

## FINITE ENERGY GROUND STATES OF THE $-|x|^n$ POTENTIALS. II

S. GARIDI<sup>1,a</sup>, R.J. LOMBARD<sup>2</sup>, R. MEZHOU<sup>1,b</sup>

<sup>1</sup>Département de Physique, Faculté des Sciences,  
Université M'Hamed Bougara de Boumerdes,  
route de la gare ferroviaire, 35000 Boumerdes, Algeria

<sup>2</sup>IJClab, Orsay,  
F-91406 Orsay Cedex, France

Corresponding author: lombard@ijclab.in2p3.fr

<sup>a</sup>Laboratoire de Physique des Particules et de Physique Statistique,  
(LPPPS) ENS-KOUBA, Algeria

*E-mail:* s.garidi@univ-boumerdes.dz

<sup>b</sup>Laboratoire Revêtement Matériaux et Environnement LRME,  
Bloc de Recherche, Campus nord, Université M'Hamed Bougara, de Boumerdes, 35000, Algeria

*E-mail:* r.mezhoud@univ-boumerdes.dz

*Received December 12, 2022*

*Abstract.* In a preceding paper (here referred as I) [1], we have shown that infinitely negative potentials at large distances have finite ground state energy if use is made of complex wave functions. We revisit this work in the case of the  $-|x|^n$  potentials. We first reconsider the ground state energies. Then we discuss the first excited state. Finally, for the  $n = 4$  case, the 4 first levels are calculated, and compared to the results of Bender and Boettcher [2].

*Key words:* Quantum mechanics, complex wave functions, asymptotically negative potentials.

### 1. INTRODUCTION

In the preceding paper (I) [1], we have shown that potentials infinitely negative at large distances admit finite energy states provided that use is made of complex wave functions. A finite scalar product exists ensuring the normalisation of the wave functions. Our work was dedicated to the ground states of a few analytical examples, and of the  $-|x|^n$  potentials.

In the present paper, we complete our study of the  $-|x|^n$  potentials. First we discuss and bring a correction to the ground states energies. Then we study the first excited state, and check the orthogonality of the wave functions. Finally, for the  $n = 4$  case, we determine the 4 first levels, and compared our results to those of Bender and Boettcher ([2]).

The paper is organised as follows. The basic elements of our method are sketched. Then we display results for the chosen specific examples.

## 2. SKETCHING THE METHOD

Considering the  $D = 1$  dimensional space with  $\hbar = 2m = 1$ , we start with the Schrödinger equation for the state  $k$

$$\left[-\frac{d^2}{dx^2} + U(x)\right]\Psi_k(x) = E_k\Psi_k(x). \quad (1)$$

Let us consider solutions of the form

$$\Psi_k(x) = R_k(x)\exp[i\Phi_k(x)], \quad R_k(x) \geq 0. \quad (2)$$

It leads to

$$\begin{aligned} -R_k''(x) - i\Phi_k''(x)R_k(x) - 2iR_k'(x)\Phi_k'(x) + (\Phi_k'(x))^2R_k(x) + U(x)R_k(x) \\ = E_kR_k(x). \end{aligned} \quad (3)$$

Requiring real eigenvalues implies

$$2\Phi_k'(x)R_k'(x) + \Phi_k''(x)R_k(x) = 0. \quad (4)$$

This is a linear differential equation for the derivative of the phase. Let us write

$$g_k(x) = \Phi_k'(x). \quad (5)$$

It yields

$$g_k(x) = \frac{c_k}{R_k^2(x)}, \quad (6)$$

up to possible integration constants  $c_k$ , that we fix to unity. We are left with a highly non-linear differential equation for the modulus :

$$-R_k''(x) + U(x)R_k(x) + \frac{1}{R_k^4(x)}R_k(x) = E_kR_k(x). \quad (7)$$

The modulus function  $R_k(x)$  has to be a real positive function. Furthermore, following Bender and Boettcher [2], their study of a special class of PT-symmetric potentials (see also [3–7]), the scalar product is defined by

$$\int_{-\infty}^{\infty} \Psi_k^2(x)dx = \int_{-\infty}^{\infty} R_k^2(x)\exp[2i\Phi_k(x)]dx. \quad (8)$$

This definition ensures finite norms of the wave functions in an unusual way. It corresponds to an extension of the concept of probability in the complex plane [8]. The exponential term oscillates rapidly as the phase function becomes large. Thus, beyond a given point, the contributions to the norm average to zero. Consequently, there is no need for  $R_k(x)$  to decay exponentially at large distances.

### 3. GROUND STATES OF THE $-|x|^n$ POTENTIALS

Equation (7) is integrated *via* a Runge-Kutta algorithm. It requires to fix the boundary conditions. In our preceding work, for the ground state, we took  $R_0(0) = 1.0$  and  $R_0'(0) = 0.0$ . If the latter condition is justified because  $R_0(x)$  is an even function, the chosen value for  $R_0(0)$  is not necessarily valid. Actually, it was suggested by our analytical and semi-analytical examples, and the usual treatment of the Schrödinger equation. However, this is not generally valid, especially in the case of a non-linear Schrödinger equation. The purpose of the present work is precisely to revisit this boundary condition.

Concerning the behaviour at large distances,  $R_0(x)$  being positive definite, we search for a function decaying to zero at infinity with a zero derivative. As it is well known this goal is difficult to reach exactly, so only approximate solutions are obtained.

This ambiguity is amplified in our case by the non-linear term. Because of this non-linearity, most of the module functions undergo small oscillations around a smooth average curve. It generates a range of energy values with *a priori* acceptable module functions. We retain the module function closest to zero at the largest distance, and corresponding to the minimum of fluctuations.

There is another criterion allowing to select the correct solution. For each acceptable case, the  $1/R_0^4(x)$  term can be used as an effective potential and brought in an ordinary Schrödinger equation. The searched solution is given by the module function for which the two ways of determining the energy yield the same result.

We have applied this method to the ground states of the  $-|x|^n$  potentials for  $2 \leq n \leq 5$ , and letting  $R_0(0)$  to be a free parameter. The results are noticeably different from those quoted in I, especially for low  $n$  values. They are displayed in Table 1.

Note that for  $n = 4$ , the present result is in good agreement with the value given by Bender and Boettcher [2], whereas in I we quote a value about 10 % larger.

Table 1

Ground state  $R_0(0)$  and energies for  $-|x|^n$  potentials.

n	2.0	2.5	3.0	3.5	4.0	4.5	5
$R_0(0)$	0.606	0.855	0.937	1.00	1.051	1.097	1.139
$E_0$	7.485	2.207	1.769	1.57	1.477	1.42	1.385

#### 4. EXCITED STATES

The method has been extended to excited states, in particular to the lowest one. Contrarily to the usual case of hermitian potentials, we cannot rely on the number of nodes of the wave function to fix the quantum number  $k$ . The module function  $R_k(x)$  is a positive definite function whatever  $k$ . Thus, the states are ordered according to the successive energies obtained by the integration of the Runge-Kutta equation, relying on the criteria stated above.

Our results are displayed in Table 2. We note a sensible decrease of  $R_1(0)$  compared to the ground state values.

The criteria we have used for selecting the module function reach a limitation when  $R_k(x)$  extends over very large distances. This is occurring for low  $n$  values, because, as it can be checked, the asymptotic behaviour of the module function is given by

$$\frac{R_k(0)}{(1. + R_k^4(0)|x|^{n/4})^{1/4}}. \quad (9)$$

It may also happen for excited states. In such cases, the fluctuations are gradually diminishing with  $R_k(0)$ , and a proper solution cannot be defined. The energy difference between the Runge-Kutta and the Schrödinger estimates can help but is also not sufficient. In fact we have realised that the number of zeros of the module function derivative beyond the point  $x = 0$ , is a way to classify the excited states.

This provides us with an efficient criterion. When the fluctuations cannot select the proper solution, we choose the lowest  $R_k(0)$  value corresponding to the largest distance of the  $k$ th zero of the derivative. It may not give the exact solution but at least we get a lower limit for  $E_k$ .

In the present case, we have faced this difficulty for the ground state of  $n = 2$  and the first excited state of  $n = 3$ . As it can be seen from Tables 1 and 2. This situation results in large values of the energy compared to the other cases.

Table 2

Values of  $R_1(0)$  and energies for the first excited state of  $-|x|^n$  potentials.

n	3	3.5	4.0	4.5	5
$R_1(0)$	0.4006	0.605	0.6325	0.6375	0.6356
$E_1$	38.83	7.402	6.006	5.591	5.396

4.1. THE  $-x^4$  CASE

The results of Bender and Boettcher [2] for the case  $-x^4$  provide us with a good case to check our method to higher levels. It has been applied up to  $k = 3$ . The results are displayed in Table 3. The agreement is of the 0.1 % level. It gives us further confidence in our method.

Table 3

Energies of the first excited states of the  $-x^4$  potential compared to the values obtained by Bender and Boettcher [2].

	$E_0$	$E_1$	$E_2$	$E_3$
present results	1.477	6.006	11.803	18.476
Bender and Boettcher	1.477	6.003	11.802	18.459

## 4.2. NORMS AND ORTHOGONALITY

Concerning the norm, it is given by Eq. (8). However, it turns out that the integral get a negative sign, for all investigated cases when  $k$  is odd. To remedy this default, we remark that by analogy with the usual case, it is possible to make the real parts of  $\Psi_k(x)$  even or odd according to the parity of  $k$ . This is obtained by a simple shift of the odd  $k$  wave functions

$$\Psi_k(x) \rightarrow e^{i\pi/2}\Psi_k(x) \text{ for } k \text{ odd} . \quad (10)$$

This shift is sufficient to ensure all norms to be positive.

The orthogonality of wave functions is checked by

$$\int_{-\infty}^{\infty} \Psi_k(x)\Psi_{k'}(x)dx . \quad (11)$$

According to the phase shift of the odd  $k$  wave function, the integral is real or imaginary for  $k$  and  $k'$  for the same or different parities, respectively. The absolute value of the integral is found of the order of 1 %. It underlines the numerical sensibility of dealing with the wave functions.

## 5. CONCLUSIONS

The present paper corrects and completes our previous work on aspects of the quantum mechanics in the complex plane, *i.e.* when the wave function is complex function [1]. It concerns the  $-|x|^n$  potentials for  $2 \leq n \leq 5$ .

We show the role of the module function value at the origin. An important point for non-linear differential equations. Besides the ground states we discuss the first excited state. In the  $n = 4$  case, we have determined the spectrum up to  $k = 3$ , and compared our values to the results of Bender and Boettcher [2]. The comparison gives us confidence in our method. We have also seen its limitations.

#### REFERENCES

1. R.J. Lombard, S. Garidi and R. Mezhoud, Rom. J. Phys. **67**, 104 (2022).
2. C.M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
3. D. Mihalache, Rom. Rep. Phys. **67**, 1383 (2015).
4. B. Liu, L. Li and D. Mihalache, Rom. Rep. Phys. **67**, 802 (2015).
5. D. Mihalache, Rom. Rep. Phys. **69**, 403 (2017).
6. C.M. Bender, Rep. Prog. Phys. **70**, 205 (2007).
7. C.M. Bender, *PT Symmetry in Quantum and Classical Physics*, World Scientific (2019).
8. Chang Dong Yang and Shiang-Yi Han, *Extending Quantum Probability from Real Axis to Complex Plane*, Entropy (Basel) (2021). PMC 7915924.