INVESTIGATING THE REVISITED GENERALIZED STOCHASTIC POTENTIAL-KDV EQUATION: FRACTIONAL TIME-DERIVATIVE AGAINST PROPORTIONAL TIME-DELAY

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Abstract. In this study, we revisit the stochastic potential-KdV equation by considering the time-derivative of Caputo-type and the time-coordinate to be affected by proportional delay. We apply two approaches namely, fractional Maclaurin series and novel Homotopy perturbation method, to construct closed-form solutions to the proposed model. A graphical scheme is conducted to study the simulations of fractional time-derivative against the proportional delay-time. Finally, we investigate the influence of the model's coefficients on the propagation of the obtained solutions.

Key words: Stochastic potential-KdV; Caputo derivative;
Fractional Maclaurin series; Homotopy perturbation method.

1. INTRODUCTION

The stochastic potential-KdV (spKdV) equation is a nonlinear model that describes the propagation of nonlinear photons and optical solitons, and arises in the field of plasmas and electrical circuits. The mathematical formulation of spKdV equation reads as:

$$\phi_t + \alpha\phi_x + \beta\left(\phi_x\right)^2 + \gamma\phi_{xxx} = 0, \tag{1}$$

where $\phi = \phi(x,t)$ is the unknown function, α is the stochastic parameter, and β, γ are the nonlinearity and dispersion parameters. Equation (1) was first proposed by Alhami and Alquran and only explicit solutions of lumps, breathers, and multi-solitons types are obtained by using the Cole-Hopf transformation and the simplified Hirota method [1].

The main goal of the current work is to further investigate some physical aspects to (1) by promoting its time-derivative to Caputo-type and restrict the time-coordinate by proportional delay. The revisited spKdV equation takes the following form:

$$D_t^{\lambda}\phi(x,\mu t) + \alpha\phi_x(x,t) + \beta\left(\phi_x(x,t)\right)^2 + \gamma\phi_{xxx}(x,t) = 0, \tag{2}$$

where $0 < \lambda \leq 1$ refers to the order of the Caputo-derivative, $0 < \mu \leq 1$ is the proportional-delay factor, and $D_t^{\lambda}\psi(x,t)$ is defined by [2, 3]

$$D_t^{\lambda}\phi(x,t) = \frac{1}{\Gamma(1-\lambda)}\int_0^t \frac{\frac{\partial\phi(x,\tau)}{\partial\tau}}{(t-\tau)^{\lambda}}d\tau. \tag{3}$$

The fact that there is no single mathematical method capable to find explicit solutions to nonlinear equations with a fractional derivative, alternatively, we use analytical-numerical approaches to obtain analytic-numerical solutions to such fractional problems.

The contributions of this work is threefold. First, we find closed-form solutions to spKdV equation by adapting the fractional Maclaurin series and the Homotopy perturbation techniques. Second, we compare the effect of the fractional derivatives against the delay time on the shape of the attaining waveform motion. Finally, we conduct graphical schemes to study the effect of the model's coefficients on the dynamics of the obtained solutions.

For the past three decades, there has been a great effort in developing mathematical approaches to treat the presence of fractional derivatives. There have been numerical and analytical methods for obtaining approximate solutions to fractional and classical equations. Examples of such methods are, operational matrix method [4–6], collocation methods [7, 8], finite-difference methods [9, 10], reproducing kernel approaches [11, 12], different forms of fractional power series [13–19], the Homotopy perturbation technique and its updates [20–24], combined Laplace transform and fractional power series [25, 26] and many others [27–35]. With regard to the methods used in solving problems involving the delay time, we advise readers to view [36–39] and the references therein.

The organization of this work is as follows. In Sec. 2, we find a closed-form solution to the revisited spKdV equation (2) *via* using the fractional Maclaurin series method and then investigate the influence of the parameter β. In Sec. 3, the alternative Homotopy perturbation approach will be used to solve (2) and study the influence of the parameter α. Also, we compare the findings obtained by the proposed approaches to validate their implementations and accuracy. Finally, some recommendations will be given in Sec. 4.

2. FRACTIONAL MACLAURIN SERIES

In this Section, we recall some preliminaries related to the topic of fractional Maclaurin series that will be used in this paper.

Definition 2.1 *The fractional Maclaurin series (FMS) in (x,t)-plane is given by*

$$\sum_{n=0}^{\infty} A_n(x)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)} = A_0(x) + A_1(x)\frac{t^{\lambda}}{\Gamma(\lambda+1)} + A_2(x)\frac{t^{2\lambda}}{\Gamma(2\lambda+1)} + \dots \tag{4}$$

Theorem 2.2 *Assume $\phi(x,t)$ has a FMS representation*

$$\phi(x,t)=\sum_{n=0}^{\infty}A_n(x)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)}, \quad a<x<b, \;\; 0\leq t<R. \tag{5}$$

Then, $A_n(x)=D_t^{n\lambda}\phi(x,0)$, where $A_n(x)$ is bounded and R refers to the radius of convergence.

Theorem 2.3 *Assume $\phi(x,t)$ has a FMS representation, then*

$$D_t^{\lambda}\phi(x,t)=\sum_{n=0}^{\infty}A_{n+1}(x)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)}. \tag{6}$$

$$D_t^{\lambda}\phi(x,\mu t)=\sum_{n=0}^{\infty}A_{n+1}(x)\mu^{(n+1)\lambda}\frac{t^{n\lambda}}{\Gamma(n\lambda+1)}. \tag{7}$$

$$\phi_x(x,t)=\sum_{n=0}^{\infty}A_n'(x)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)}. \tag{8}$$

$$\phi_{xxx}(x,t)=\sum_{n=0}^{\infty}A_n'''(x)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)}. \tag{9}$$

Now, we proceed by assuming that the solution of (2) has a FMS form, *i.e.*

$$\phi(x,t)=\sum_{n=0}^{\infty}A_n(x)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)}. \tag{10}$$

Then, we plug (6)-(10) in (2) to get

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}A_{n+1}(x)\mu^{(n+1)\lambda}\frac{t^{n\lambda}}{\Gamma(n\lambda+1)}+\sum_{n=0}^{\infty}\alpha A_n'(x)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)} \\ &+ \beta\left(\sum_{n=0}^{\infty}A_n'(x)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)}\right)^2+\sum_{n=0}^{\infty}\gamma A_n'''(x)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)}. \end{aligned} \tag{11}$$

By using the relationship $\left(\sum_{n=0}^{\infty}B_n\right)\left(\sum_{n=0}^{\infty}C_n\right)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_mC_{n-m}\right)$, (11) is reduced to

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty}\frac{A_{n+1}(x)\mu^{n\lambda}t^{n\lambda}}{\Gamma(n\lambda+1)}+\sum_{n=0}^{\infty}\alpha A_n'(x)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)}+\sum_{n=0}^{\infty}\gamma A_n'''(x)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)} \\ &+ \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\beta A_m'(x)A_{n-m}'(x)\Gamma(n\lambda+1)}{\Gamma(m\lambda+1)\Gamma((n-m)\lambda+1)}\right)\frac{t^{n\lambda}}{\Gamma(n\lambda+1)}. \end{aligned} \tag{12}$$

Further simplifications on (12) leads to

$$\sum_{n=0}^{\infty} C_n(x) \frac{t^{n\lambda}}{\Gamma(n\lambda+1)} = 0, \tag{13}$$

where

$$C_n(x) = \mu^{(n+1)\lambda} A_{n+1}(x) + \alpha A_n'(x) + \gamma A_n'''(x) + \sum_{m=0}^{n} \frac{\beta A_m'(x) A_{n-m}'(x) \Gamma(n\lambda+1)}{\Gamma(m\lambda+1)\Gamma((n-m)\lambda+1)}.$$

For a power series to vanish identically over any interval, each coefficient in the series must be zero. Thus, for (13) to be valid over its given domain, we deduce the following recurrence relation

$$A_{n+1}(x) = -\mu^{-(n+1)\lambda} \left(\alpha A_n'(x) + \gamma A_n'''(x) + \sum_{m=0}^{n} \frac{\beta A_m'(x) A_{n-m}'(x) \Gamma(n\lambda+1)}{\Gamma(m\lambda+1)\Gamma((n-m)\lambda+1)} \right), \tag{14}$$

where $n = 0,1,2,....$ Relation (14) is used to identify $A_n(x)$ for $n \geq 1$ in term of a given arbitrary $A_0(x)$, which can be obtained when solving (2) subject to a general initial condition of the form $\phi(x,0) = f(x)$. In fact, by considering (4) and (5), $A_0(x) = f(x)$. Next, we introduce a numerical example to validate our approach and explore the effect of the fractional-order λ and the proportional-delay μ acting independently on the propagation of spKdV equation.

2.1. EXAMPLE

Consider the following initial value problem

$$D_t^{\lambda} \phi(x,\mu t) + \alpha \phi_x(x,t) + \beta \left(\phi_x(x,t)\right)^2 + \gamma \phi_{xxx}(x,t) = 0, \;\; \phi(x,0) = \frac{6\gamma}{\beta(1+e^{-x})}. \tag{15}$$

Let $\phi_n(x,t)$ represents the first $(n+1)$-partial sum of the series solution of $\phi(x,t)$. By using the relation (14), the first few terms of $\{A_n(x)\}$ are provided in Table 1 and Table 2. To study the impact of λ and μ, we consider $\phi_4(x,t)$ as the supportive approximate solution of (15). Figure 1(a) shows the propagation of spKdV solution for different values of the fractional-order λ with no effect of the proportional-delay, *i.e.*, $\mu = 1$, whereas, Fig. 1(b) shows the propagation of spKdV solution for different values of the proportional-delay μ *versus* $\lambda = 1$, the integer-time derivative. We observe from Fig. 1 that one of the geometric explanations for the role of the fractional derivative is delaying the value of the rate of change of the function (whether it is increasing or decreasing), just like the effect of the proportional-delay in the time coordinate. On the other side, regarding the spKdV's coefficients acting on its propagation form, we study in particular the impact of the nonlinear parameter β.

Figures 1 and 2 show that upon the change of β's sign, spKdV equation reverses the monotonicity of propagating its wave-solutions.

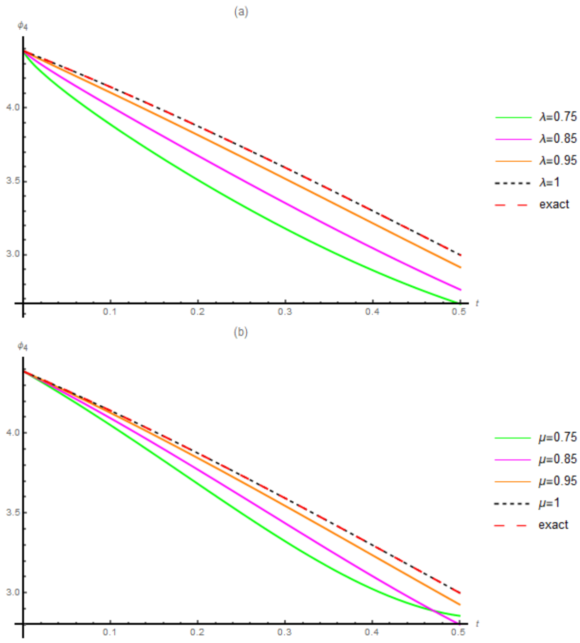

Fig. 1 – Profile solutions of $\phi_4(x,t)$. (a) The effect of the fractional-order λ when $\mu = 1$. (b) The effect of the proportional-delay μ when $\lambda = 1$. Here $\alpha = \beta = \gamma = 1$.

Table 1

The first four coefficients of the fractional Maclaurin series solution for Example 1: $\alpha = \beta = \gamma = 1$.

$A_0(x)$	$\frac{6}{e^{-x}+1}$
$A_1(x)$	$-\frac{12e^x\mu^{-\lambda}}{(e^x+1)^2}$
$A_2(x)$	$-\frac{24e^x(e^x-1)\mu^{-3\lambda}}{(e^x+1)^3}$
$A_3(x)$	$-\frac{48e^x\mu^{-6\lambda}\left(3e^x(e^x-1)^2\mu^{\lambda}\Gamma(2\lambda+1)+\left(-8e^x+6e^{2x}-8e^{3x}+e^{4x}+1\right)\Gamma(\lambda+1)^2\right)}{(e^x+1)^6\Gamma(\lambda+1)^2}$

3. HOMOTOPY PERTURBATION

The advantage of this method is that a general homotopic form can be defined for fractional nonlinear problems, where it does not deal mainly with the involved fractional derivative, but *via* considering its anti-derivative. Then, we write the solution as a power series in term of an auxiliary parameter called the perturbation. Based

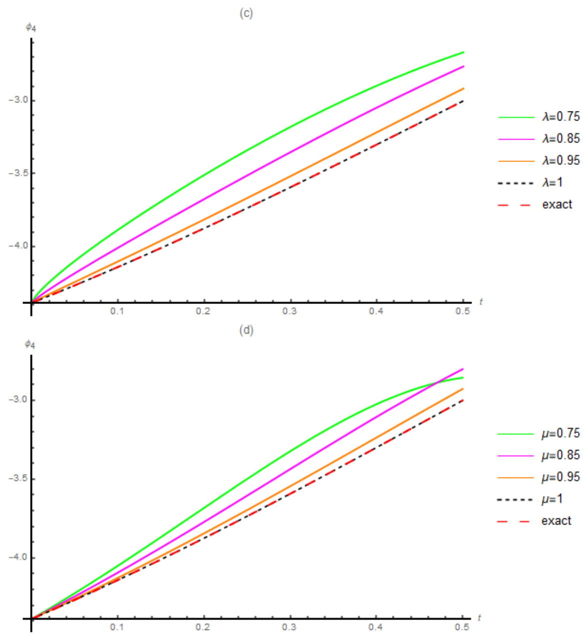

Fig. 2 – Profile solutions of $\phi_4(x,t)$. (c) The effect of the fractional-order λ when $\mu = 1$. (d) The effect of the proportional-delay μ when $\lambda = 1$. Here $\alpha = 1,\ \beta = -1,\ \gamma = 1$.

Table 2

The first four coefficients of the fractional Maclaurin series solution
for Example 1: $\alpha = 1, \beta = -1, \gamma = 1$.

$A_0(x)$	$-\frac{6}{e^{-x}+1}$
$A_1(x)$	$\frac{12e^{x}\mu^{-\lambda}}{(e^{x}+1)^2}$
$A_2(x)$	$\frac{24e^{x}(e^{x}-1)\mu^{-3\lambda}}{(e^{x}+1)^3}$
$A_3(x)$	$\frac{48e^{x}\mu^{-6\lambda}\left(3e^{x}(e^{x}-1)^2\mu^{\lambda}\Gamma(2\lambda+1)+\left(-8e^{x}+6e^{2x}-8e^{3x}+e^{4x}+1\right)\Gamma(\lambda+1)^2\right)}{(e^{x}+1)^6\Gamma(\lambda+1)^2}$

on the defined homotopy form, an iterative relationship can be drawn to identify the terms of the series solution.

Now, we define the following homotopy form regarding the spKdV equation given in (2)

$$D_t^{\lambda}\phi(x,\mu t) = -p\left(\alpha\phi_x(x,t) + \beta\left(\phi_x(x,t)\right)^2 + \gamma\phi_{xxx}(x,t)\right), \qquad (16)$$

where $0 \le p \le 1$, is the perturbation parameter. Then, we decompose $\phi(x,t)$ as a power series in p, *i.e.*,

$$\phi(x,t) = \sum_{i=0}^{\infty} \phi_i(x,t)p^i = \phi_0(x,t) + \phi_1(x,t)p + \phi_2(x,t)p^2 + ... \tag{17}$$

Substitution of (17) in (16) gives

$$\begin{aligned} \sum_{i=0}^{\infty} D_t^{\lambda}\phi_i(x,\mu t)p^i &= -\alpha \sum_{i=0}^{\infty} \phi_{x,i}(x,t)p^{i+1} - \beta p \left(\sum_{i=0}^{\infty} \phi_{x,i}(x,t)p^i \right)^2 \\ &- \gamma \sum_{i=0}^{\infty} \phi_{xxx,i}(x,t)p^{i+1}, \end{aligned} \tag{18}$$

where $\phi_{x,i}(x,t) = \frac{\partial \phi_i(x,t)}{\partial x}$ and $\phi_{xxx,i}(x,t) = \frac{\partial^3 \phi_i(x,t)}{\partial x^3}$. Applying the product of two infinite series, we write (18) as

$$\begin{aligned} \sum_{i=0}^{\infty} D_t^{\lambda}\phi_i(x,\mu t)p^i &= -\alpha \sum_{i=0}^{\infty} \phi_{x,i}(x,t)p^{i+1} - \beta \sum_{i=0}^{\infty}\sum_{n=0}^{i} \phi_{x,i-n}(x,t)\phi_{x,n}(x,t)p^{i+1} \\ &- \gamma \sum_{i=0}^{\infty} \phi_{xxx,i}(x,t)p^{i+1}. \end{aligned} \tag{19}$$

Unifying the above four series in terms of its index-counter and the power of the parameter p, one can verify the following formula

$$\begin{aligned} 0 &= D_t^{\lambda}\phi_0(x,\mu t) \\ &+ \sum_{i=1}^{\infty} \left(D_t^{\lambda}\phi_i(x,\mu t) + \alpha\phi_{x,i-1}(x,t) + \gamma\phi_{xxx,i-1} + \beta \sum_{n=0}^{i-1} \phi_{x,i-1-n}\phi_{x,n} \right) p^i. \end{aligned} \tag{20}$$

To determine the terms of (20) subject to $\phi(x,0) = f(x)$, we solve the following two problems

$$D_t^{\lambda}\phi_0(x,\mu t) = 0, \;\; \phi_0(x,0) = f(x),$$

and

$$D_t^{\lambda}\phi_i(x,\mu t) = -\left(\alpha\phi_{x,i-1}(x,t) + \beta \sum_{n=0}^{i-1} \phi_{x,i-1-n}(x,t)\phi_{x,n}(x,t) + \gamma\phi_{xxx,i-1}(x,t) \right), \tag{21}$$

subject to the initial condition $\phi_i(x,0) = 0, \;\; i \geq 1$. From (21), one can verify the following outputs:

$$\phi_0(x,t) = f(x),$$

and

$$\phi_1(x,t) = -\mu^{-\lambda}\left(\alpha f'(x) + \beta\left(f'(x)\right)^2 + \gamma f'''(x)\right)\frac{t^\lambda}{\Gamma(\lambda+1)}. \tag{22}$$

Now, we plug $i = 2$ in (21) to get

$$\phi_2(x,\mu t) = -J_t^\lambda\left[\alpha\phi_{x,1}(x,t) + \beta\sum_{n=0}^{1}\phi_{x,1-n}(x,t)\phi_{x,n}(x,t) + \gamma\phi_{xxx,1}(x,t)\right]$$

$$= -\left(\alpha\phi_{x,1}(x,1) + \beta\sum_{n=0}^{1}\phi_{x,1-n}(x,1)\phi_{x,n}(x,1) + \gamma\phi_{xxx,1}(x,1)\right)\frac{\Gamma(\lambda+1)t^{2\lambda}}{\Gamma(2\lambda+1)}, \tag{23}$$

where J_t^λ is the anti-fractional derivative of the Caputo operator D_t^λ. We should point out here that $D_t^r t^s = \frac{\Gamma(s+1)}{\Gamma(s-r+1)}t^{s-r}$ and $J_t^r t^s = \frac{\Gamma(s+1)}{\Gamma(s+r+1)}t^{s+r}$. By rescaling the time-coordinate in (23), we get

$$\phi_2(x,t) = -\left(\alpha\phi_{x,1}(x,1) + \beta\sum_{n=0}^{1}\phi_{x,1-n}(x,1)\phi_{x,n}(x,1) + \gamma\phi_{xxx,1}(x,1)\right) \times\frac{\Gamma(\lambda+1)t^{2\lambda}\mu^{-2\lambda}}{\Gamma(2\lambda+1)}. \tag{24}$$

Based on (21) and (24), the general i^{th}-term of the Homotopy series solution to spKdV equation takes the following form

$$\phi_i(x,t) = -\left(\alpha\phi_{x,i-1}(x,1) + \beta\sum_{n=0}^{i-1}\phi_{x,i-1-n}(x,1)\phi_{x,n}(x,1) + \gamma\phi_{xxx,i-1}(x,1)\right) \times\frac{\Gamma((i-1)\lambda+1)t^{i\lambda}\mu^{-i\lambda}}{\Gamma(i\lambda+1)}. \tag{25}$$

By (25), the closed form solution of spKdV equation is recognized.

3.1. VALIDATIONS AND COMPARISON STUDY

Here, we validate the accuracy of the proposed methods in approximating the solution of spKdV equation. Let $h_k(x,t)$ represent the k^{th}-order of the Homotopy solution against $\phi_k(x,t)$, the k^{th}-order of the fractional Maclaurin series. For the purpose of comparing our findings, we consider the following numerical test

$$D_t^\lambda\phi(x,\mu t) + \phi_x(x,t) + (\phi_x(x,t))^2 + \phi_{xxx}(x,t) = 0,\ \ \phi(x,0) = \frac{6}{1+e^{-x}}. \tag{26}$$

The exact solution of (26) is $\phi(x,t) = \frac{6}{e^{2t-x}+1}$. Using the proposed approaches, we perform the accuracy analysis for the cases of $\lambda = 0.9,\ \mu = 0.5$, and $\lambda = \mu = 1$ to reach at the following findings:

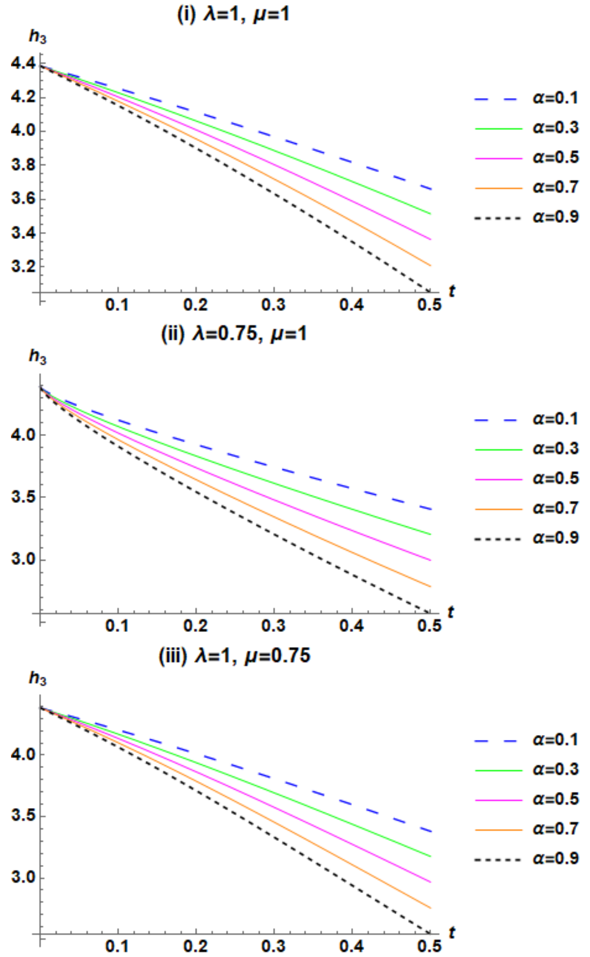

Fig. 3 – Profile solutions of $h_3(x,t)$. (i) $\lambda = 1,\ \mu = 1$. (ii) $\lambda = 0.75,\ \mu = 1$. (iii) $\lambda = 1,\ \mu = 0.75$. Here $\alpha = \beta = \gamma = 1$.

Case I: $(\lambda = 0.9,\ \mu = 0.5)$:

$$h_3(1,t) = 29.5476t^{2.7} - 8.45187t^{1.8} - 4.57772t^{0.9} + 4.38635 = \phi_3(1,t).$$

Case II: $(\lambda = \mu = 1)$:

$$h_3(1,t) = -\frac{8\left(e\left(1-4e+e^2\right)\right)t^3}{(1+e)^4} - \frac{12((e-1)e)t^2}{(1+e)^3} - \frac{12et}{(1+e)^2} + \frac{6e}{1+e}.$$

$$= \phi_3(2,t) = \phi(2,t) - O(t)^4. \tag{27}$$

Based on the last numerical example (26)-(27), we can say that both methods produce the same analytical solution, which indicates the correctness of their implementation. In comparison with the explicit solution of the spKdV equation in the absence of both fractional derivative and the delay time, all three solutions are identical, which is a strong evidence of the effectiveness of these methods.

3.2. INFLUENCE OF THE STOCHASTIC PARAMETER

In this part, by considering the third-order Homotopy solution $h_3(x,t)$, we study the effect of the stochastic parameter α on the dynamics of spKdV solution with/without the presence of both fractional-derivative and delay-time. In order to achieve this goal, we plot the profiles of solutions for different values of α within three categories; without the effect of both the fractional-derivative and the delay-time, under the effect of fractional-derivative only, and under the effect of delay-time only, see Fig. 3. We conclude from this graphical analysis that the stochastic parameter affects the wave's height of the spKdV solution.

4. CONCLUSION

In conclusion, we have presented the stochastic potential–KdV equation under the influence of both fractional time-derivative and the proportional time-delay. The same closed-form solution has been obtained to the revisited spKdV *via* using two different approaches. Based on the supportive approximate power series solution, we studied the effect of the fractional-derivative only ($\mu = 1$), then we have studied the effect of the delay-time only ($\lambda = 1$). With aid of graphical analysis, we conclude that the fractional derivative can be approximated by exposing the time coordinate with proportional delay, *i.e.*, $D_t^{\lambda}\phi(x,t) \approx \phi(x,\lambda t)$ whenever λ is close to 1. This finding has been observed based on some numerical examples. Thus, an interesting topic for future research would be to provide theoretical justifications. Finally, the dynamics of spKdV solutions has been visualized graphically by studying the impact of both stochastic and the nonlinearity parameters.

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