NONLINEAR WOBLING OF EVEN-EVEN NUCLEI

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Abstract. This work is an inquiry into the role played by classical nonlinearities/quantum anharmonicities in the excitation of wobble modes of a triaxial even-even nucleus.

In the first part of the paper the free rotation at high spin of an asymmetric top around an axis near the $x_3$-axis is separated into three uncoupled nonlinear oscillators, each described by the unforced and undamped Duffing equation. In order to analyze and visualize the wobbling as a sub-class of rigid body dynamics, the analysis of the asymptotic solutions of a nonlinear oscillator is carried out in the framework of the Krylov-Bogoliubov-Mitropolski method that is suitable to calculate an approximate periodic solution for the stationary state of a second-order differential equation with a weak cubic perturbation. The consequences of this analysis are discussed in the particular case of the triaxial nucleus $^{130}$Ba.

In the second part of the paper a classical form of the triaxial rotor Hamiltonian in terms of a tilt angle and its conjugate momentum is supplied using the so-called Nodvik representation. Expanding in powers of the tilt angle up to quartic terms, the Hamiltonian is splitted into a part describing the high-spin rotation around the third axis and an anharmonic oscillator which essentially relates to the wobbling motion. Upon quantization according to the Pauli prescription the first- and second-order contributions to the wobbler energy within the framework of stationary perturbation theory is obtained.

Key words: Triaxial nuclei; wobbling motion; nonlinear vibrations.

1. INTRODUCTION

Wobbling is one of the main fingerprints of triaxial nuclei. Nonetheless the macroscopic counterpart of this type of motion is known since a long time in classical mechanics: the triaxiality of a rigid body plays a significant role on its free rotation [1, 2]. In the case of rigid celestial bodies (e.g. Earth, Mars) a conspicuous type of precession is represented by the small amplitude polar motion (polbody) which consists of oscillations of the angular momentum axis with respect to one of the body’s intrinsic axis [3]. Another example which applies to non-rigid celestial bodies is the Chandler wobble that accounts satisfactorily for the lengthening of the period, predicted for a rigid Earth to be 305 days, to the observed 434.3 days. The current...
view is that this free nutation or Eulerian precession of the Earth is excited due to
the gravitational coupling between atmosphere and oceans and consists of a slow
precession about the ecliptic pole [4]. It has also been realized that for Venus, large
internal mass anomalies produce appreciable wobble motion [5].

The asymmetric rotator model was for the first time evoked in quantum me-
chanics by Casimir in connection with the rotation of molecules [6, 7] and later on
Davydov and Filippov assumed that some nuclei exhibiting collective rotational fea-
tures can be described by asymmetric rotators (see the literature in refs. [8] and
[9]). These are analogous to classical rigid bodies with three moments of inertia
\( \mathcal{J}_1 \neq \mathcal{J}_2 \neq \mathcal{J}_3 \) rotating about the center of mass and kinetic energy given by

\[
T = \sum_k \frac{M_k^2}{2 \mathcal{J}_k}
\]

where \( M_k = \mathcal{J}_k \omega_k \) is the angular momentum around the \( k \)-axis of the body-fixed
(intrinsic) system and \( \omega_k \) is the corresponding angular velocity.

In the case of the free rotation of an asymmetric top, the wobbling motion
appears as a slight perturbation of a stable mode of rotation, such as for example the
rotation strictly around the intrinsic third axis (\( M_3 \approx |M| \)) that due to a perturbation
at \( t = 0 \) acquires small non-vanishing components of the angular momentum on the
first and second axis, such that \( 0 < M_1(0), M_2(0) \ll M_3 \) [2]. Under the assumption
that over a long period of time the components \( M_{1,2}(t) \) fulfil the same conditions as
their initial values, i.e. the rotation is stable, and solving the Euler equations in the
linear approximation, it can be readily established that the perturbed variables are
subjected to a periodic motion characterized by the same circular frequency

\[
\omega_w = M_3 \sqrt{\left( \frac{1}{\mathcal{J}_1} - \frac{1}{\mathcal{J}_3} \right) \left( \frac{1}{\mathcal{J}_2} - \frac{1}{\mathcal{J}_3} \right)}
\]

The wobbling motion in an even-even nucleus (simple wobbler) was for first time
evoked in the textbook of Bohr and Mottelson in reference to the yrast states
of lowest energy for a given spin \(|M| \equiv I \gg 1 \) [10]. The frequency obtained via a
contraction of the raising and lowering angular momentum operators provides the
same expression as in the classical case (2). In more recent times the application of
Boson expansions on the angular momentum operators confirmed this result with the
proviso that the total spin is slightly corrected according to the substitution
\( I \rightarrow I - \frac{1}{2} \) [11]. It should be also mentioned that in very recent times a significant number
of papers has been dedicated to the investigation of this collective type of motion
from which I mention Refs. [12–16]. To the date the simple wobbling mode was
validated experimentally in even-even and odd-\( A \) nuclei, the most recent reported
case being \(^{187}\text{Au} \) [17]. Its manifestation in even-even nuclei is the subject of a very
recent investigation of rotation in $^{130}$Ba [18].

The Euler equations for the perturbed asymmetric rotor, worked out in the lowest order of the fluctuations of the components $M_1$ and $M_2$, predict that these two variables execute harmonic oscillations with the same frequency $\omega_1$. The purpose of the present investigation is to go beyond this approximation and analyze the free rotation of an asymmetric top around an axis near the $x_3$-axis by retaining first- and second-order terms in the weak rotation around the intrinsic $x_1$-axis and $x_2$-axis of the body-fixed system. Whereas in the first part of the paper the perturbational analysis of the nonlinear wobbler is performed in the framework of classical mechanics, in the second part of the paper the quantization of the Hamiltonian describing this type of oscillations is carried out and the wobbling energy is obtained via a direct stationary perturbations calculation.

2. NONLINEAR OSCILLATOR CLASSICAL REALIZATION OF THE TRIAXIAL ROTOR MODEL

The Euler equations for the torque-free motion of a triaxial rigid body are

\begin{align*}
J_1 \dot{\omega}_1 &= (J_2 - J_3)\omega_2 \omega_3 \\
J_2 \dot{\omega}_2 &= (J_3 - J_1)\omega_3 \omega_1 \\
J_3 \dot{\omega}_3 &= (J_1 - J_2)\omega_1 \omega_2
\end{align*}

Above, $\omega$ is the angular velocity vector and in what follows we assume that the intrinsic axes are ordered so that $J_3 > J_2 > J_1$. The rotational energy (1) and total angular momentum

\begin{align*}
2T &= J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2, \\
I^2 &= (J_1\omega_1)^2 + (J_2\omega_2)^2 + (J_3\omega_3)^2
\end{align*}

are constants of motion as can be checked directly by working out the Euler equations. Differentiating the equations of motion (3-5) with respect to time and substituting the $\dot{\omega}$'s on the right hand side leads to

\begin{align*}
\ddot{\omega}_i &= \frac{J_j - J_k}{J_i} \left[ \frac{J_i - J_j}{J_k} \omega_j^2 + \frac{J_k - J_i}{J_j} \omega_k^2 \right] \omega_i, \quad (i,j,k - \text{cyclic})
\end{align*}

Since two out of the three $\omega$'s can be expressed in terms of the constants of motion $2T$ and $I$, the three-component equations (7) can be put in the final form of a Duffing free undamped oscillator [19, 20].

\begin{align*}
\ddot{\omega}_i + A_i \omega_i + B_i \omega_i^3 &= 0
\end{align*}

where

\begin{align*}
A_i &= \frac{(J_i - J_j)(2J_kT - I^2) + (J_k - J_i)(I^2 - 2J_jT)}{J_1J_2J_3},
\end{align*}
\[ B_i = \frac{2(J_j - J_i)(J_k - J_i)}{J_k J_j} \]  

Assuming that initially the rotational kinetic energy is stored in pure rotation with maximum angular momentum around the \( x_3 \)-axis, i.e. \( M_3 = I \),  
\[ T = \frac{I^2}{2J_3}, \]  
the \( A \)-coefficients can thereby be re-expressed as  
\[ A_1 = A_2 = \omega_w^2, \quad A_3 = -2\omega_w^2 \]

If the ratio \( B_{1,2}/A_{1,2} \) is small, the cubic term can be neglected in the dynamical equations for \( \omega_1 \) and \( \omega_2 \) and therefore it can be easily checked that they obey the harmonic oscillator equation for the traditional wobbling mode  
\[ \ddot{\omega}_{1,2} + \omega_w^2 \omega_{1,2} = 0 \]

If we take for example \( \mathcal{J}_1/J_3 = 1/4 \), \( \mathcal{J}_2/J_3 = 1/3 \), this approximation is valid when \( \omega_w^2 \gg 3\omega_{1,2}^2/2 \), a condition that is safely fulfilled for rapid rotations around the \( x_3 \)-axis. This linearization is no longer justified for the equation describing the third component of the angular velocity, \( \omega_3 \), since \( B_3/B_1 = 8 \) ! Nevertheless for the above choice of the initial kinetic energy, which implies the initial values for the angular velocity \( \omega_1(0) = \omega_2(0) = 0 \), at any later time \( M_1(t) = M_2(t) = 0 \). This can be easily inferred from the exact solutions of the Euler equations (3-5) [2]:  
\[ \omega_1(t) = \sqrt{\frac{2\mathcal{J}_3 T - I^2}{\mathcal{J}_1(\mathcal{J}_3 - \mathcal{J}_1)}} \, \text{cn}(\sigma t, k), \quad \omega_2(t) = \sqrt{\frac{2\mathcal{J}_3 T - I^2}{\mathcal{J}_2(\mathcal{J}_3 - \mathcal{J}_2)}} \, \text{sn}(\sigma t, k), \]
\[ \omega_3(t) = \sqrt{\frac{I^2 - 2\mathcal{J}_1 T}{\mathcal{J}_3(\mathcal{J}_3 - \mathcal{J}_1)}} \, \text{dn}(\sigma t, k) \]

where  
\[ \sigma = \sqrt{\frac{(I^2 - 2\mathcal{J}_1 T)(\mathcal{J}_3 - \mathcal{J}_2)}{\mathcal{J}_1 \mathcal{J}_2 \mathcal{J}_3}}, \quad k = \sqrt{\frac{2\mathcal{J}_3 T - I^2}{I^2 - 2\mathcal{J}_1 T}} \, \sqrt{\frac{\mathcal{J}_3 - \mathcal{J}_2}{\mathcal{J}_2 - \mathcal{J}_1}} \]

and \( \text{cn}, \text{sn} \) and \( \text{dn} \) are \textit{Jacobi elliptic functions} of argument \( \sigma t \) and modulus \( k \) [21].

In the lowest-order approximation used to derive the wobbling mode, the largest part of rotational motion is stored in the third component of the angular momentum vector \( \mathbf{M} \), i.e \( M_1(0) \ll M_3(0) \) and \( M_2(0) = 0 \). The next order of approximation is derived by using the asymptotic representation of elliptic functions [21], expand in powers of the obviously small parameter \( \delta = M_1(0)/I \), and keep in the expansion.
terms up to $O(\delta^2)$. The kinetic energy thereby reads
\[ T = \frac{M_1^2}{2\mathcal{J}_1} + \frac{M_2^2}{2\mathcal{J}_3} \approx I_1^2 + \frac{1}{2} \left( \frac{1}{\mathcal{J}_1} - \frac{1}{\mathcal{J}_3} \right) I_1^2 \] (15)
where $I_1 = M_1(0)$. The $A$-coefficient in the nonlinear equation (8) is modified to
\[ A_1 \equiv \dot{\omega}_w^2 = \omega_w^2 + \left[ \left( \frac{1}{\mathcal{J}_1} - \frac{1}{\mathcal{J}_3} \right)^2 + \frac{1}{\mathcal{J}_3^2} \left( \frac{1}{\mathcal{J}_2} - \frac{1}{\mathcal{J}_1} \right) \right] I_1^2 \] (16)
For small initial values of $M_1$ the second term is significantly smaller than the first. In this order of approximation the three components of the angular momentum corresponding to the circular frequencies (14) are
\[ M_1(t) \approx I_1 \cos \omega_w t, \quad M_2(t) \approx I_1 \sqrt{\mathcal{J}_1(\mathcal{J}_3 - \mathcal{J}_1)} \sin \omega_w t, \]
\[ M_3(t) \approx I_1 \left[ 1 - \frac{1}{2} \delta^2 \left( 1 + \frac{\mathcal{J}_2 - \mathcal{J}_1}{\mathcal{J}_3 - \mathcal{J}_2} \cdot \frac{\mathcal{J}_3}{\mathcal{J}_1} \right) \right] \sin^2 \omega_w t \] (17)
Let me focus on equation (8) for the first component of the angular velocity. It was earlier pointed out that the parameter $\varepsilon \equiv B_1/\omega_w^2$ is small compared to the square of free oscillations frequency $\omega_w^2$. Under such circumstances a qualitative analysis of the asymptotic solution of the unforced Duffing oscillator for a single of freedom based on the Krylov-Bogolyubov-Mitropolski method [22, 23] can be carried out. Details are provided in the Appendix and below I write the approximate form of $M_1(t) = \mathcal{J}_1 \omega_1(t)$ and the modified frequency $\Omega_1$ in the $\varepsilon^2$-order of the perturbation series
\[ M_1(t) \approx I_1 \left\{ \cos \Omega_1 t + \frac{B_1}{32} \left( \frac{I_1}{\mathcal{J}_1 \omega_w} \right)^2 \left[ 1 - \frac{21 B_1}{32} \left( \frac{I_1}{\mathcal{J}_1 \omega_w} \right)^2 \right] \cos 3\Omega_1 t \right. \]
\[ + \frac{B_1^2}{1024} \left( \frac{I_1}{\mathcal{J}_1 \omega_w} \right)^4 \cos 5\Omega_1 t \left. \right\} \] (18)
where the amplitude $I_1$ is determined from the initial conditions. In the same order of approximation the frequency $\Omega_1$ is given by the expression
\[ \Omega_1^2 \approx \frac{\omega_w^2}{2} \left\{ 1 + \frac{3}{4} B_1 \left( \frac{I_1}{\mathcal{J}_1 \omega_w} \right)^2 + \frac{3}{128} B_1^2 \left( \frac{I_1}{\mathcal{J}_1 \omega_w} \right)^4 \right\} \] (19)
The average value of $M_1^2$ over a period $2\pi/\omega_w$ is in the order $O(\varepsilon^2)$:
\[ \bar{M}_1^2 \equiv \frac{1}{T} \int_0^T M_1^2(t) dt \approx \frac{1}{2} \bar{I}_1^2 \left[ 1 + \frac{1}{2024} B_1^2 \left( \frac{I_1}{\mathcal{J}_1 \omega_w} \right)^2 \right] \] (20)
Fig. 1 – Time evolution of the $x_1$-component of the angular momentum for $^{130}$Ba in the linear (full line) and nonlinear case (dashed line) for two selections of $\mathcal{J}_k$: $(\mathcal{J}_k)_{RR}$ (left panel) and $0.45(\mathcal{J}_k)_{RR}$ and $\delta = 0.1$.

The expression of $M_2(t)$ and its average is obtained by the same token, the label “1” being replaced by the label “2” in eqs. (18) and (19). The third component, $M_3(t)$, results straightforwardly from the angular momentum conservation law (6).

Fig. 2 – Same as in Fig.1 for $M_3$.

To asses the role of nonlinearities, the rotational state characterized by the active nucleon configuration $\nu[514]9/2^-[404]7/2^+$, with deformation parameters $\beta = 0.23$, $\gamma = 11.7^\circ$ in the even-even triaxial nucleus $^{130}$Ba is considered according to ref. [18]. Rigid-body values are chosen for the moments of inertia as given by the Lund formula

$$\mathcal{J}_{RR} = \frac{\mathcal{J}_0}{1 + \left(\frac{5}{16}\pi\right)\frac{1}{2}\beta_2} \left[1 - \left(\frac{5}{4}\pi\right)\frac{1}{2}\beta_2 \cos \left(\gamma + \frac{2}{3}k\pi\right)\right]$$

with $\mathcal{J}_0 = 21 h^2$/MeV.

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1.3. FROM CLASSICAL NONLINEARITIES TO QUANTUM ANHARMONICITIES

A convenient way to treat the stability of the triaxial rotator was introduced by Nodvik, which consists of a special canonical representation of the angular momentum [24]. In this representation the components of the angular momentum in the intrinsic system are given by

\[ M_1 = (I^2 - p_\theta^2)^{1/2} \sin \theta, \quad M_2 = p_\theta, \quad M_3 = (I^2 - p_\theta^2)^{1/2} \cos \theta \]  

(22)

where \( \theta \) is the tilt angle and \( p_\theta \) its conjugate momentum, satisfying the Poisson bracket relation \( \{ \theta, p_\theta \} = 1 \). The above definition is suited to describe oscillations around the tilted equilibrium in the \( x_1 - x_3 \) plane. Consequently the rotational kinetic energy (1) will include a \( \theta \)-dependent term as well a \( p_\theta \) one

\[ T = \frac{1}{2} J_3 I^2 + \frac{1}{2} \left( \frac{1}{J_2} - \frac{1}{J_3} \right) \sin^2 \theta \, p_\theta^2 + \frac{1}{2} \left( \frac{1}{J_1} - \frac{1}{J_3} \right) I^2 \sin^2 \theta \]  

(23)

The first term represents the precession energy or energy of rotation with maximum angular momentum projection on the intrinsic \( x_3 \) axis. The second term describes the well-known nutation of an asymmetric top about the rotating equilibrium that in the lowest order of the tilt angle corresponds to the wobbling motion [10]. To analyze the nonlinear dynamics of Nodvik’s triaxial rotator, a Taylor series expansion in the coordinate \( \theta \) is performed such that the energy acquires the perturbational form

\[ H' = T - \frac{1}{2} J_3 I^2 \approx H_w + \delta H' \]  

(24)

where \( H_w \) contains the quadratic term that generates the linear wobbling harmonic oscillations, and \( \delta H' \) accounts for the perturbation series that I assume in this case to be truncated at the second-order level. Using from now on the short-hand notations

\[ \frac{1}{J_{13}} = \frac{1}{J_1} - \frac{1}{J_3}, \quad \frac{1}{J_{23}} = \frac{1}{J_2} - \frac{1}{J_3} \]  

(25)

*Note that the same representation was introduced in Celestial Mechanics a century ago by Andoyer [25] and since then extensively used in that field.
and the definition of the wobbling circular frequency given in eq. (2), the two pieces of the Hamiltonian (24) are rewritten as follows:

\[ H_w = \frac{1}{2} J_{23} p_\theta^2 + \frac{1}{2} J_{23} \omega_w^2 \theta^2 \]  

(26)

\[ \delta H' = -\frac{1}{2} J_{13} \theta^2 p_\theta^2 - \frac{1}{6} J_{23} \omega_w^2 \theta^4 \]  

(27)

The Pauli prescription [26] for quantizing the Hamiltonian (24) leads to the operator form

\[ \hat{H}' = -\frac{\hbar^2}{2 J(\theta)} \left[ \frac{d^2}{d\theta^2} - \frac{1}{2} \frac{d}{d\theta} \ln J(\theta) \right] + V(\theta) \]  

(28)

where

\[ J(\theta) = J_{23} \left( 1 - \frac{J_{23}}{J_{13}} \theta^2 \right) \]  

(29)

is a coordinate-dependent moment of inertia and

\[ V(\theta) = \frac{1}{2} J_{23} \omega_w^2 \theta^2 \left( 1 - \frac{1}{3} \theta^2 \right) \]  

(30)

is the quadratic+cuartic potential in the tilt angle. The Schrödinger equation with coordinate-dependent mass parameter is tackled like in Ref. [27], where a scale transformation was proposed in the form

\[ \xi = \int_0^\theta d\theta' \sqrt{J(\theta')} \]  

(31)

The transformed Schrödinger equation acquires a new form with the salutary feature that the first-order derivative is absent:

\[ \left\{ -\frac{\hbar^2}{2} \frac{d^2}{d\xi^2} + U(\xi) \right\} \varphi(\xi) = E \varphi(\xi) \]  

(32)

The potential energy and the wave function \( \psi(\theta) \) in the original representation are transformed as follows

\[ U(\xi) = V(\theta(\xi)), \quad \varphi(\xi) = \psi(\theta(\xi)) \]  

(33)

To express the connection between the old \( \theta \) and the new variable \( \xi \) in a simpler form I use the fact that the variable \( \theta \) assumes small values, and I am thus allowed to expand \( \xi(\theta) \) around \( \theta_0 = 0 \),

\[ \xi(\theta) \approx \xi(0) + \theta \xi'(0) + \frac{1}{2!} \theta^2 \xi''(0) + \ldots \]  

(34)

Choosing \( p_\theta = J_{23} \dot{\theta} \) and discarding cubic or larger power terms in the expansion (34)

\[ \xi(\theta) \approx J_{23}^{1/2} \theta \]  

(35)
The Hamiltonian in the $\xi$-coordinate representation corresponds to a harmonic oscillator perturbed by a quartic anharmonicity

$$\hat{H}' = -\frac{\hbar^2}{2} \frac{d^2}{d\xi^2} + \frac{1}{2} \omega_w^2 \xi^2 - \frac{1}{6} J_{13} \omega_w^2 \xi^4$$ (36)

I note in passing that the role of anharmonicities in the quantized asymmetric rotor Hamiltonian in the presence of appreciable total spin was discussed long time ago [28–32]. Within the zero-order stationary perturbation theory the energy of the wobbler characterized by the number $n_w$ of wobbling phonons excited on the yrast state [10] reads

$$E_{n_w}^{(0)} = \hbar \omega_w \left( n_w + \frac{1}{2} \right)$$ (37)

Due to the quartic anharmonicity, the energy will be decreased by the first- and second-order perturbation theory corrections (see for example [26])

$$\Delta E_{n_w}^{(1)} = -\frac{\hbar^2}{8 J_{23}} (2n_w^2 + 2n_w + 1)$$ (38)

$$\Delta E_{n_w}^{(2)} = -\left( \frac{\hbar^2}{6 J_{23}} \right)^2 \frac{1}{8 \hbar \omega_w} (34n_w^3 + 51n_w^2 + 59n_w + 21)$$ (39)

The expectation value of $M_1^2$ in a state of quantum numbers $I$, $M$ and $n_w$ is

$$\langle \psi(\theta) | \hat{M}_1^2 | \psi(\theta) \rangle \approx [I(I+1)\langle \psi(\theta) | \theta^2 | \psi(\theta) \rangle - \langle \psi(\theta) | \hat{p}_\theta^2 | \psi(\theta) \rangle]$$ (40)

Using the stationary perturbation theory in the first order the corresponding diagonal matrix element on the $n_w$-phonon state reads

$$(M_1^2)_{n_w,n_w} \approx \frac{\hbar I(I+1)}{2 J_{23} \omega_w} \left\{ 2n_w + 1 + \frac{\hbar}{J_{23} \omega_w} \left( n_w^2 + n_w + \frac{1}{2} \right) \right\}$$ (41)

In Fig. 3 the energies of the wobble vibrations in the harmonic case (full lines) and with account of anharmonicities in the second-order perturbation theory (dashed line), are plotted as a function of the spin $I$ for a triaxial rotor with $J_1 = 31.4 \, \hbar^2 / \text{MeV}$, $J_2 = 42 \, \hbar^2 / \text{MeV}$ and $J_3 = 125 \, \hbar^2 / \text{MeV}$. The corrections due to anharmonicities gain in importance with increasing phonon number and are visible at low spins when the approximation used in separating the wobbling Hamiltonian breaks down.

Before closing this section we note that in Ref. [33] an attempt was made in order to take into account the anharmonicities in the quantum picture of wobbling motion. For that end the author added \textit{ex-abrupto} a potential term of the type $a \sin^2 \theta$ and explored the role played by the anharmonicities by varying the control parameter $a$. Compared to Oi’s work, the approach presented above has the advantage of not introducing additional parameters for it separates straightforwardly the tilt angle
dependent hamiltonian from the rigid-rotator one by resorting to the particular form of the Nodvik representation.

4. CONCLUSIONS

While the wobble motion of triaxial nuclei is generally understood nowadays in the framework of rigid rotator model, the Earth’s Chandler wobble generation mechanism consists in adding also to the rotational energy of the Earth’s rigid parts the contribution of gravitation and elastic strain. In this paper I showed that an irrotational fluid admixture to the inertia moments enhance the role of nonlinear behavior of wobble dynamics. In the quantum picture proposed in this study, the anharmonic contributions to the excitation of wobble modes increase with the phonon quantum number $n_w$. Thus, in a macroscopic framework I stress the importance of the interplay between the rigid rotor and irrotational fluid picture of inertia moments as well of the surface effects which to the date have not been discussed in the nuclear wobbling context.

An interesting perspective related to this type of collective rotation in nuclear systems is represented by its possible occurrence in non-axial symmetric scission configurations or high spin metastable states in nuclear isomers [34]. The angular momentum generation in binary [35–37] or ternary fission [38] was extensively discussed mainly for low-energy fission. It would be of certain interest to investigate the possibility of wobbling modes occurrence in highly deformed nuclear systems consisting of two or three clusters.

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APPENDIX. STATIONARY SOLUTION OF Eq. (8) IN THE KRYLOV-BOGO LIUBOV-MITROPOLSKY METHOD

The approximate solution of the undamped and unforced oscillator perturbed by a cubic term was extensively discussed in the literature (see for example [19, 20]). I present below the derivation of the stationary solution in the so called quasi-harmonic case, where the perturbed oscillations are nearly sinusoidal. This method was proposed in the thirties by Krylov and Bogoliubov [22] and assumes in the particular case of a cubic perturbation that the parameter multiplying the non-linear term characterizes the smallness of the deviation from the harmonic case.

Consider the first component, for $\omega_1$, of eq. (8). To keep the exposition simple, a dimensionless coordinate $x(t) = \omega_1(t)/\omega_1(0)$ is introduced. The initial conditions are $x(0) = 1$ and $\dot{x}(0) = 0$. From the structure of the coefficient $B_1$ and the smallness of the ratio $I_1/I$, it is easy to ascertain the fact that $\varepsilon = B_1 I_1^2 / \mathcal{F}_1$ is a small parameter. Hence, the equation satisfied by the variable describing the oscillating system with a cubic perturbation reads

$$\ddot{x} + \omega_w^2 x + \varepsilon x^3 = 0 \quad (42)$$

According to the analysis outlined in the book by Bogolyubov and Mitropolis (see Chap.1, §2 in Ref. [23]), which is actually a refinement of the Lindstedt-Poincaré method, the solution of the above equation is assumed to be of the form $x = X(\Omega t + \phi)$, where $X(\psi)$ is a periodic function with period $2\pi$ of the angular variable $\psi = \Omega t + \phi$ that satisfies the equation

$$\Omega^2 \frac{d^2 X}{d\psi^2} + \omega_w^2 X + \varepsilon X^3 = 0 \quad (43)$$

Note that by enforcing the condition of periodicity the annoying presence of secular terms is avoided. The solution $X(\psi)$ and the square of the frequency $\Omega$ are next expanded in powers of the small parameter $\varepsilon$

$$X(\psi) = \sum_{n=0}^{\infty} \varepsilon^n X_n(\psi), \quad \Omega^2 = \sum_{n=0}^{\infty} \varepsilon^n \omega_n(\psi) \quad (44)$$

Substituting (44) into (43) and equating to zero the coefficients multiplying the various powers of $\varepsilon$, the following infinite coupled set of second-order linear differential equations is obtained:
\( \varpi_0 \frac{d^2 X_0}{d\psi^2} + \omega_w^2 X_0 = 0 \)

\( \varpi_0 \frac{d^2 X_1}{d\psi^2} + \omega_w^2 X_1 = -X_0^3 - \varpi_1 \frac{d^2 X_0}{d\psi^2} \)

\( \varpi_0 \frac{d^2 X_2}{d\psi^2} + \omega_w^2 X_2 = -3X_0^2 X_1 - \varpi_2 \frac{d^2 X_0}{d\psi^2} - \varpi_1 \frac{d^2 X_1}{d\psi^2} \)

\( \varpi_0 \frac{d^2 X_3}{d\psi^2} + \omega_w^2 X_3 = -3X_0^2 X_2 - 3X_0 X_1^2 - \varpi_3 \frac{d^2 X_0}{d\psi^2} - \varpi_2 \frac{d^2 X_1}{d\psi^2} - \varpi_1 \frac{d^2 X_2}{d\psi^2} \)

From the first equation of the above set it can be easily inferred that

\[ X_0(\psi) = \cos \psi, \quad \varpi_0 = \omega_w^2 \]  

Substitution of these values in the second equation of the system (45) results in

\[ \omega_w^2 \left( \frac{d^2 X_1}{d\psi^2} + X_1 \right) = \left( \varpi_1 - \frac{3}{4} \right) \cos \psi - \frac{1}{4} \cos 3\psi \]  

By requiring the periodicity of the function \( X_1(\psi) \), the coefficient multiplying the term containing the first harmonic of the argument \( \psi \) vanish

\[ X_1 = \frac{1}{32\omega_w^2} \cos 3\psi, \quad \varpi_1 = \frac{3}{4} \]  

Substituting next \( X_0, X_1, \varpi_0 \) and \( \varpi_1 \), the third equation of (45) takes the form

\[ \omega_w^2 \left( \frac{d^2 X_2}{d\psi^2} + X_2 \right) = \left( \varpi_2 - \frac{3}{128\omega_w^2} \right) \cos \psi + \frac{21}{128\omega_w^2} \cos 3\psi - \frac{3}{128\omega_w^2} \cos 5\psi \]  

By constraining again to zero the coefficient multiplying \( \cos \psi \)

\[ X_2 = -\frac{21}{1024\omega_w^4} \cos 3\psi + \frac{1}{1024\omega_w^4} \cos 5\psi, \quad \varpi_2 = \frac{3}{128\omega_w^2} \]  

Hence, truncating the series (44) at this level, i.e. including terms up to \( \varepsilon^2 \) the approximate stationary solution of (43) can be put in the approximate form

\[ X(\psi) \approx \cos \psi + \frac{\varepsilon}{32\omega_w^2} \cos 3\psi - \frac{21\varepsilon^2}{1024\omega_w^4} \cos 3\psi + \frac{\varepsilon^2}{1024\omega_w^4} \cos 5\psi \]  

whereas the modified frequency is expressed as

\[ \Omega^2 \approx \omega_w^2 + \frac{\varepsilon}{4} + \frac{3}{128\omega_w^2} \varepsilon^2. \]
REFERENCES