THE DYNAMICAL BEHAVIORS OF BECK'S COLUMN AND TWO-LINK ROBOT ARM *VIA* GENERALIZATION OF THE SYLVESTER MATRIX EQUATIONS

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Received November 15, 2020

Abstract. In this paper, we introduce a class of generalized Sylvester matrix equations, which is frequently used in control systems. The presented matrix equation is the generalization of classical equations and the general form of previous generalizations in this field. By using the Kronecker product of matrices, we propose a method for solving the introduced matrix equations. To support the presented matrix equations and provided solution, we investigate some practical examples, such as a double inverted pendulum and two–link robot arm with two fractional dampers.

Key words: Sylvester matrix equations, Kronecker product, block pulse functions, operational matrix, Beck's column, two-link robot arm.

1. INTRODUCTION

In recent decades, the Sylvester equations have been a favorite of researchers and can be found in many scientific contexts such as control theory, noninteracting control, analysis and design of control systems, complex–valued linear periodic systems, perturbation analysis and some other scientific and engineering fields [1–13]. The various approaches have been proposed to solve the Sylvester equations, such as the projection [14], extended Arnoldi [8, 15], least squares [13, 16–23], Hessenberg and gradiant based schemes [10, 20, 24–27]. A number of scholars have proposed some methods for solving the Sylvester matrix equation (SME) [14, 15, 24, 25, 28],

$$AX + XB = C.$$

The generalized SMEs

$$AXB \pm CXD = E,$$

has been studied in [16–19]. Some other researchers have proposed the various approaches to solve the classical and generalized of the $*$-SME [13, 20, 21, 29, 30],

$$AX \pm X^{*}B = C \;\; , \;\; AXB \pm CX^{*}E = F,$$

respectively, where $*$ being either the transpose (T) or the conjugate transpose $(*)$. Also, some schemes have been used to investigate the coupled SMEs [22, 26, 31–36],

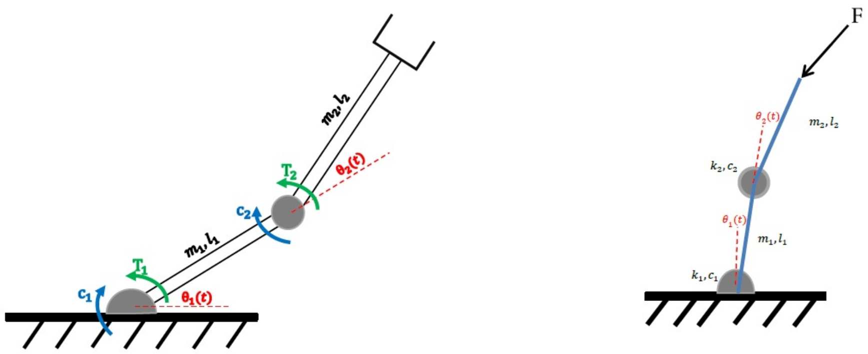
$$\begin{cases} A_1XB_1 + C_1YE_1 = F_1, \\ A_2XB_2 + C_2YE_2 = F_2. \end{cases}$$

The above system is related to the generalized coupled SMEs. For $B_1 = C_1 = B_2 = C_2 = I$, where I denotes the identity matrix, the above relation is related to the classical coupled SMEs. In all of the above equations, X, Y are unknown matrices and A, $A_{1,2}$, B, $B_{1,2}$, C, $C_{1,2}$, E, $E_{1,2}$, F, $F_{1,2}$ are known matrices and the mentioned matrices have suitable sizes. We present the following class of generalized SMEs,

$$A \begin{bmatrix} XE_1 \\ YE_2 \end{bmatrix} + B \begin{bmatrix} XF_1 \\ YF_2 \end{bmatrix} + C \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \tag{1}$$

where $X, Y \in \mathbb{R}^{1\times N}$ are unknown matrices and $G_{1,2} \in \mathbb{R}^{1\times N}$, $A, B, C \in \mathbb{R}^{2\times 2}$, $E_{1,2}, F_{1,2} \in \mathbb{R}^{N\times N}$ are known matrices. Equation (1) is equivalent to the SMEs, for $E_1 = E_2 = B = I_N, F_1 = F_2 = F$ and it is equivalent to the generalized SMEs for $E_1 = E_2 = E, F_1 = F_2 = F$. This kind of equations are found in many engineering issues such as the two–link robot arm and double inverted pendulum.

In this paper, we consider the two–link robot arm and the double inverted pendulum



(a) (b)

Fig. 1 – (a) The two–link robot arm and (b) the double inverted pendulum.

shown in Fig. 1, that by converting the dynamic equations obtained from them into matrix equations, we see that it is similar to the equation (1). We propose a method for solving equation (1), obviously by using that we can solve Sylvester equations in the classical and generalized form.

The remain of this paper is organized as follows. In Section 2, we introduce some preliminaries and use them to solve the equations (1). This section consists of two subsection, fractional calculus and approximation of functions using the block pulse function. Section 3 explains our proposed method for solving matrix equations (1). In Section 4 we present some dynamical systems that verify the application of the suggested generalized matrix equations. The conclusion is represented in the last section.

2. PRELIMINARIES

We give some essential definitions of fractional calculus and approximation of functions.

2.1. FRACTIONAL CALCULUS

In this paper, we use the fractional Riemann–Liouville integral and Caputo derivative of order $n \in \mathbb{R}_+$ on $f(t) \in L_1[a,b]$, with denoting $\mathfrak{I}_a^n f(t)$ and ${}^C\mathfrak{D}_a^n$, respectively [37].

Theorem 1. *Let* $n \geq 0$, $m = \lceil n \rceil$ *and* $f \in A^m[a,b]$, *then*

$$\mathfrak{I}_a^n \, {}^C\mathfrak{D}_a^n f(t) = f(t) - \sum_{k=0}^{m-1} \frac{D^k f(a)}{k!} \Big(t-a\Big)^k. \tag{2}$$

Proof. See [37]. □

2.2. BLOCK PULSE FUNCTIONS

The block pulse function $\Phi_m(t)$ of these basis functions is defined as

$$\Phi_m(t) = [\phi_0(t), \phi_1(t), \cdots, \phi_{m-1}(t)]^T,$$

such that

$$\phi_i(t) = \begin{cases} 1, & \frac{iT}{m} \leq t < \frac{(i+1)T}{m}, \\ 0, & \text{otherwise}, \end{cases} \qquad i = 0, 1, \ldots, m-1$$

where $t \in [0, T)$. The block pulse functions have the following characteristics,

- Disjointness

$$\phi_i(t)\phi_j(t) = \begin{cases} \phi_i(t), & i = j \\ 0, & \text{otherwise} \end{cases},$$

- Discrete Orthogonality Relationship

$$\sum_{l=0}^{m-1} \phi_i(t_l)\phi_j(t_l) = \delta_{ij}.$$

2.2.1. Expansion of functions

The function $f(t)$ can be expanded by the block pulse basis functions as follows,

$$f(t) \approx \sum_{i=0}^{m-1} f_i \phi_i(t) = [f_0, f_1, \cdots, f_{m-1}]\Phi(t), \tag{3}$$

where

$$f_i = \sum_{l=0}^{m-1} \phi_i(t_l) f(t_l) = f(t_i), \qquad i = 0, 1, \cdots, m-1.$$

t_is, are called the collocation points and are defined as $t_i \in [ih, (i+1)h)$ and $h = \frac{T}{m}$. Therefore, we can rewrite the relation (3) as

$$f(t) \approx F\Phi(t),$$

where $F = [f(t_0), f(t_1), \ldots, f(t_{m-1})]$.

2.2.2. Operational matrix

In this part, we introduce the types of operational matrix of the block pulse functions. The block pulse integration operational matrix $\mathbf{P}$ is defined as

$$\mathbf{P} = h \begin{bmatrix} \frac{1}{2} & 1 & 1 & \cdots & 1 \\ 0 & \frac{1}{2} & 1 & \cdots & 1 \\ 0 & 0 & \frac{1}{2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \frac{1}{2} \end{bmatrix}.$$

Considering the initial conditions of zero, the nth order derivative approximation of function $f(t)$ is

$$f^{(n)}(t) \approx F\left(\mathbf{P}^{-1}\right)^n \Phi(t).$$

Also, the fractional block pulse integration operational matrix $\mathbf{P}^\alpha$ is defined as

$$\mathbf{P}^\alpha = \frac{h^\alpha}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \zeta_1 & \zeta_2 & \zeta_3 & \cdots & \zeta_{m-1} \\ 0 & 1 & \zeta_1 & \zeta_2 & \cdots & \zeta_{m-2} \\ 0 & 0 & 1 & \zeta_1 & \cdots & \zeta_{m-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \zeta_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

where

$$\zeta_i = (i+1)^{\alpha+1} - 2i^{\alpha+1} + (i-1)^{\alpha+1}, \quad i = 1, 2, \ldots, m-1.$$

Considering relation (2) and the initial conditions of zero, approximation of fractional–order derivative of function f(t) is

$$^{C}\mathfrak{D}^{\alpha} f(t) \approx F\mathbf{D}^{\alpha}\Phi(t),$$

where $\mathbf{D}^{\alpha} = \left(\mathbf{P}^{\alpha}\right)^{-1}$ and can be obtained as follows

$$\mathbf{D}^{\alpha} = \frac{\Gamma(\alpha+2)}{h^{\alpha}} \begin{bmatrix} 1 & d_1 & d_2 & d_3 & \cdots & d_{m-1} \\ 0 & 1 & d_1 & d_2 & \cdots & d_{m-2} \\ 0 & 0 & 1 & d_1 & \cdots & d_{m-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & d_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

where

$$d_k = -\Big(\sum_{i=0}^{k-1} \zeta_i d_{k-i} + \zeta_k\Big), \quad k = 1, 2, \ldots, m-1.$$

3. PROPOSED METHOD

In this section, we introduce our presented method for solving a class of the generalized SMEs which appear in mechanical problems. These equations are more general than the generalized SMEs, and can be converted into the SMEs of classical and generalized forms by considering the certain states. Therefore, we can solve the classical and previous generalized SMEs by using the presented method. Firstly, we need to prove the following theorem.

Theorem 2. *Let $A \in \mathbb{R}^{2\times 2}$, $P_1, P_2 \in \mathbb{R}^{N\times N}$, $\mathfrak{M} \in \mathbb{R}^{2N\times 2N}$ are known matrices and $X, Y \in \mathbb{R}^{1\times N}$ are unknown matrices. Then, we have,*

$$A \begin{bmatrix} XP_1 \\ YP_2 \end{bmatrix} \equiv \mathfrak{M} \begin{bmatrix} X^T \\ Y^T \end{bmatrix},$$

where

$$\mathfrak{M} = A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes P_1^T + A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes P_2^T,$$

and $\otimes$ denotes the Kronecker product of matrices.

Proof. Let

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ X=[x_1,x_2,\ldots,x_N],\ Y=[y_1,y_2,\ldots,y_N],$$

$$P_1=\begin{bmatrix} p^1_{11} & p^1_{12} & \cdots & p^1_{1N} \\ p^1_{21} & p^1_{22} & \cdots & p^1_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p^1_{N1} & p^1_{N2} & \cdots & p^1_{NN} \end{bmatrix},\ P_2=\begin{bmatrix} p^2_{11} & p^2_{12} & \cdots & p^2_{1N} \\ p^2_{21} & p^2_{22} & \cdots & p^2_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p^2_{N1} & p^2_{N2} & \cdots & p^2_{NN} \end{bmatrix},$$

Then, we have

$$A\begin{bmatrix} XP_1 \\ YP_2 \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} [x_1,x_2,\ldots,x_N]\begin{bmatrix} p^1_{11} & p^1_{12} & \cdots & p^1_{1N} \\ p^1_{21} & p^1_{22} & \cdots & p^1_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p^1_{N1} & p^1_{N2} & \cdots & p^1_{NN} \end{bmatrix} \\ \\ [y_1,y_2,\ldots,y_N]\begin{bmatrix} p^2_{11} & p^2_{12} & \cdots & p^2_{1N} \\ p^2_{21} & p^2_{22} & \cdots & p^2_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p^2_{N1} & p^2_{N2} & \cdots & p^2_{NN} \end{bmatrix} \end{bmatrix}$$

$$=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \sum_{i=1}^{N} x_i p^1_{i1} & \sum_{i=1}^{N} x_i p^1_{i2} & \cdots & \sum_{i=1}^{N} x_i p^1_{iN} \\ \sum_{i=1}^{N} y_i p^2_{i1} & \sum_{i=1}^{N} y_i p^2_{i2} & \cdots & \sum_{i=1}^{N} y_i p^2_{iN} \end{bmatrix}$$

$$=\begin{bmatrix} \sum_{i=1}^{N} a_{11}x_i p^1_{i1}+a_{12}y_i p^2_{i1} & \sum_{i=1}^{N} a_{11}x_i p^1_{i2}+a_{12}y_i p^2_{i2} & \cdots & \sum_{i=1}^{N} a_{11}x_i p^1_{iN}+a_{12}y_i p^2_{iN} \\ \sum_{i=1}^{N} a_{21}x_i p^1_{i1}+a_{22}y_i p^2_{i1} & \sum_{i=1}^{N} a_{21}x_i p^1_{i2}+a_{22}y_i p^2_{i2} & \cdots & \sum_{i=1}^{N} a_{21}x_i p^1_{iN}+a_{22}y_i p^2_{iN} \end{bmatrix}$$

$$=\begin{bmatrix} \mathcal{A}_{1\times N} \\ \mathcal{B}_{1\times N} \end{bmatrix} \tag{4}$$

Now, by converting the above matrix into a vector with $2N$ components, we obtain

$$\begin{bmatrix}\mathcal{A}^T\\ \mathcal{B}^T\end{bmatrix} = \begin{bmatrix} \sum_{i=1}^N a_{11}x_ip_{i1}^1 + a_{12}y_ip_{i1}^2 \\ \sum_{i=1}^N a_{11}x_ip_{i2}^1 + a_{12}y_ip_{i2}^2 \\ \vdots \\ \sum_{i=1}^N a_{11}x_ip_{iN}^1 + a_{12}y_ip_{iN}^2 \\ \sum_{i=1}^N a_{21}x_ip_{i1}^1 + a_{22}y_ip_{i1}^2 \\ \sum_{i=1}^N a_{21}x_ip_{i2}^1 + a_{22}y_ip_{i2}^2 \\ \vdots \\ \sum_{i=1}^N a_{21}x_ip_{iN}^1 + a_{22}y_ip_{iN}^2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}p_{11}^1 & a_{11}p_{21}^1 & \cdots & a_{11}p_{N1}^1 & a_{12}p_{11}^2 & a_{12}p_{21}^2 & \cdots & a_{12}p_{N1}^2 \\ a_{11}p_{12}^1 & a_{11}p_{22}^1 & \cdots & a_{11}p_{N2}^1 & a_{12}p_{12}^2 & a_{12}p_{22}^2 & \cdots & a_{12}p_{N2}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{11}p_{1N}^1 & a_{11}p_{2N}^1 & \cdots & a_{11}p_{NN}^1 & a_{12}p_{1N}^2 & a_{12}p_{2N}^2 & \cdots & a_{12}p_{NN}^2 \\ a_{21}p_{11}^1 & a_{21}p_{21}^1 & \cdots & a_{21}p_{N1}^1 & a_{22}p_{11}^2 & a_{22}p_{21}^2 & \cdots & a_{22}p_{N1}^2 \\ a_{21}p_{12}^1 & a_{21}p_{22}^1 & \cdots & a_{21}p_{N2}^1 & a_{22}p_{12}^2 & a_{22}p_{22}^2 & \cdots & a_{22}p_{N2}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{21}p_{1N}^1 & a_{21}p_{2N}^1 & \cdots & a_{21}p_{NN}^1 & a_{22}p_{1N}^2 & a_{22}p_{2N}^2 & \cdots & a_{22}p_{NN}^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}P_1^T & a_{12}P_2^T \\ a_{21}P_1^T & a_{22}P_2^T \end{bmatrix} \begin{bmatrix} X^T \\ Y^T \end{bmatrix} = \left(\begin{bmatrix} a_{11}P_1^T & 0 \\ a_{21}P_1^T & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12}P_2^T \\ 0 & a_{22}P_2^T \end{bmatrix} \right) \begin{bmatrix} X^T \\ Y^T \end{bmatrix}$$

$$= \left(\begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} \otimes P_1^T + \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix} \otimes P_2^T \right) \begin{bmatrix} X^T \\ Y^T \end{bmatrix}$$

$$= \left(A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes P_1^T + A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes P_2^T \right) \begin{bmatrix} X^T \\ Y^T \end{bmatrix} = \mathfrak{M} \begin{bmatrix} X^T \\ Y^T \end{bmatrix}.$$

□

Using Theorem 2, we can represent the following corollary.

Corollary 1. *Let* $A,B,C \in \mathbb{R}^{2\times 2}$*,* $E_{1,2},F_{1,2} \in \mathbb{R}^{N\times N}$*,* $G_{1,2} \in \mathbb{R}^{1\times N}$ *are known matrices and* $X,Y \in \mathbb{R}^{1\times N}$ *are unknown matrices. To solve the matrix equations* (1)*,* *we rewrite these matrix equations as following*

$$\mathfrak{M}_A \begin{bmatrix} X^T \\ Y^T \end{bmatrix} + \mathfrak{M}_B \begin{bmatrix} X^T \\ Y^T \end{bmatrix} + \mathfrak{M}_C \begin{bmatrix} X^T \\ Y^T \end{bmatrix} = \begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix}$$

where

$$\mathfrak{M}_A = A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes E_1^T + A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes E_2^T,$$

$$\mathfrak{M}_B = B \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes F_1^T + B \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes F_2^T,$$

$$\mathfrak{M}_C = C \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_N + C \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes I_N.$$

So, for solving matrix equations (1) *we have*

$$\Big(\mathfrak{M}_A + \mathfrak{M}_B + \mathfrak{M}_C\Big) \begin{bmatrix} X^T \\ Y^T \end{bmatrix} = \begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix}.$$

Remark 1. *The presented method in Theorem 2 can be generalized. In other words, we can solve the generalization of matrix equations* (1) *by using the proposed method.*

Theorem 3. *Let $A \in \mathbb{R}^{m\times m}$, $P_i \in \mathbb{R}^{N\times N}$, for $i = 1,2,3,\ldots,m$, $\mathfrak{M} \in \mathbb{R}^{mN\times mN}$ are known matrices and $X_i \in \mathbb{R}^{1\times N}$, for $i = 1,2,3,\ldots,m$, are unknown matrices. Then, we have,*

$$A \begin{bmatrix} X_1P_1 \\ X_2P_2 \\ X_3P_3 \\ \vdots \\ X_mP_m \end{bmatrix} \equiv \mathfrak{M} \begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \\ \vdots \\ X_m^T \end{bmatrix},$$

where

$$\mathfrak{M} = \sum_{i=1}^{m} AD_i \otimes P_i^T,$$

and $D_i = [d_{ij}]_{m\times m}$ is a diagonal matrix whose elements are all zero except $d_{ii} = 1$.

Proof. Proof of this theorem is similar to proof of Theorem 2. □

4. NUMERICAL SIMULATIONS

Mathematical model of many engineering issues is equations system [38–42] and in many of these issues we can convert equations system into fractional–order

equations system by using fractional calculus [43–54]. Then, to analysis engineering issues, we converting equations system into matrix equation.
In this section, to support matrix equations (1), we present some practical examples. First, we convert the problem to matrix equations (1), then we can find solution of matrix equations by applying the proposed method.
Example 1. *Consider the Beck's column, we can model it via a double inverted pendulum that has two fractional dampers (see Fig. 2). The linearized equations*

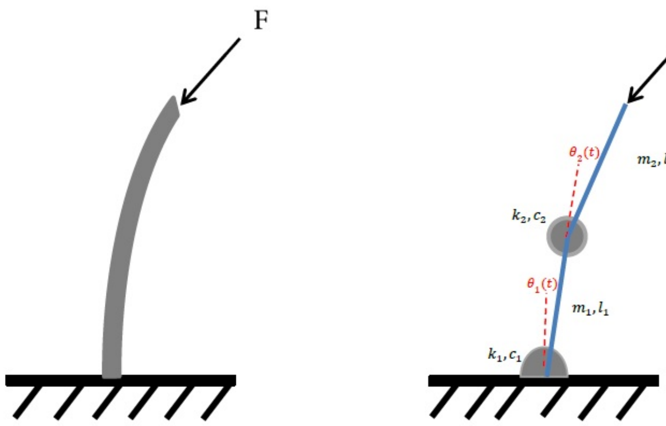


Fig. 2 – The Beck's column and a double inverted pendulum that has two fractional dampers.

around the zero angle related to model of Beck's column are as follows [55]

$$A\begin{bmatrix}\ddot{\theta}_1(t)\\ \ddot{\theta}_2(t)\end{bmatrix}+B\begin{bmatrix}\theta_1(t)\\ \theta_2(t)\end{bmatrix}+C\begin{bmatrix}{}^{C}\mathfrak{D}^{\alpha}\theta_1(t)\\ {}^{C}\mathfrak{D}^{\beta}\theta_2(t)\end{bmatrix}=\begin{bmatrix}F_1(t)\\ F_2(t)\end{bmatrix}, \tag{5}$$

where $A,B,C\in\mathbb{R}^{2\times 2}$ *are known matrices,* $F_1(t),F_2(t)$ *are known functions of time and* $\theta_1(t),\theta_2(t)$ *are unknown functions of time. Using the operatinal matrix and function expansion, we can rewrite equation* (5) *with zero initial conditions,*

$$A\begin{bmatrix}\Theta_1\left(\mathbf{P}^{-1}\right)^2\Phi(t)\\ \Theta_2\left(\mathbf{P}^{-1}\right)^2\Phi(t)\end{bmatrix}+B\begin{bmatrix}\Theta_1\Phi(t)\\ \Theta_2\Phi(t)\end{bmatrix}+C\begin{bmatrix}\Theta_1\mathbf{D}^{\alpha}\Phi(t)\\ \Theta_2\mathbf{D}^{\beta}\Phi(t)\end{bmatrix}\approx\begin{bmatrix}\mathbf{F}_1\Phi(t)\\ \mathbf{F}_2\Phi(t)\end{bmatrix},$$

where $\theta_i(t)\approx\Theta_i\Phi(t)$, $F_i(t)\approx\mathbf{F}_i\Phi(t)$, $i=1,2$. *By simplifying the* $\Phi(t)$ *from both sides of the above relation, we have*

$$A\begin{bmatrix}\Theta_1\left(\mathbf{P}^{-1}\right)^2\\ \Theta_2\left(\mathbf{P}^{-1}\right)^2\end{bmatrix}+B\begin{bmatrix}\Theta_1\\ \Theta_2\end{bmatrix}+C\begin{bmatrix}\Theta_1\mathbf{D}^{\alpha}\\ \Theta_2\mathbf{D}^{\beta}\end{bmatrix}\approx\begin{bmatrix}\mathbf{F}_1\\ \mathbf{F}_2\end{bmatrix}. \tag{6}$$

Clearly, matrix equations (6) *equivalent to matrix equations* (1). *Then, using the proposed method in Theorem 2, we obtain*

$$\mathfrak{M}_A \begin{bmatrix} \Theta_1^T \\ \Theta_2^T \end{bmatrix} + \mathfrak{M}_B \begin{bmatrix} \Theta_1^T \\ \Theta_2^T \end{bmatrix} + \mathfrak{M}_C \begin{bmatrix} \Theta_1^T \\ \Theta_2^T \end{bmatrix} \approx \begin{bmatrix} \mathbf{F}_1^T \\ \mathbf{F}_2^T \end{bmatrix},$$

where

$$\mathfrak{M}_A = A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \left((\mathbf{P}^{-1})^2\right)^T + A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left((\mathbf{P}^{-1})^2\right)^T,$$

$$\mathfrak{M}_B = B \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes I_N + B \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes I_N,$$

$$\mathfrak{M}_C = C \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \left(\mathbf{D}^{\alpha}\right)^T + C \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left(\mathbf{D}^{\beta}\right)^T.$$

Therefore, we have

$$\left(\mathfrak{M}_A + \mathfrak{M}_B + \mathfrak{M}_C\right) \begin{bmatrix} \Theta_1^T \\ \Theta_2^T \end{bmatrix} \approx \begin{bmatrix} \mathbf{F}_1^T \\ \mathbf{F}_2^T \end{bmatrix}.$$

Example 2. *In this example, we consider two-link robot arm with two fractional dampers (see Fig. 1a). The linearized equations of motion of two-link arm are as follows,*

$$A \begin{bmatrix} \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{bmatrix} + B \begin{bmatrix} \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \end{bmatrix} + C \begin{bmatrix} {}^C\mathfrak{D}^{\alpha}\theta_1(t) \\ {}^C\mathfrak{D}^{\beta}\theta_2(t) \end{bmatrix} = \begin{bmatrix} T_1(t) \\ T_2(t) \end{bmatrix}, \tag{7}$$

where $A, B, C \in \mathbb{R}^{2\times 2}$ *are known matrices,* $T_1(t), T_2(t)$ *are known functions and* $\theta_1(t), \theta_2(t)$ *are unknown functions. Similarly, consideration the initial conditions of zero, we can rewrite equation* (7)*,*

$$A \begin{bmatrix} \Theta_1\left(\mathbf{P}^{-1}\right)^2 \\ \Theta_2\left(\mathbf{P}^{-1}\right)^2 \end{bmatrix} + B \begin{bmatrix} \Theta_1\mathbf{P}^{-1} \\ \Theta_2\mathbf{P}^{-1} \end{bmatrix} + C \begin{bmatrix} \Theta_1\mathbf{D}^{\alpha} \\ \Theta_2\mathbf{D}^{\beta} \end{bmatrix} \approx \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix},$$

where $T_i(t) \approx \mathbf{T}_i\Phi(t),\ i = 1, 2$. *Therefore, according to Theorem 2, we have*

$$\left(\mathfrak{M}_A + \mathfrak{M}_B + \mathfrak{M}_C\right) \begin{bmatrix} \Theta_1^T \\ \Theta_2^T \end{bmatrix} \approx \begin{bmatrix} \mathbf{T}_1^T \\ \mathbf{T}_2^T \end{bmatrix},$$

where

$$\mathfrak{M}_A = A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \left((\mathbf{P}^{-1})^2\right)^T + A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left((\mathbf{P}^{-1})^2\right)^T,$$

$$\mathfrak{M}_B = B \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \left(\mathbf{P}^{-1}\right)^T + B \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left(\mathbf{P}^{-1}\right)^T,$$

$$\mathfrak{M}_C = C\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\otimes\left(\mathbf{D}^{\alpha}\right)^T + C\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\otimes\left(\mathbf{D}^{\beta}\right)^T.$$

5. CONCLUSION

In the last few decades, Sylvester equations have been a popular of scholars and can be seen in many scientific subjects. In this paper, we considered a class of generalized Sylvester matrix equations that is seen in the process of solving some applied problems. This type of matrix equations are more general than the generalized Sylvester matrix equations. Also, we proposed a method to solving these matrix equations. Using this method, we can solve Sylvester matrix equations of classical and generalized forms. Our proposed method can lead to other studies to investigate the behavior of dynamic systems that the required model corresponds to the matrix equation studied in this paper.

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