EXPONENTIAL STABILIZATION OF SPIN-$\frac{1}{2}$ SYSTEMS BASED ON SWITCHING STATE FEEDBACK

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Abstract. We propose a novel switching state feedback strategy to exponentially stabilize quantum spin-$\frac{1}{2}$ systems in this paper. In order to obtain the faster state exponential convergence, the state space is divided into two subspaces, and two different continuous state feedback, which are applied in the corresponding subspaces respectively, compose the switching state feedback, under which the state convergence is faster than that under the continuous state feedback individually.

Key words: Exponential stabilization, spin-$\frac{1}{2}$ systems, state feedback.

1. INTRODUCTION

In the second quantum revolution, the stabilization of quantum states is one of the important tasks in quantum technologies, and the closed-loop feedback control, which needs the measured information, is recommended to use for the better robustness. Considering the collapse of quantum states caused by measurement operations, Belavkin proposed the quantum filter theory [1], based on which the global stabilizations of various quantum systems [2, 3] had been achieved by using different control strategies to overcome the convergence obstacle caused by the symmetric structure of state space. Particularly, the exponential stabilization of quantum states is of importance due to the rapidity of quantum manipulations. For this goal, the continuous state feedback in exponential form was designed by Liang et al. for quantum spin-$\frac{1}{2}$ systems in [4]. Different from state feedback, Cardona et al. proposed the switching noise-assisted feedback to achieve the exponential stabilization of $N$-level quantum systems in [5], from which we proposed two continuous noise-assisted feedback strategies for the same goal in [6]. On the basis of the state feedback in [4], a switching state feedback is designed to improve the state convergence rate for quantum spin-$\frac{1}{2}$ systems in this paper. We design a new continuous state feedback, which is different from that in [4], and divide the state space into two state subspaces. Then, the two continuous state feedback are used in the different state subspaces respectively, and compose the switching state feedback, under which the state convergence is faster than that under the state feedback designed in [4].

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2. DESIGN OF SWITCHING STATE FEEDBACK

From quantum filter theory, under continuous measurement and state feedback, quantum spin-$\frac{1}{2}$ systems satisfy the following Itô stochastic differential equation [4]

$$
d\rho_t = \left(-\frac{i\omega_{eg}}{2}[\sigma_z, \rho_t] - \frac{i\eta M}{2}[\sigma_y, \rho_t] + \frac{M}{4}(\sigma_z\rho_t\sigma_z - \rho_t)\right)dt$$

$$+ \frac{\sqrt{\eta M}}{2}(\sigma_z\rho_t + \rho_t\sigma_z - 2\text{tr}(\rho_t\sigma_z)\rho_t)\,dW_t$$

(1)

where, $\rho_t$ is quantum state and evolves in the state space $\mathcal{S} = \{ \rho \in \mathbb{C}^{2 \times 2} : \rho = \rho^\dagger \geq 0$, $\text{tr}(\rho) = 1 \}$, in which $\rho^\dagger$ is the conjugate transpose of $\rho$, $\text{tr}(\rho)$ denotes the trace of $\rho$; $M$ and $\omega_{eg}$ are the interaction strength and the energies difference, respectively; $W_t$ is a Wiener process; $\eta \in (0, 1]$ is the measurement efficiency; $u^{sf}_t(\rho_t)$ is the state feedback; $\sigma_z$ and $\sigma_y$ denote the Pauli matrices; $[\sigma_z, \rho_t] = \sigma_z\rho_t - \rho_t\sigma_z$.

However, when $u^{sf}_t(\rho_t) = 0$, the system state will converge to arbitrary equilibrium $\bar{\rho}$ with the probability $\text{tr}(\rho_0\bar{\rho})$ from the initial state $\rho_0$ [4]. Thus, the exogenous control inputs need to be applied to make the system state converge to the target state instead of arbitrary equilibrium. For this goal, we propose a novel state feedback strategy, which is denoted as switching state feedback in this paper, to exponentially stabilize the target state for quantum spin-$\frac{1}{2}$ systems with faster state convergence than that under the continuous state feedback in [4]. The equilibria of quantum spin-$\frac{1}{2}$ system (1) are the eigenstates, i.e., the excited state $\rho_e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and ground state $\rho_g = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The ground state $\rho_g$ is selected as the target state in this section, and we ignore the case in which $\rho_e$ is the target state due to the similarity.

The continuous state feedback in [4] is designed as

$$u^{cf1}_t(\rho_t) = \alpha V^\beta(\rho_t) - \mu \text{tr}(i[\sigma_y, \rho_t]\rho_f)$$

(2)

where, $\rho_f$ is the target state, $V(\rho_t) = \sqrt{1 - \text{tr}(\rho_t\rho_f)}$, $\mu \geq 0, \beta \geq 1$ and $0 < \alpha < \frac{\eta M^2}{(1 - \lambda)^{\frac{1 - \beta}{2}}}$ with $\lambda \in (0, 1)$. Based on (2), we design another continuous state feedback as

$$u^{cf2}_t(\rho_t) = -\alpha \text{tr}(i[\sigma_y, \rho_t]\rho_f)V^\beta(\rho_t) - \mu \text{tr}(i[\sigma_y, \rho_t]\rho_f)$$

(3)

i.e., replacing $\alpha$ in $u^{cf1}_t(\rho_t)$ with $-\alpha \text{tr}(i[\sigma_y, \rho_t]\rho_f)$. The state feedback (2) and (3) are denoted as continuous state feedback I and continuous state feedback II.

Considering that the state convergence rates under the state feedback (2) and (3) are different, we propose the switching state feedback by combining (2) with (3). The idea of switching state feedback is that the state space is divided into two state subspaces, which are denoted as $\{\Psi_1, \Psi_2\}$, and the state convergence under (3) is faster than that under (2) when the state is located in $\Psi_1$, while the state convergence
under (2) is faster when the state is located in $\Psi_2$, so that the state convergence under switching state feedback is faster than that under the continuous state feedback (2) and (3) individually.

On the other hand, the infinitesimal generator $A$ associated with the quantum spin-$\frac{1}{2}$ system (1) under state feedback $u_{sf}^\rho_t(\rho_t)$ acts on $V(\rho_t)$ as $AV(\rho_t) = \nu_s^\rho_t(\rho_t)\text{tr}(i[\sigma_y,\rho_t]\rho_f) - \frac{\eta M}{2}\text{tr}^2(\rho_t\rho_f) V(\rho_t)$. Due to $\text{tr}(i[\sigma_y,\rho_t]\rho_f) = 0$, $AV(\rho_t) = 0$ such that $AV(\rho_t) \leq -C V(\rho_t)$ with $C > 0$ does not hold for $\forall \rho_t \in S$, i.e., there does not exist any state feedback to make the target state $\rho_f$ globally exponentially stable. Thus, we consider the exponential stabilization of quantum spin-$\frac{1}{2}$ systems under state feedback in a state subspace $D_\lambda = \{ \rho \in S : \lambda < \text{tr}(\rho\rho_f) \leq \lambda \}$ first. Based on the idea of switching state feedback and state subspace $D_\lambda$, we have Theorem 1.

**Theorem 1.** For quantum spin-$\frac{1}{2}$ systems, if the state feedback in (1) is designed as

$$
u_s^\rho_t(\rho_t) = K(\rho_t)V^\beta(\rho_t) - \mu \text{tr}(i[\sigma_y,\rho_t]\rho_f)$$

(4)

with $K(\rho_t) = \begin{cases} -\alpha \text{tr}(i[\sigma_y,\rho_t]\rho_f), & \text{if } \rho_t \in \Psi_1 \\ \alpha, & \text{if } \rho_t \in \Psi_2 \end{cases}$, where $\Psi_1 = \{ \rho : \text{tr}^2(i[\sigma_y,\rho_t]\rho_f) + \text{tr}(i[\sigma_y,\rho_t]\rho_f) > \epsilon \} \cap D_\lambda$ with $\epsilon > 0$, $\Psi_2 = D_\lambda \setminus \Psi_1$, then $E[V(\rho_t)] \leq e^{-\gamma t} V(\rho_0)$ with the convergence rate $\gamma \geq \frac{\eta M}{2} - \frac{\eta}{2}(1 - \lambda)^{\beta-1}$ for $\forall \rho_0, \rho_t \in D_\lambda$, i.e., the target state $\rho_f$ is exponentially stable in $D_\lambda$. In particular, the state convergence under the switching state feedback (4) is faster than that under the continuous state feedback (2) and (3) individually in the state subspace $D_\lambda$.

**Proof.** When $K(\rho_t) = -\alpha \text{tr}(i[\sigma_y,\rho_t]\rho_f)$ and $K(\rho_t) = \alpha$, the infinitesimal generator $A$ associated with (1) acts on $V(\rho_t)$ are given by

$$AV(\rho_t) = -\frac{1}{2} \left( \eta M \text{tr}^2(\rho_t\rho_f) + \frac{\mu \text{tr}^2(i[\sigma_y,\rho_t]\rho_f)}{2V^2(\rho_t)} + \frac{\alpha}{2} \text{tr}^2(i[\sigma_y,\rho_t]\rho_f) V^{\beta-2}(\rho_t) \right) V(\rho_t)$$

(5)

$$AV(\rho_t) = -\frac{1}{2} \left( \eta M \text{tr}^2(\rho_t\rho_f) + \frac{\mu \text{tr}^2(i[\sigma_y,\rho_t]\rho_f)}{2V^2(\rho_t)} - \frac{\alpha}{2} \text{tr}(i[\sigma_y,\rho_t]\rho_f) V^{\beta-2}(\rho_t) \right) V(\rho_t)$$

(6)

respectively. Compare (5) and (6), if $\alpha \text{tr}^2(i[\sigma_y,\rho_t]\rho_f) > -\alpha \text{tr}(i[\sigma_y,\rho_t]\rho_f)$, i.e., $\rho_t \in \Psi_1$, the state convergence is faster when $K(\rho_t) = -\alpha \text{tr}(i[\sigma_y,\rho_t]\rho_f) V^\beta(\rho_t)$ than that when $K(\rho_t) = \alpha$. Thus, the state convergence is faster under the switching state feedback (4) than that under the state feedback (2) when $\rho_t \in \Psi_1$, while the convergence is the same when $\rho_t \in \Psi_2$, such that the state convergence under the switching state feedback (4) is faster in the state subspace $D_\lambda$. The comparison of
switching state feedback (4) and state feedback (3) is similar, so we ignore it here. Therefore, the state convergence under switching state feedback (4) is faster than that under state feedback (2) and (3) individually in the state subspace $D_{\lambda}$.

When $\rho_t \in D_{\lambda}$, as described in [4], (5) becomes $AV'(\rho_t) \leq -\frac{nM}{2}\lambda^2 V'(\rho_t)$, from which one can obtain that $\mathbb{E}[V(\rho_t)] \leq V(\rho_0) e^{-\gamma t} \leq V(\rho_0) e^{-\frac{nM}{2}\lambda^2 t}, \forall \rho_0, \rho_t \in D_{\lambda}$ based on Theorem 3 in [5], which means that $\rho_f$ is exponentially stable under state feedback (2) due to that $V(\rho_t) = 0$ if and only if $\rho_t = \rho_f$, and the convergence rate $\gamma \geq \frac{nM}{2}\lambda^2$ when $\rho_t \in D_{\lambda}$. On the other hand, for quantum spin-$\frac{1}{2}$ systems, the system state $\rho_t$ can be characterized by Bloch sphere coordinates $(x_t, y_t, z_t) \in B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ as $\rho_t = \frac{1}{2} \sigma_x + y \frac{1}{2} \sigma_y + z \sigma_z$, where $\sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbb{I}_2$ is the identity matrix. Thus, $\text{tr}(i[\sigma_y, \rho_t]) = x_t \leq 2V(\rho_t)$, which makes (6) become $AV'(\rho_t) \leq -\frac{1}{2} \left( \eta M \lambda^2 - \alpha (1 - \lambda) \frac{\beta - 1}{2} \right) V(\rho_t)$. After the similar discussion, one can obtain that $\mathbb{E}[V(\rho_t)] \leq V(\rho_0) e^{-\gamma t} \leq V(\rho_0) e^{-\left( \frac{nM}{2}\lambda^2 - \frac{\alpha}{2} (1 - \lambda) \frac{\beta - 1}{2} \right) t}, \forall \rho_0, \rho_t \in D_{\lambda}$, which means that $\rho_f$ is exponentially stable under state feedback (3), and $\gamma \geq \frac{nM}{2}\lambda^2 - \frac{\alpha}{2} (1 - \lambda) \frac{\beta - 1}{2}$ when $\rho_t \in D_{\lambda}$. Thus, under switching state feedback (4), the convergence rate $\gamma \geq \frac{nM}{2}\lambda^2$ when $\rho_t \in \Psi_1$, while $\gamma \geq \frac{nM}{2}\lambda^2 - \frac{\alpha}{2} (1 - \lambda) \frac{\beta - 1}{2}$ when $\rho_t \in \Psi_2$, so that the convergence rate $\gamma \geq \frac{nM}{2}\lambda^2 - \frac{\alpha}{2} (1 - \lambda) \frac{\beta - 1}{2}$ when $\rho_t \in D_{\lambda}$ due to $\frac{\alpha}{2} (1 - \lambda) \frac{\beta - 1}{2} > 0$. The proof is complete.

As described in Theorems 1, the target state $\rho_f$ is exponentially stable in state subspace $D_{\lambda}$ under the switching state feedback (4). In order to make the system state $\rho_t$ exponentially converge in the entire state space $S$ instead of state subspace $D_{\lambda}$, we can have the aid of noise-assisted feedback. In other words, we can use noise-assisted feedback in state subspace $S \setminus D_{\lambda}$ and use switching state feedback (4) in state subspace $D_{\lambda}$. The idea and method are similar to that of combined feedback as shown in [7], i.e., use state subspaces $S \setminus D_{\lambda}$ and $D_{\lambda}$ to replace the state subspaces $\Psi_1$ and $\Psi_2$ in [7], respectively, and use switching state feedback (4) rather than state feedback (2), so we ignore the technical details here, which can be found in [7].

3. NUMERICAL SIMULATIONS

We use numerical simulations to verify the effectiveness of switching state feedback (4), and compare switching state feedback (4) with the two continuous state feedback (2) and (3) to present the superiority. The eigenstate $\rho_c$ is chosen as the target state $\rho_f$, and the initial state is set as $\rho_0 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Let the physical
parameters $\omega_{eg} = 0$, $\eta = 0.5$, $M = 1$, and the control parameters $\alpha = 0.5$, $\beta = 8$, $\mu = 5$, $\epsilon = 0.3$ in switching state feedback (4). The experiment results of 10 sample trajectories are shown in Fig. 1. From Fig. 1, we can see that the system state $\rho_t$ converges to the target state $\rho_e$ from $\rho_0$, which verifies the effectiveness of switching state feedback (4). Moreover, we also put the results under state feedback (2) and (3) in Fig. 1, from which one can see that the state convergence is faster under switching state feedback (4) than that under state feedback (2) and (3), which is in accordance with the theoretical results in Theorem 1, and presents the superiority of switching state feedback (4).

![Fig. 1 – Exponential stabilization of $\rho_e$ under the switching state feedback (4), continuous state feedback (2) and (3). (a) The curves of $V(\rho_t)$; (b) The semi-log version of (a).](image)

### 4. CONCLUSIONS

In this paper, we proposed the switching state feedback to exponentially stabilize eigenstates for quantum spin-$\frac{1}{2}$ systems. The exponential convergence under switching state feedback and the superiority relative to the continuous state feedback in [4] were proved, and also verified in numerical simulations.

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### REFERENCES