AN ANALYTICAL STUDY OF $(2+1)$-DIMENSIONAL PHYSICAL MODELS EMBEDDED ENTIRELY IN FRACTAL SPACE

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Abstract. In this article, we analytically furnish the solution of $(2+1)$-dimensional fractional differential equations, with distinct fractal-memory indices in all coordinates, as a trivariate $(\alpha,\beta,\gamma)-$fractional power series representation. The method is tested on several physical models with inherited memories. Moreover, a version of Taylor's theorem in fractal three-dimensional space is presented. As a special case, the solutions of the corresponding integer-order cases are extracted by letting $\alpha, \beta, \gamma \to 1$, which indicates to some extent for a sequential memory.

Key words: Memory index (fractional derivative); Fractional partial differential equations; Solutions in closed form.

1. INTRODUCTION

The prominence of non-integer derivatives is related to the appearance of particular materials that is affected by a memory during deformation (called memory index) [1]. This memory can be interpreted as a chaotic or harmonic behavior during a short time subject to different types of local initial conditions. To study such behavior, several phenomena that are modeled by differential equations have been adapted to comprise fractional derivatives and, consequently, some classical approaches have been adapted to solve these equations [2–24].

Mainly, most of these fractional models have been studied for the case that a particular variable possesses memory index. However, to the best of our knowledge, no hybrid fractional models with inherited fractional derivatives in all variable coordinates are considered.

Inspired by this idea, we here present an analytical representation in terms of a new power series expansion to study and address the influence of the fractional derivative on all variable coordinates involved in the model. Then, we exploit the corresponding Taylor power series scheme to evolve closed-form and approximate analytic solutions for various $(2+1)-$dimensional fractional differential equations with embedded memory index in all coordinates.

In the literature, diversified definitions of fractional derivative have been suggested by several authors. However, in this work, we consider the fractional derivative in the Caputo sense which is known as

$$\mathcal{D}_t^{\alpha}\left[\psi(x,y,t)\right] = \frac{\partial^{\alpha}\psi(x,y,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial\psi(x,y,\tau)}{\partial\tau}\frac{d\tau}{(t-\tau)^{\alpha}}, \tag{1}$$

where $0<\alpha<1$ is the Caputo derivative order. An explicit application of the Caputo definition gives

$$\mathcal{D}_t^{\alpha}\left[t^{a}\right] = \begin{cases} \dfrac{\Gamma(a+1)}{\Gamma(a-\alpha+1)}\,t^{a-\alpha} & ; a>0 \\ 0 & ; a=0. \end{cases} \tag{2}$$

For further details about the fractional calculus and its applications, we refer the readers to [25, 26].

2. REPRESENTATION WITH TRIFOLD CAPUTO DERIVATIVE ORDERING

In this Section, we provide an analytic representation for the solution of $(2+1)$-dimensional fractional differential equations with embedded memory index in all coordinates. This representation is based on a new version of fractional power series expansion that involves trifold Caputo fractional derivative parameters, namely $\alpha,\beta,\gamma\in(0,1]$. As a result, the Taylor's theorem regarding this form is reformulated by means of mixed Caputo derivatives.

Definition 2.1. An $(\alpha,\beta,\gamma)-$fractional power series representation ($(\alpha,\beta,\gamma)-$FPS) around $(0,0,0)$ is a power series in the form

$$\begin{aligned}\sum_{\substack{n+m+k=0\\ n,m,k\in\mathbb{N}_0}}^{\infty}\lambda_{nmk}\,t^{n\alpha}x^{m\beta}y^{k\gamma} &= \underbrace{\lambda_{000}}_{n+m+k=0} + \underbrace{\lambda_{100}\,t^{\alpha}+\lambda_{010}\,x^{\beta}+\lambda_{001}\,y^{\gamma}}_{n+m+k=1} + \cdots \\ &\quad + \underbrace{\sum_{i=0}^{N}\sum_{j=0}^{i}\lambda_{N-i,i-j,j}\,t^{(N-i)\alpha}x^{(i-j)\beta}y^{j\gamma}}_{n+m+k=N} + \cdots,\end{aligned} \tag{3}$$

where $\alpha,\beta,\gamma\in(0,1]$, $(x,y,t)\geq(0,0,0)$ are variables, and λ_{nmk}'s are real constants.

We remark here that the expansion (3) can be alternatively expressed as

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{m}\lambda_{n-m,m-k,k}\,t^{(n-m)\alpha}x^{(m-k)\beta}y^{k\gamma}. \tag{4}$$

Lemma 2.2. *Suppose $\psi(x,y,t)$ has an $(\alpha,\beta,\gamma)-$FPS representation for $(x,y,t)\in[0,\mathcal{R}_x)\times[0,\mathcal{R}_y)\times[0,\mathcal{R}_t)$. If $\mathcal{D}_t^{i\alpha}\mathcal{D}_x^{j\beta}\mathcal{D}_y^{r\gamma}\left[\psi(x,y,t)\right]$ are continuous on $(0,\mathcal{R}_x)\times$*

$(0,\mathcal{R}_y)\times(0,\mathcal{R}_t)$ *for* $(i,j,r)\in\mathbb{N}_0^3$, *then*

$$\mathcal{D}_t^{i\alpha}\mathcal{D}_x^{j\beta}\mathcal{D}_y^{r\gamma}\left[\psi(x,y,t)\right]=\\\sum_{n+m+k=0}^{\infty}\lambda_{n+i,m+j,k+r}\frac{\Gamma\big((n+i)\alpha+1\big)\Gamma\big((m+j)\beta+1\big)\Gamma\big((k+r)\gamma+1\big)}{\Gamma(n\alpha+1)\Gamma(m\beta+1)\Gamma(k\gamma+1)}t^{n\alpha}x^{m\beta}y^{k\gamma}. \tag{5}$$

Proof. The proof is by induction on the parameters (i,j,r). For the base case $(i,j,r)=(1,1,1)$, by term-by-term Caputo differentiation with taking into account (2) and then shifting the indices, we have

$$\begin{aligned}
\mathcal{D}_t^{\alpha}\mathcal{D}_x^{\beta}\mathcal{D}_y^{\gamma}\big[\psi(x,y,t)\big] &= \mathcal{D}_t^{\alpha}\mathcal{D}_x^{\beta}\left[\sum_{\substack{n+m+k=1\\k\geq 1}}^{\infty}\lambda_{nmk}\frac{\Gamma\big(k\gamma+1\big)}{\Gamma((k-1)\gamma+1)}t^{n\alpha}x^{m\beta}y^{(k-1)\gamma}\right]\\
&=\mathcal{D}_t^{\alpha}\mathcal{D}_x^{\beta}\left[\sum_{n+m+k=0}^{\infty}\lambda_{n,m,k+1}\frac{\Gamma\big((k+1)\gamma+1\big)}{\Gamma(k\gamma+1)}t^{n\alpha}x^{m\beta}y^{k\gamma}\right]\\
&=\mathcal{D}_t^{\alpha}\left[\sum_{\substack{n+m+k=1\\m\geq 1}}^{\infty}\lambda_{n,m,k+1}\frac{\Gamma\big(m\beta+1\big)\Gamma\big((k+1)\gamma+1\big)}{\Gamma((m-1)\beta+1)\Gamma(k\gamma+1)}t^{n\alpha}x^{(m-1)\beta}y^{k\gamma}\right]\\
&=\mathcal{D}_t^{\alpha}\left[\sum_{n+m+k=0}^{\infty}\lambda_{n,m+1,k+1}\frac{\Gamma\big((m+1)\beta+1\big)\Gamma\big((k+1)\gamma+1\big)}{\Gamma(m\beta+1)\Gamma(k\gamma+1)}t^{n\alpha}x^{m\beta}y^{k\gamma}\right]\\
&=\sum_{\substack{n+m+k=1\\n\geq 1}}^{\infty}\lambda_{n,m+1,k+1}\frac{\Gamma\big(n\alpha+1\big)\Gamma\big((m+1)\beta+1\big)\Gamma\big((k+1)\gamma+1\big)}{\Gamma\big((n-1)\alpha+1\big)\Gamma(m\beta+1)\Gamma(k\gamma+1)}t^{(n-1)\alpha}x^{m\beta}y^{k\gamma}\\
&=\sum_{n+m+k=0}^{\infty}\lambda_{n+1,m+1,k+1}\frac{\Gamma\big((n+1)\alpha+1\big)\Gamma\big((m+1)\beta+1\big)\Gamma\big((k+1)\gamma+1\big)}{\Gamma(n\alpha+1)\Gamma(m\beta+1)\Gamma(k\gamma+1)}t^{n\alpha}x^{m\beta}y^{k\gamma}.
\end{aligned} \tag{6}$$

Now, suppose that the assertion is valid for (i,j,r). Then, we need to show its validity when $(i+1,j,r)$, $(i,j+1,r)$, and $(i,j,r+1)$. It suffices to show the case $(i+1,j,r)$ and the other cases are each processed in a similar fashion because of the commutativity of the independent operators $\left\{\mathcal{D}_t^{\alpha},\mathcal{D}_x^{\beta},\mathcal{D}_y^{\gamma}\right\}$. By term-by-term Caputo differentiation and then exchanging the index $n-1$ by n, we have

$$\mathcal{D}_t^{(i+1)\alpha}\mathcal{D}_x^{j\beta}\mathcal{D}_y^{r\gamma}\big[\psi(x,y,t)\big]=\mathcal{D}_t^{\alpha}\Big[\mathcal{D}_t^{i\alpha}\mathcal{D}_x^{j\beta}\mathcal{D}_y^{r\gamma}\big[\psi(x,y,t)\big]\Big]=$$

$$= \mathcal{D}_t^{\alpha}\left[\sum_{n+m+k=0}^{\infty} \lambda_{n+i,m+j,k+r} \frac{\Gamma\big((n+i)\alpha+1\big)\Gamma\big((m+j)\beta+1\big)\Gamma\big((k+r)\gamma+1\big)}{\Gamma(n\alpha+1)\Gamma(m\beta+1)\Gamma(k\gamma+1)} t^{n\alpha}x^{m\beta}y^{k\gamma}\right] = \sum_{\substack{n+m+k=1\\ n\geq 1}}^{\infty} \lambda_{n+i,m+j,k+r}$$

$$\frac{\Gamma\big((n+i)\alpha+1\big)\Gamma\big((m+j)\beta+1\big)\Gamma\big((k+r)\gamma+1\big)}{\Gamma((n-1)\alpha+1)\Gamma(m\beta+1)\Gamma(k\gamma+1)} t^{(n-1)\alpha}x^{m\beta}y^{k\gamma} =$$

$$= \sum_{n+m+k=0}^{\infty} \lambda_{n+i+1,m+j,k+r} \frac{\Gamma\big((n+i+1)\alpha+1\big)\Gamma\big((m+j)\beta+1\big)\Gamma\big((k+r)\gamma+1\big)}{\Gamma(n\alpha+1)\Gamma(m\beta+1)\Gamma(k\gamma+1)} t^{n\alpha}x^{m\beta}y^{k\gamma}, \quad (7)$$

as required. □

The next result presents the Taylor's theorem in fractal three-dimensional (3D) space as a consequence of Lemma 2.2.

Theorem 2.3. *Suppose* $\psi(x,y,t)$ *has an* $(\alpha,\beta,\gamma)-$*FPS representation for* $(x,y,t) \in [0,\mathcal{R}_x)\times[0,\mathcal{R}_y)\times[0,\mathcal{R}_t)$. *If* $\mathcal{D}_t^{n\alpha}\mathcal{D}_x^{m\beta}\mathcal{D}_y^{k\gamma}\big[\psi(x,y,t)\big]$ *are continuous on* $(0,\mathcal{R}_x)\times(0,\mathcal{R}_y)\times(0,\mathcal{R}_t)$ *for* $(n,m,k)\in\mathbb{N}_0^3$, *then*

$$\psi(x,y,t) = \sum_{n+m+k=0}^{\infty} \frac{\mathcal{D}_t^{n\alpha}\mathcal{D}_x^{m\beta}\mathcal{D}_y^{k\gamma}\big[\psi(x,y,t)\big]\big|_{(x,y,t)=(0,0,0)}}{\Gamma(n\alpha+1)\Gamma(m\beta+1)\Gamma(k\gamma+1)} t^{n\alpha}x^{m\beta}y^{k\gamma}. \quad (8)$$

Proof. The assertion follows immediately by substituting $(x,y,t)=(0,0,0)$ into (5). □

3. FRACTIONAL PHYSICAL MODELS WITH $(\alpha,\beta,\gamma)-$MEMORY INDICES

We aim in this Section to present a closed-form series solution for a class of $(2+1)-$dimensional physical models with embedded memory index in all coordinates. The $(\alpha,\beta,\gamma)-$FPS (3) will be utilized for this purpose together with the Taylor series solution scheme. For validation and comparison purposes, we choose to recall four models (diffusion, wave-like, telegraph, and Burgers') that have been recently solved in fractal 2D space [27]. In all our illustrative models, we assume that $\alpha,\beta,\gamma \in (0,1]$ and $x,y,t \geq 0$.

Example 1. Consider the following initial value $(\alpha,\beta,\gamma)-$fractional diffusion problem:

$$\begin{aligned} &\mathcal{D}_t^{\alpha}\big[\psi(x,y,t)\big] = \mathcal{D}_x^{2\beta}\big[\psi(x,y,t)\big] + \mathcal{D}_y^{2\gamma}\big[\psi(x,y,t)\big] \\ &\psi(x,y,0) = E_\beta\big(x^\beta\big)E_\gamma\big(y^\gamma\big), \end{aligned} \quad (9)$$

where $E_\beta(\cdot)$ is the Mittag-Leffler function. By assuming that equation (9) has a solution form of (3), we have from the initial condition that $\lambda_{0mk} = \frac{1}{\Gamma(m\beta+1)\Gamma(k\gamma+1)}$ for all $m,k \geq 0$. Now, substituting the pertinent formulas (5) into (9) and collecting terms with common powers will give the following recursion formula

$$\frac{\Gamma\big((n+1)\alpha+1\big)}{\Gamma\big(n\alpha+1\big)}\lambda_{n+1,m,k} - \frac{\Gamma\big((m+2)\beta+1\big)}{\Gamma\big(m\beta+1\big)}\lambda_{n,m+2,k} - \frac{\Gamma\big((k+2)\gamma+1\big)}{\Gamma\big(k\gamma+1\big)}\lambda_{n,m,k+2} = 0. \tag{10}$$

Consequently, this leads to

$$\lambda_{nmk} = \frac{2^n}{\Gamma\big(n\alpha+1\big)\Gamma\big(m\beta+1\big)\Gamma\big(k\gamma+1\big)}. \tag{11}$$

Plugging (11) back in (3) gives the closed-form solution

$$\begin{aligned}\psi(x,y,t) &= \sum_{n+m+k=0}^{\infty}\frac{2^n}{\Gamma\big(n\alpha+1\big)\Gamma\big(m\beta+1\big)\Gamma\big(k\gamma+1\big)}t^{n\alpha}x^{m\beta}y^{k\gamma}\\ &= \left(\sum_{m=0}^{\infty}\frac{x^{m\beta}}{\Gamma\big(m\beta+1\big)}\right)\left(\sum_{k=0}^{\infty}\frac{y^{k\gamma}}{\Gamma\big(k\gamma+1\big)}\right)\left(\sum_{n=0}^{\infty}\frac{\big(2t^\alpha\big)^n}{\Gamma\big(n\alpha+1\big)}\right)\\ &= E_\beta\big(x^\beta\big)E_\gamma\big(y^\gamma\big)E_\alpha\big(2t^\alpha\big).\end{aligned} \tag{12}$$

In particular, when $\beta,\gamma \to 1$, we obtain the time-fractional solution $\psi(x,y,t) = e^{x+y}E_\alpha\big(2t^\alpha\big)$ that is analogous to the attained one *via* the fractional reduced differential transform method (RDTM) [28] and the homotopy perturbation method (HPM) [29]. We also point out here that, when $\gamma \to 1$, the form (12) reduced to $\psi(x,y,t) = e^y E_\beta\big(x^\beta\big)E_\alpha\big(2t^\alpha\big)$, which is analogous to the solution obtained recently by using a hybrid method of Taylor's power series in fractal 2D space (TPSM) [27]. Finally, when $\alpha,\beta,\gamma \to 1$, we have the exact solution $\psi(x,y,t) = e^{x+y+2t}$ of the classical integer diffusion equation.

Example 2. Consider the following initial value $(\alpha,\beta,\gamma)-$fractional hyperbolic wave-like problem:

$$\begin{aligned}&\mathcal{D}_t^{2\alpha}\big[\psi(x,y,t)\big] = \tfrac{1}{12}\Big(x^{2\beta}\mathcal{D}_x^{2\beta}\big[\psi(x,y,t)\big] + y^{2\gamma}\mathcal{D}_y^{2\gamma}\big[\psi(x,y,t)\big]\Big)\\ &\psi(x,y,0) = x^{4\beta},\ \ \mathcal{D}_t^{\alpha}\big[\psi(x,y,0)\big] = y^{4\gamma}.\end{aligned} \tag{13}$$

Plugging the initial conditions into the ansatz (3) gives $\lambda_{040} = 1,\ \lambda_{104} = \frac{1}{\Gamma(\alpha+1)}$, and $\lambda_{0mk} = \lambda_{1mk} = 0$ otherwise. Upon inserting the pertinent formulas (5) into the equation (13) and identifying the coefficients of similar powers, we have the recursion formula

$$\frac{\Gamma\big((n+2)\alpha+1\big)}{\Gamma\big(n\alpha+1\big)}\lambda_{n+2,m,k}-\frac{1}{12}\left(\frac{\Gamma\big(m\beta+1\big)}{\Gamma\big((m-2)\beta+1\big)}-\frac{\Gamma\big(k\gamma+1\big)}{\Gamma\big((k-2)\gamma+1\big)}\right)\lambda_{n,m,k}=0, \quad (14)$$

for $n \geq 0$, and $m, k \geq 2$. This implies that

$$\begin{aligned} \lambda_{2n,4,0} &= \frac{\Gamma^n\big(4\beta+1\big)}{12^n\,\Gamma^n\big(2\beta+1\big)\,\Gamma\big(2n\alpha+1\big)}, \\ \lambda_{2n+1,0,4} &= \frac{\Gamma^n\big(4\gamma+1\big)}{12^n\,\Gamma^n\big(2\gamma+1\big)\,\Gamma\big((2n+1)\alpha+1\big)}, \\ \lambda_{n,m,k} &= 0, \qquad \text{otherwise.} \end{aligned} \quad (15)$$

Thus, the $(\alpha,\beta,\gamma)-$series solution of (13) has the form

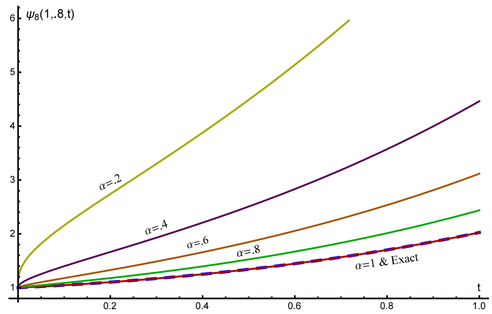
$$\psi(x,y,t) = \sum_{n=0}^{\infty}\frac{\Gamma^n\big(4\beta+1\big)t^{2n\alpha}x^{4\beta}}{12^n\,\Gamma^n\big(2\beta+1\big)\,\Gamma\big(2n\alpha+1\big)}+\sum_{n=0}^{\infty}\frac{\Gamma^n\big(4\gamma+1\big)t^{(2n+1)\alpha}y^{4\gamma}}{12^n\,\Gamma^n\big(2\gamma+1\big)\,\Gamma\big((2n+1)\alpha+1\big)}=x^{4\beta}\sum_{n=0}^{\infty}\left(\frac{\Gamma\big(4\beta+1\big)}{12\,\Gamma\big(2\beta+1\big)}\right)^n\frac{t^{2n\alpha}}{\Gamma\big(2n\alpha+1\big)}+y^{4\gamma}\sum_{n=0}^{\infty}\left(\frac{\Gamma\big(4\gamma+1\big)}{12\,\Gamma\big(2\gamma+1\big)}\right)^n\frac{t^{(2n+1)\alpha}}{\Gamma\big((2n+1)\alpha+1\big)}. \quad (16)$$

When $\gamma \to 1$, we have the same solution obtained by using TPSM [27]. Also, when $\alpha,\beta,\gamma \to 1$, we obtain the exact solution $\psi(x,y,t) = x^4\cosh(t) + y^4\sinh(t)$ for the classical integer wave-like equation, which is in complete agreement with the attained one *via* the Adomian decomposition method (ADM) [30] and the He's homotopy perturbation method (HsHPM) [31].

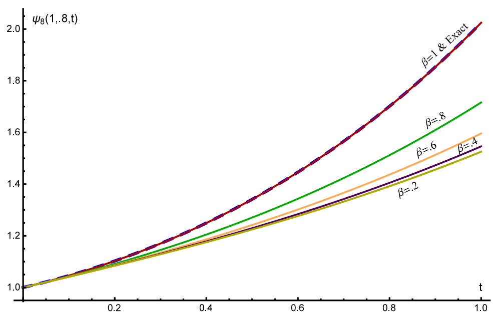
Figures 1, 2 and 3 illustrate the cross-sections behavior of the 8-th partial sum solution approximation of fractional hyperbolic wave-like problem. The relation on the impact of these fractional derivatives is sequentially mapping to the integer-order state.

Example 3. Consider the following initial value $(\alpha,\beta,\gamma)-$fractional hyperbolic telegraph problem:

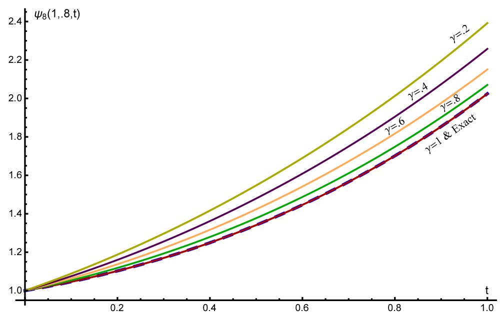
$$\mathcal{D}_t^{2\alpha}\big[\psi(x,y,t)\big]+2\mathcal{D}_t^{\alpha}\big[\psi(x,y,t)\big]+\psi(x,y,t)=\frac{1}{2}\Big(\mathcal{D}_x^{2\beta}\big[\psi(x,y,t)\big]+\mathcal{D}_y^{2\gamma}\big[\psi(x,y,t)\big]\Big), \quad \psi(x,y,0)=\sinh_\beta\big(x^\beta\big)\sinh_\gamma\big(y^\gamma\big), \quad \mathcal{D}_t^{\alpha}\big[\psi(x,y,0)\big]=-2\sinh_\beta\big(x^\beta\big)\sinh_\gamma\big(y^\gamma\big), \quad (17)$$



(a) $\beta=\gamma=1$.



(b) $\alpha=\gamma=1$.



(c) $\alpha=\beta=1$.

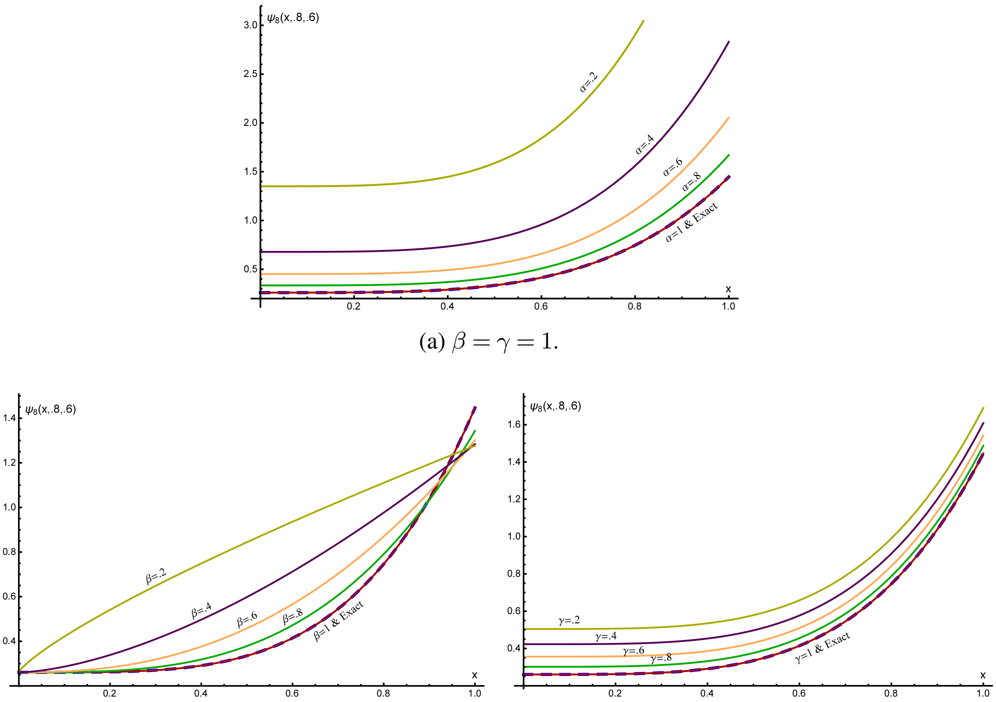
Fig. 1 – Cross-section behavior of the approximate solution $\psi_8(1,.8,t)$ for the wave-like model (13) at varied values of α, β, $\gamma \in (0,1]$ and $t \in [0,1]$.

where $\sinh_\beta\left(x^\beta\right)=\sum_{i=0}^{\infty}\frac{x^{(2i+1)\beta}}{\Gamma((2i+1)\beta+1)}$. From the initial conditions, we obtain

$$\begin{aligned}
\lambda_{0,2m+1,2k+1} &= \frac{1}{\Gamma\big((2m+1)\beta+1\big)\Gamma\big((2k+1)\gamma+1\big)}\\
\lambda_{1,2m+1,2k+1} &= \frac{-2}{\Gamma\big(\alpha+1\big)\Gamma\big((2m+1)\beta+1\big)\Gamma\big((2k+1)\gamma+1\big)}\\
\lambda_{0mk} &= \lambda_{1mk}=0, \qquad \text{otherwise.}
\end{aligned} \tag{18}$$

Now, inserting the pertinent formulas (5) into the equation (17) and gathering the coefficients of same powers, leads to the following recursion formula

$$\begin{aligned}
&\frac{\Gamma\big((n+2)\alpha+1\big)}{\Gamma\big(n\alpha+1\big)}\lambda_{n+2,m,k}+\frac{2\Gamma\big((n+1)\alpha+1\big)}{\Gamma\big(n\alpha+1\big)}\lambda_{n+1,m,k}+\lambda_{n,m,k}=\\
&\frac{\Gamma\big((m+2)\beta+1\big)}{2\Gamma\big(m\beta+1\big)}\lambda_{n,m+2,k}+\frac{\Gamma\big((k+2)\gamma+1\big)}{2\Gamma\big(k\gamma+1\big)}\lambda_{n,m,k+2}.
\end{aligned} \tag{19}$$



(a) $\beta = \gamma = 1$.

(b) $\alpha = \gamma = 1$.

(c) $\alpha = \beta = 1$.

Fig. 2 – Cross-section behavior of the approximate solution $\psi_8(x, .8, .6)$ for the wave-like model (13) at varied values of α, β, $\gamma \in (0,1]$ and $x \in [0,1]$.

By solving (19) recursively, we get

$$\lambda_{n,2m+1,2k+1} = \frac{(-2)^n}{\Gamma\big(n\alpha+1\big)\Gamma\big((2m+1)\beta+1\big)\Gamma\big((2k+1)\gamma+1\big)}, \tag{20}$$

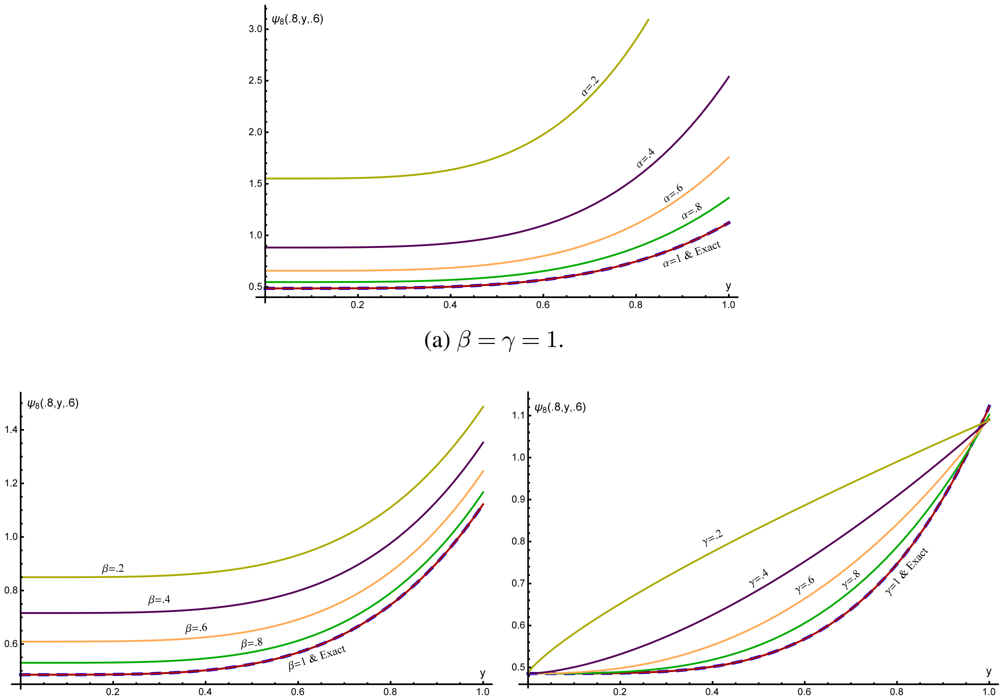
and $\lambda_{nmk} = 0$ otherwise. Plugging (20) back in (3) grants the closed-form solution

$$\begin{aligned}\psi(x,y,t) &= \sum_{n+m+k=0}^{\infty} \frac{(-2)^n t^{n\alpha} x^{(2m+1)\beta} y^{(2k+1)\gamma}}{\Gamma\big(n\alpha+1\big)\Gamma\big((2m+1)\beta+1\big)\Gamma\big((2k+1)\gamma+1\big)} \\ &= \left(\sum_{m=0}^{\infty} \frac{x^{(2m+1)\beta}}{\Gamma\big((2m+1)\beta+1\big)}\right)\left(\sum_{k=0}^{\infty} \frac{y^{(2k+1)\gamma}}{\Gamma\big((2k+1)\gamma+1\big)}\right)\left(\sum_{n=0}^{\infty} \frac{(-2t^{\alpha})^n}{\Gamma\big(n\alpha+1\big)}\right) \\ &= \sinh_\beta\left(x^\beta\right)\sinh_\gamma\left(y^\gamma\right)E_\alpha\left(-2t^\alpha\right).\end{aligned} \tag{21}$$

In particular, when $\gamma \to 1$, the equation (21) reduces to

$$\psi(x,y,t) = \sinh_\beta\left(x^\beta\right)\sinh\left(y\right)E_\alpha\left(-2t^\alpha\right),$$

which is the same solution obtained by using TPSM [27]. Further, if $\alpha, \beta, \gamma \to 1$, we



(a) $\beta = \gamma = 1$.

(b) $\alpha = \gamma = 1$.

(c) $\alpha = \beta = 1$.

Fig. 3 – Cross-section behavior of the approximate solution $\psi_8(.8, y, .6)$ for the wave-like model (13) at varied values of $\alpha, \beta, \gamma \in (0,1]$ and $y \in [0,1]$.

get the exact solution $\psi(x,y,t) = \sinh(x)\sinh(y)e^{-2t}$ for the associated classical telegraph equation and this is identical to the solution gained by using RDTM [32].

Example 4. Finally, consider the following nonlinear initial value $(\alpha, \beta, \gamma)-$fractional Burgers' problem:

$$\begin{aligned} &\mathcal{D}_t^{\alpha}\left[\psi(x,y,t)\right] = \mathcal{D}_x^{2\beta}\left[\psi(x,y,t)\right] + \mathcal{D}_y^{2\gamma}\left[\psi(x,y,t)\right] + \psi(x,y,t)\,\mathcal{D}_x^{\beta}\left[\psi(x,y,t)\right] \\ &\psi(x,y,0) = x^{\beta} + y^{\gamma}. \end{aligned} \tag{22}$$

Starting by substituting the initial condition into the expression (3), we get $\lambda_{010} = 1$, $\lambda_{001} = 1$, and $\lambda_{0mk} = 0$ otherwise. Now, by inserting the relevant formulas (5) into the equation (22) and solving the resulting recursion formulas, we have

$$\begin{aligned} \lambda_{n01} &= \lambda_{n10}, \quad && n \geq 0, \\ \lambda_{nmk} &= 0, && \text{otherwise}, \end{aligned} \tag{23}$$

where the first few coefficients λ_{n10} are recursively given by

$$\begin{aligned}
\lambda_{010} &= 1\\
\lambda_{110} &= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)}\\
\lambda_{210} &= \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(2\alpha+1)}\Big(2\lambda_{010}\lambda_{110}\Big)\\
\lambda_{310} &= \frac{\Gamma(2\alpha+1)\Gamma(\beta+1)}{\Gamma(3\alpha+1)}\Big(\lambda_{110}^2+2\lambda_{010}\lambda_{210}\Big)\\
\lambda_{410} &= \frac{\Gamma(3\alpha+1)\Gamma(\beta+1)}{\Gamma(4\alpha+1)}\Big(2\lambda_{010}\lambda_{310}+2\lambda_{110}\lambda_{210}\Big)\\
\lambda_{510} &= \frac{\Gamma(4\alpha+1)\Gamma(\beta+1)}{\Gamma(5\alpha+1)}\Big(\lambda_{210}^2+2\lambda_{010}\lambda_{410}+2\lambda_{110}\lambda_{310}\Big)\\
\lambda_{610} &= \frac{\Gamma(5\alpha+1)\Gamma(\beta+1)}{\Gamma(6\alpha+1)}\Big(2\lambda_{010}\lambda_{510}+2\lambda_{110}\lambda_{410}+2\lambda_{210}\lambda_{310}\Big)\\
\lambda_{710} &= \frac{\Gamma(6\alpha+1)\Gamma(\beta+1)}{\Gamma(7\alpha+1)}\Big(\lambda_{310}^2+2\lambda_{010}\lambda_{610}+2\lambda_{110}\lambda_{510}+2\lambda_{210}\lambda_{410}\Big).
\end{aligned} \tag{24}$$

Generally, we have

$$\begin{aligned}
\lambda_{n10} &= \frac{\Gamma\big((n-1)\alpha+1\big)\Gamma(\beta+1)}{\Gamma(n\alpha+1)}\sum_{i=1}^{k}2\lambda_{i-1,1,0}\lambda_{n-i,1,0}, && n=2k,\\
\lambda_{n10} &= \frac{\Gamma\big((n-1)\alpha+1\big)\Gamma(\beta+1)}{\Gamma(n\alpha+1)}\left(\lambda_{k,1,0}^2+\sum_{i=1}^{k}2\lambda_{i-1,1,0}\lambda_{n-i,1,0}\right), && n=2k+1.
\end{aligned} \tag{25}$$

Thus, the $N-$th partial sum solution approximation of (22) is

$$\begin{aligned}
\psi_N(x,y,t) &= \sum_{n+m+k=0}^{N}\lambda_{nmk}t^{n\alpha}x^{m\beta}y^{k\gamma} = \sum_{n=0}^{N}\lambda_{n10}t^{n\alpha}x^{\beta}+\sum_{n=0}^{N}\lambda_{n01}t^{n\alpha}y^{\gamma}\\
&= \Big(x^{\beta}+y^{\gamma}\Big)\sum_{n=0}^{N}\lambda_{n10}\,t^{n\alpha}.
\end{aligned} \tag{26}$$

We point out here that if $\gamma \to 1$, we have the same solution obtained by using TPSM [27]. Perceptibly, $\lambda_{n10} = 1$ when $\alpha,\beta,\gamma \to 1$. Hence, the closed-form solution of Burgers' integer-order is

$$\psi(x,y,t) \;=\; (x+y)\sum_{n=0}^{\infty}t^n \;=\; \frac{x+y}{1-t} \tag{27}$$

provided that $0 \le t < 1$. This is analogous to the solution attained by using the ADM, HPM, and RDTF [33].

4. CONCLUSION

In this article, we have provided an analytical technique to investigate the effect of a trifold memory index resulted from the presence of the temporal-spatial fractional derivatives. The solutions of $(2+1)-$dimensional fractional diffusion, wave-like, telegraph, and Burgers' equations are given in terms of a new closed-form series solution, which involves the aforementioned $(\alpha,\beta,\gamma)-$memory indices. By allowing α,β, and γ approach 1, we obtained the exact solutions of the corresponding integer-order equations. Thus, we may deduce that the proposed solution form for the differential equations with trifold fractional derivative ordering is effective and reflects, in some sense, sequential memories.

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