrt{C_2^2 - 4\lambda_2}\right), \\
& \mu = \frac{2\lambda^2 C_1^2 - C_2^2 + 2\lambda_2 - C_2\sqrt{C_2^2 - 4\lambda_2}}{8C_1^2}.
\end{aligned} \tag{28}$$

For similarity and simplicity, we will construct the identical solution for only one of these two results say the first which is

$$A_1 = \frac{-2C_1}{\lambda_1},\ A_0 = \frac{-2\lambda\lambda_1 C_1 + C_2\lambda_2 - \lambda_1\sqrt{C_2^2 - 4\lambda_2}}{2\lambda_1^2},$$

$$\lambda_3 = \frac{1}{4}\left(-C_2\lambda_1 - \lambda_1\sqrt{C_2^2 - 4\lambda_2}\right), \quad \mu = \frac{2\lambda^2 C_1^2 - C_2^2 + 2\lambda_2 + C_2\sqrt{C_2^2 - 4\lambda_2}}{8C_1^2}.$$

This result can be simplified by choose definite values of the parameters to be

$$\begin{aligned} A_1 = -2, \quad A_0 = 1, \quad \lambda_1 = C_1 = C_2 = 1, \\ \lambda_2 = 0.125, \quad \lambda_3 = -0.4, \quad \lambda = -0.9, \quad \mu = 0.4. \end{aligned} \tag{29}$$

Thus the solution is $\varphi(\zeta) = A_0 + A_1\left(\frac{G'}{G}\right)$,

where

$$\begin{aligned} \frac{G'}{G} &= \left(\frac{\sinh\zeta + 2\cosh\zeta}{\cosh\zeta + 2\sinh\zeta}\right) - 02, \\ \varphi(\zeta) &= 1 - \left\{\left(\frac{2\sinh\zeta + 4\cosh\zeta}{\cosh\zeta + 2\sinh\zeta}\right) - 0.4\right\}. \end{aligned} \tag{30}$$

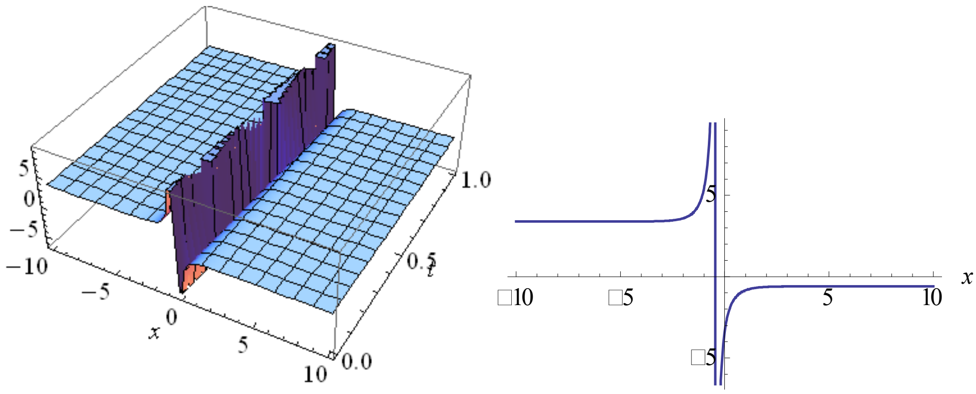

Fig. 2 – The draw of Eq.(30) in 2D and 3D with values: $A_0 = C_0 = C_1 = C_2 = \lambda_1 = X_0 = d_1 = 1$, $\lambda_2 = 0.125$, $\lambda_3 = -0.4$, $A_1 = -2$, $\lambda = -0.9$, $\mu = 0.4$, $l_1 = 1$, $l_2 = 2$, for $2Dt = 0.1$.

4. THE VARIATIONAL ITERATION METHOD

Considering the following differential equation

$$LS + NS = f(\zeta). \tag{31}$$

The above equation contains the inhomogeneous function $f(\zeta)$ and the linear and the nonlinear operators L, N.

The correction functional to equation (20) according to the VIM is

$$S_{m+1}(\zeta) = S_m(\zeta) + \int_0^{\zeta} \gamma(t)(LS_m(t) + N\tilde{S}_m(t) - g(t))\mathrm{d}t, \tag{32}$$

with restricted variation $\tilde{S}_m$ and its means $\delta\tilde{S}_m$, γ is a general Lagrange's multiplier which is crucial, critical in this technique, may be a constant or function and will be used to determination of the consecutive guesstimatations of the solution $S(\zeta)$. The zeros guesstimatation S_0 can be any choosy function. Moreover, the initial values $S(0), S'(0)$ are used to determine zeros guesstimatation S_0. Hence, the solution is $S(\zeta) = \lim_{\zeta\to\infty} S_m(\zeta)$.

The various forms to equation (31) in terms of Lagrange multipliers γ in the framework of the VIM can be admitted as follow:

For any first order differential equation in the form,

$$S' + q(\zeta)S = p(\zeta),\ S(0) = \rho. \tag{33}$$

The Lagrange multipliers $\gamma = -1$ and the improvement repetition formula is

$$S_{m+1}(\zeta) = S_m(\zeta) - \int_0^{\zeta} (S'_m(t) + q(t)S_m(t) - p(t))\mathrm{d}t. \tag{34}$$

For any second order differential equation in the form,

$$S''(\zeta) + cS'(\zeta) + dh(\zeta) = g(\zeta),\ \ S(0) = \rho, S'(0) = \eta. \tag{35}$$

The Lagrange multipliers $\lambda = t - x$, and the improvement repetition formula is

$$S_{m+1}(\zeta) = S_m(\zeta) + \int_0^{\zeta} (t-x)(S''_m(t) + cS'_m(t) + dS_m - g(t))\mathrm{d}t. \tag{36}$$

For any third order differential equation in the form

$$S'''(\zeta) + cS''(\zeta) + dS'(\zeta) + eS(\zeta) = g(\zeta),\ S(0) = \rho,\ S'(0) = \eta,\ S''(0) = \sigma. \tag{37}$$

The Lagrange multipliers $\lambda = -\frac{1}{2!}(t-x)^2$, and the improvement repetition formula is

$$S_{m+1}(\zeta) = S_m(\zeta) - \frac{1}{2!}\int_0^{\zeta}(t-x)^2(S_m'''(t) + cS_m''(t) + dS_m'(t) + eS_m - g(t))\mathrm{d}t\,. \tag{38}$$

Finally, for any general ODE of the form

$$\begin{gathered} S^{(m)} + f(S', S'', S'''..., S^{(m-1)}) = g(\zeta),\ S(0) = \rho_0, \\ S'(0) = \rho_1,\ S''(0) = \rho_2..., S^{m-1}(0) = \rho_{m-1}. \end{gathered} \tag{39}$$

The general form of Lagrange multipliers $\lambda = \frac{(-1)^m}{(m-1)!}(t-x)^{m-1}$, and the improvement repetition formula is

$$S_{m+1}(\zeta) = S_m(\zeta) + \frac{(-1)^m}{(m-1)!}\int_0^{\zeta}(t-x)^{m-1}\left(S^{(m)} + f(S', S'', S''', ..., S^{(m-1)}) - g(t)\right)\mathrm{d}t. \tag{40}$$

Furthermore, the zeros guesstimatation $S_0(\zeta)$ is

$$S_0(\zeta) = S_0(0) + S'(0)\zeta + \tfrac{1}{2!}S''(0)\zeta^2 + \tfrac{1}{3!}S'''(0)\zeta^3....... + \tfrac{1}{(m-1)!}S^{m-1}(0)\zeta^{m-1} \tag{41}$$

and m is the order of the ODE.

Now, we will implement the VIM to construct the numerical solutions identical to the travelling wave solutions achieved by PPAM and $\left(\frac{G'}{G}\right)$ techniques to equation (2) mentioned above:

(I) Firstly for the exact solution that achieved using the PPAM whish is $\varphi(\zeta) = \frac{-2}{2+e^{\zeta}}$. According to the initial condition, then

$$\varphi(0) = -0.7, \varphi'(0) = 0.2\,,\quad \varphi_0(\zeta) = \varphi(0) + \zeta\varphi'(0),\ \varphi_0(\zeta) = -0.7 + 0.2\zeta\,, \tag{42}$$

$$\begin{aligned} &\varphi_1(\zeta) = \varphi_0(\zeta) - \int_0^{\zeta}\left(\varphi_0'' - \varphi_0\varphi_0' + 2\varphi_0 - 0.5\varphi_0^2 - \varphi_0'\right)\mathrm{d}t, \\ &\varphi_1 = -0.7 + 0.2\zeta - \int_0^{\zeta}[-0.2(-0.7+0.2t) + 2(-0.7+0.2t) - \\ &\quad -0.5(-0.7+0.2t)^2 - 0.2]\mathrm{d}t, \\ &\varphi_1 = -0.7 + 1.9\zeta - 0.3\zeta^2 + 0.01\zeta^3. \end{aligned} \tag{43}$$

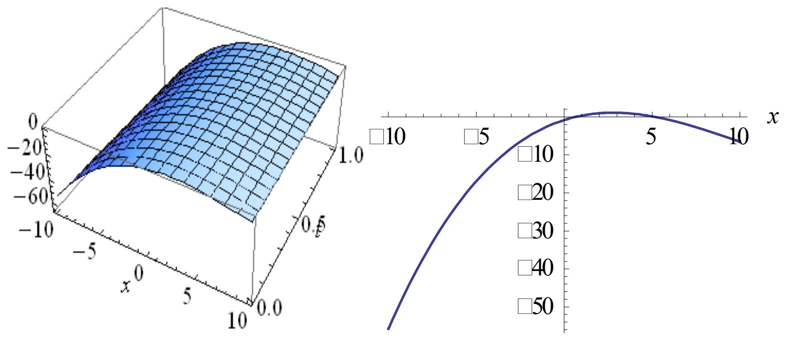

Fig. 3 – The draw of Eq.(43) in 2D and 3D with values:
$A_0 = -1$, $C_0 = C_1 = C_2 = \lambda_1 = \lambda_2 = \lambda_3 = X_0 = d_1 = 1$, $A_1 = -1$, $A = -1$, $N = -1$, for $2Dt = 0.1$.

(II) Secondly for the exact solution that achieved using the $(\frac{G'}{G})$ whish is $\varphi(\zeta) = 1 - \left\{ \left(\frac{2\sinh\zeta + 4\cosh\zeta}{\cosh\zeta + 2\sinh\zeta} \right) - 0.4 \right\}$.

According to the initial condition, then

$$\varphi(0) = -2.6,\ \varphi'(0) = -6,\ \ \varphi_0(\zeta) = \varphi(0) + \zeta\varphi'(0),\ \varphi_0(\zeta) = -2.6 - 6\zeta. \tag{44}$$

$$\varphi_1(\zeta) = \varphi_0(\zeta) - \int_0^{\zeta} \left(\varphi_0'' - \varphi_0\varphi_0' + 2\varphi_0 - 0.5\varphi_0^2 - \varphi_0' \right) \mathrm{d}t,$$

$$\varphi_1 = -2.6 - 6\zeta - \int_0^{\zeta} [6(-2.6 - 6t) + 2(-2.6 - 6t) - 0.5(-2.6 - 6t)^2 - 0.2]\mathrm{d}t, \tag{45}$$

$$\varphi_1 = -2.6 + 18.4\zeta + 31.8\zeta^2 + 6\zeta^3.$$

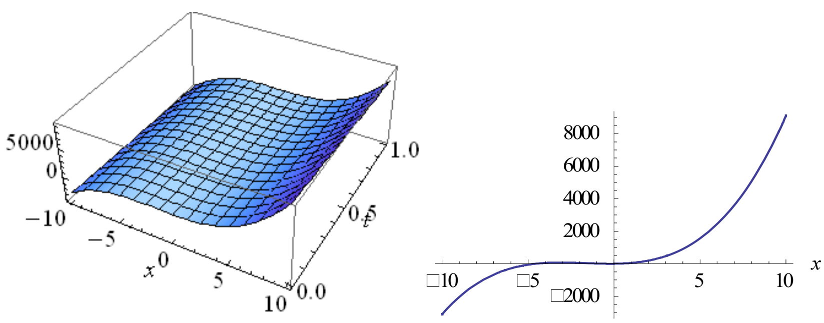

Fig. 4 – The draw of Eq.(45) in 2D and 3D with values: $A_0 = C_0 = C_1 = C_2 = \lambda_1 = X_0 = d_1 = 1$, $\lambda_2 = 0.125$, $\lambda_3 = -0.4$, $A_1 = -2$, $\lambda = -0.9$, $\mu = 0.4$, $l_1 = 1$, $l_2 = 2$, for $2Dt = 0.1$.

We can obtain the other iteration according to the next steps

$$\begin{aligned}
\varphi_2(\zeta) &= \varphi_1(\zeta) - \int_0^{\zeta} \left(\varphi_1'' - \varphi_1\varphi_1' + 2\varphi_1 - 0.5\varphi_1^2 - \varphi_1' \right) \mathrm{d}t, \\
\varphi_3(\zeta) &= \varphi_2(\zeta) - \int_0^{\zeta} \left(\varphi_2'' - \varphi_2\varphi_2' + 2\varphi_2 - 0.5\varphi_2^2 - \varphi_2' \right) \mathrm{d}t, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\varphi_{m+1}(\zeta) &= \varphi_m(\zeta) - \int_0^{\xi} \left(\varphi_m'' - \varphi_m\varphi_m' + 2\varphi_m - 0.5\varphi_m^2 - \varphi_m' \right) \mathrm{d}t,
\end{aligned} \tag{46}$$

Note that the exact solution can be achieved *via* $\varphi(\zeta) = \lim_{\zeta\to\infty} \varphi_m(\zeta)$.

5. CONCLUSION

In this work, unique analytical solution and hence unique travelling wave solution of RCDE has been verified in the beginning by using the PPAM Fig. 1. The new unique configuration of the achieved analytical solution will give ideal representation of the concentration population rate of one or more substances species distributed in space. In the same connection and parallel, other two analytical solutions have been extracted by using the (G'/G) technique, we implement only one of them Fig. 2. The achieved solutions *via* our proposed methods were not achieved previously. Also the VIM is used effectively to achieve the corresponding numerical solutions of the two analytical solutions that achieved by the PPAM and (G'/G) techniques respectively Figs. 3, 4. From our realized results, it is clear there are agreements between Figures plotting in 2D and 3D to each one of these distinct methods separately as well as the agreements between the two exact and the numerical solution.

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