

GAUGE BACKGROUNDS FROM GENERALIZED COMMUTATORS – A REVIEW –

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Received November 21, 2019

Abstract. We review the generation of the known gauge backgrounds - and of some new ones - from commutation relations. The gauge connections corresponding to electromagnetism, Yang-Mills theory and Einstein gravity are obtained by assuming specific commutation relations between the phase-space variables of a first quantized theory. Extending the procedure to noncommuting coordinates leads to new types of dynamics, which are explored. In particular, the conditions for the coexistence of an electromagnetic background and a noncommutative two-form are presented, as well as a generalized mechanism for dimensional reduction. The noncommutative deformation of a gravitational background is also constructed. A different perspective is allowed by the so-called Feynman approach to the Maxwell equations. We review it, together with its extension to the noncommutative case. Finally, a few related topics, including the Wigner problem, are briefly overviewed.

Key words: gauge fields, commutation relations, noncommutative dynamics.

1. INTRODUCTION AND SUMMARY

Alternative formulations of fundamental physical theories always present interest. A reformulation may be a better starting point for a new analytical method, or it can help conceptually by shedding light from a different angle on a previously less understood aspect. An alternative viewpoint can also suggest new paths of investigation (in this paper new types of dynamics) in the search for 'new physics'.

Gauge fields are a basic building block of today's fundamental theories of nature. They are usually introduced in a field theory context through a principle of local invariance, which enforces additional interaction terms in the field theory Lagrangian. Historically, gauge invariance appeared as a useful symmetry of the Maxwell equations but gained recognition especially after the advent of quantum mechanics. In 1929, Weyl [1] proposed it as a basic postulate, which enforced the existence of electromagnetic interactions. The subsequent generalization to non-Abelian [2] and gravitational [3] interactions, together with the ideas of spontaneous symmetry breaking and confinement, confirmed the universality of the gauge principle.

Romanian Journal of Physics **64**, 101 (2019)

Now, somewhat surprisingly, gauge backgrounds can also be introduced in a particle theory context, not *via* additional terms in the Hamiltonian/Lagrangian but through more general commutation relations. Electromagnetic, Yang-Mills and gravitational backgrounds can be seen as consequences of generalized Poisson brackets or commutation relations - instead of being introduced by modifications of the action of the theory. New types of dynamics can be obtained by further allowing the space coordinates to become noncommutative (NC), a generalization that suggests itself in an approach based on phase space commutators. Such generalized backgrounds will be explored throughout this review.

Section 2 shows that part of the content of a gauge theory - the connection and its corresponding field strength - follow by postulating appropriate nonzero commutators between the momenta, or momenta and coordinates, of a first-quantized theory. The equations of motion of a point particle subject to such a gauge background also follow. When the underlying space is commutative, the above commutators fall naturally into three classes, corresponding to electromagnetic, Yang-Mills and gravitational fields.

New backgrounds appear if one allows for non-trivial commutators among the coordinates. Section 3 extends the analysis to this “noncommutative” (NC) case and discusses some interesting consequences. The conditions under which an electromagnetic-like background and noncommutative space can coexist, for instance, are quite restrictive. A generalized mechanism for enforcing dimensional reduction [4, 5] represents another new feature. A gravitational background on the other hand is not restricted in any way by noncommutativity, although the equations of motion of the test particle become more complex. A discussion of the significance of the noncommutativity 2-form closes Section 3.

Section 4 discusses Darboux transformations, which lead to phase space coordinates obeying canonical commutation relations. In the process, however, they transfer nontrivial dynamics from the generalized commutators to the Hamiltonian written in canonical coordinates. As such, they are the natural set-up for studying the relation between the present point of view and the usual way to introduce dynamics.

Section 5 moves to a slightly different viewpoint, due to Feynman [6]. Although it does not assume the existence of a Hamiltonian from the beginning, in the end it reaches conclusions similar to those of Section 2. To uncover new types of dynamics, the Feynman approach requires as well the introduction of nontrivial space-time commutators. This extension is discussed in Section 6.

Finally, Section 7 mentions some interesting aspects that could not be discussed at length here. It includes a brief overview of the Wigner problem, in which one searches for different commutation relations leading to the *same* type of dynamics.

2. STANDARD BACKGROUNDS VIA COMMUTATORS

2.1. ELECTROMAGNETISM AND YANG-MILLS

The simplest instance in which the commutation relations imply the existence of a gauge connection and symmetry is electromagnetism, see *e.g.* [7]. To begin, consider a nonrelativistic $(d+1)$ -dimensional quantum-mechanical theory. Two elements encode the dynamics: the Hamiltonian, $\hat{H}(\hat{q}_i, \hat{p}_i)$, and the commutation relations between the phase space variables \hat{q}_i and \hat{p}_i . We will use a free Hamiltonian, $\hat{H} = \frac{1}{2m}\hat{p}^2$, and will generate nontrivial dynamics through the following commutation relations:

$$[q_i, q_j] = 0 \quad [q_i, p_j] = i\delta_{ij} \quad [p_i, p_j] = iF_{ij}(q_k). \quad (1)$$

The $[p, p]$ commutator implies the presence of a magnetic field strength F_{ij} . In the Schrödinger picture, this is seen as follows. In the coordinate basis $|q_i\rangle$ the relations (1) are represented by ($\frac{\partial}{\partial q_i} \equiv \partial_{q_i} \equiv \partial_i$)

$$\hat{q}_i|q_i\rangle = q_i|q_i\rangle, \quad \hat{p}_i|q_i\rangle = (-i\partial_{q_i} - A_i(\vec{q}))|q_i\rangle, \quad (2)$$

provided

$$\partial_{q_i}A_j - \partial_{q_j}A_i = F_{ij}(q_k). \quad (3)$$

The gauge connection appeared as a consequence of the nonvanishing of the $[p, p]$ commutator. Eq. (3) is not an assumption; it is enforced by the Jacobi identity for p_i, p_j, p_k ,

$$\partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki} = 0, \quad (4)$$

which is equivalent to the sourceless Maxwell equations. The Jacobi identity also implies that

$$[[p_j, p_k], q_i] = -i\frac{\partial F_{jk}}{\partial p_i} = 0, \quad (5)$$

precluding any dependence of F on p . Thus $F(q)$ is a necessary condition, not an assumption. Due to (4), the present formalism does not accommodate magnetic monopoles. The Schrödinger equation includes a gauge field,

$$\hat{H}\Psi(q_i) = -\frac{1}{2}(-i\partial_i - A_i)^2\Psi(q_i) = E\Psi(q_i), \quad (6)$$

and gauge invariance is built in automatically. In operatorial formulation, the Heisenberg equations of motion read

$$\dot{q}_i = -i[q_i, H] = p_i \quad \dot{p}_i = -i[p_i, H] = \frac{1}{2}(F_{ij}p_j + p_jF_{ij}), \quad (7)$$

and lead to

$$\ddot{q}_i = \frac{1}{2}(F_{ij}\dot{q}_j + \dot{q}_jF_{ij}). \quad (8)$$

This is the operatorial Lorentz force law for a particle in a magnetic field. The usual $\ddot{q}_i = F_{ij}\dot{q}_j$ law is obtained by prescribing an ordering for the noncommuting operators q_i and p_i . If one works at the classical level, with Poisson brackets instead of commutators, one obtains directly $\ddot{q}_i = F_{ij}\dot{q}_j$.

The above considerations extend to relativistic quantum mechanics. Consider the operators p_μ and q_μ , $\mu = 0, 1, 2, \dots, d$. The time coordinate $q_0 = t$ is treated formally as an operator, canonically conjugated to the Hamiltonian $p_0 = H$. [We do not enter the well-known issues related to that procedure; one can recall that the discussion can also take place exclusively at the classical level.] If

$$[q_\mu, q_\nu] = 0 \quad [q_\mu, p_\nu] = i\eta_{\mu\nu} \quad [p_\mu, p_\nu] = -iF_{\mu\nu}(q), \quad (9)$$

(with metric $\eta = \text{Diag}[- + + \dots +]$) one has, in the $|q_0, \vec{q}\rangle$ basis,

$$\hat{q}_\mu |q_\sigma\rangle = q_\mu |q_\sigma\rangle, \quad \hat{p}_\mu |q_\sigma\rangle = [i\partial_\mu + A_\mu(q_\nu)] |q_\sigma\rangle, \quad (10)$$

$$\partial_{q_\mu} A_\nu - \partial_{q_\nu} A_\mu = F_{\mu\nu}(q_\rho), \quad (11)$$

the later enforced again by the Jacobi identity. The Schrödinger equation follows from $2mH\Psi = \vec{p}^2\Psi$, and the Klein-Gordon equation in an electromagnetic background from $H^2\Psi = (\vec{p}^2 + m^2)\Psi$. A more elegant way is to introduce the proper time τ , on which p_μ, q_ν depend. By studying the proper-time evolution of p_μ, q_μ under the fictitious Hamiltonian $2\bar{H}_\tau = p_\mu^2$, one obtains the particle equation of motion $\frac{d^2 q_\mu}{d\tau^2} = \frac{1}{2}(F_{\mu\nu}q_\nu + q_\nu F_{\mu\nu})$.

The commutation relations fix the sourceless part of the Maxwell equations (the Bianchi identity, or existence of a connection), together with the equation of motion of a particle situated in an electromagnetic field (Lorentz law). The Maxwell equations with sources require extra assumptions, but can be inferred quite naturally: the Schrödinger equation (6) displays a divergenceless current j_μ , which shares this property with $\partial_\nu F_{\mu\nu}$. This suggests that, up to a multiplicative factor,

$$\partial_\mu F_{\mu\nu} = j_\nu. \quad (12)$$

This would account for the remaining pair of Maxwell equations, although one sees that their 'derivation' becomes quite heuristic. We were however able to obtain in a legitimate way the sourceless Maxwell equations and the Lorentz force law.

To allow for non-Abelian connections in the former procedure, we introduce multi-component wave functions labelled by an internal index $a = 1, 2, \dots, n$,

$${}^{ab}\Psi_b = iF_{\mu\nu}^{ab}\Psi_b. \quad (13)$$

The commutation relations (13) can be represented by

$$p_\mu^{ab} = -i\partial_\mu \delta^{ab} - A_\mu^s T_s^{ab}, \quad (14)$$

where T_s^{ab} are matrices that span the group space to which $F_{\mu\nu}^{ab}$ belongs as far as its action related to the indices a and b is concerned. Eqs. (13,14) require that

$$F_{\mu\nu}^{ab} = \partial_\mu A_\nu^s T_s^{ab} - \partial_\nu A_\mu^s T_s^{ab} + A_\mu^s A_\nu^t [T_s, T_t]^{ab} \quad (15)$$

and the Jacobi identity implies that F does not depend on p , and that $\partial_\rho F_{\mu\nu}^{ab} + \partial_\nu F_{\rho\mu}^{ab} + \partial_\mu F_{\nu\rho}^{ab} = 0$. The operatorial equations of motion for a test particle are derived like in the abelian case, and read

$$\ddot{q}_\mu^{ab} = \frac{1}{2}(F_{\mu\nu}^{ac} p_\nu^{cb} + p_\nu^{ac} F_{\mu\nu}^{cb}). \quad (16)$$

Further discussion of the non-Abelian case will be given elsewhere; we shall not focus on it in this review.

2.2. GRAVITY

Another possibility for the $[p, p]$ commutator is to have the wave function carry a space-time index. The commutation relation is then

$$[p_\mu, p_\nu] \Psi_\rho = i R_{\mu\nu\rho}^\sigma \Psi_\sigma. \quad (17)$$

Obviously, $R_{\mu\nu\rho}^\sigma$ is intended to be the Riemann tensor. One may realize (17) via $p_\mu = i D_\mu$, where

$$D_\mu \Psi_\nu = (\partial_\mu \delta_\nu^\sigma + i \Gamma_{\mu\nu}^\sigma) \Psi_\sigma. \quad (18)$$

(We assume that the covariant derivative is Leibniz.) Eqs. (17,18) lead to

$$R_{\lambda\mu\nu}^\kappa = \partial_\mu \Gamma_{\lambda\nu}^\kappa - \partial_\nu \Gamma_{\lambda\mu}^\kappa + i \Gamma_{\sigma\mu}^\kappa \Gamma_{\lambda\nu}^\sigma - i \Gamma_{\sigma\nu}^\kappa \Gamma_{\lambda\mu}^\sigma \quad (19)$$

with R satisfying all the usual symmetry properties (including Bianchi, again derived from Jacobi). However, due to the extra i , the R in (19) is complex, not the real (in both senses) R we are after. The mismatch is due to the real gravity connection appearing at the classical level, via $\partial \rightarrow \partial + \Gamma$, without the extra i the approach (17) introduces*. We conclude that $[p, p]$ commutators cannot generate the usual Einstein connection.

There is however a different, quite natural, way to introduce the effect of a gravitational background, by postulating

$$[q^\mu, p^\nu] = i g^{\mu\nu}(q^\sigma) \quad [q^\mu, q^\nu] = 0 \quad [p_\mu, p_\nu] = 0, \quad (20)$$

with $g^{\mu\nu}$ symmetric[†] and nondegenerate, and $p_\kappa \equiv g_{\kappa\mu}(q^\sigma) p^\mu$. The inverse of $g^{\mu\nu}$ is $g_{\mu\nu}$: $g^{\mu\nu}(q^\sigma) g_{\nu\kappa}(q^k) = \delta_\kappa^\mu$. Now q^μ, p^ν depend on the proper time τ , and $\frac{dq^\mu}{d\tau} \equiv \dot{q}^\mu$.

*The opposite problem arises in Weyl's first attempt to derive electromagnetism from gauge invariance [8]. His classical approach, based on a conformal extension of general relativity, lacked a crucial i . In trying to derive gravity from the quantum-mechanical commutation relations (17) one extra i appears.

[†]A nonsymmetric g can also be used, by taking $p^k \equiv \frac{1}{2}(g^{km} + g^{mk})p_m$ in the following.

The Hamiltonian is taken to be

$$H = \frac{1}{2}p_\mu p^\mu = \frac{1}{2}p_\mu p_\nu g^{\mu\nu}(q^\rho), \quad (21)$$

with a suitable ordering implied. The main point is to distinguish between upper and lower indices. Because $[q^\mu, p_\nu] = \delta_\nu^\mu$, $[\cdot, p_\nu]$ can be used as a derivative operator on functions $f(q^\mu)$. Since the nonconstancy of $g^{\mu\nu}(q^\kappa)$ may lead to operator ordering issues we start classically, with the following Poisson brackets:

$$\{q^\mu, p^\nu\} = g^{\mu\nu}(q^\kappa), \quad \{q^\mu, q^\nu\} = 0, \quad \{p_\mu, p_\nu\} = 0, \quad (22)$$

(q and p are now classical variables, not operators). $\{q^\mu, p_\nu\} = \delta_\nu^\mu$ implies

$$\{f(q^\mu), p_\nu\} = \frac{\partial f}{\partial q^\nu} \equiv \partial_\nu f. \quad (23)$$

Half of the Hamilton equations give

$$\dot{q}^\kappa = \{q^\kappa, H\} = \{q^\kappa, \frac{1}{2}p_\mu p_\nu g^{\mu\nu}(q^\sigma)\} = p^\kappa, \quad (24)$$

whereas the second half of them reads

$$\ddot{q}^\kappa = \{p^\kappa, H\} = \{p_\lambda g^{\lambda\kappa}(q^\sigma), \frac{1}{2}p_\mu p_\nu g^{\mu\nu}(q^\sigma)\}. \quad (25)$$

Using the relation $\partial_\kappa g^{\mu\nu} = -(\partial_\kappa g_{\sigma\tau})g^{\mu\sigma}g^{\nu\tau}$ and Eqs. (23,24) one obtains [7]

$$\ddot{q}^\kappa + \Gamma_{\mu\nu}^\kappa \dot{q}^\mu \dot{q}^\nu = 0. \quad (26)$$

Above, Γ is the usual Christoffel connection,

$$\Gamma_{\mu\nu}^\kappa = \frac{1}{2}g^{\kappa\lambda}(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}). \quad (27)$$

Eqs. (26,27) describe the motion of a point-particle in a gravitational background, represented by the metric $g^{\kappa\mu}(q)$.

At this point, the Riemann tensor can be defined through

$$R_{\lambda\mu\nu}^\kappa = \partial_\mu \Gamma_{\lambda\nu}^\kappa - \partial_\nu \Gamma_{\lambda\mu}^\kappa + \Gamma_{\sigma\mu}^\kappa \Gamma_{\lambda\nu}^\sigma - \Gamma_{\sigma\nu}^\kappa \Gamma_{\lambda\mu}^\sigma \quad (28)$$

and all its symmetries follow in the usual way. The same applies to gauge invariance. In contrast to electromagnetism and the unsuccessful attempt (17), the symmetries and the Bianchi identity of (28) do not follow from the Jacobi identities, which are in fact identically satisfied, as straightforward calculation shows [7]. Thus no constraints are imposed on the metric, which is a genuine Einstein gravity background.

To proceed to the quantum-mechanical level, one has to prescribe an ordering, be it left, right, symmetric, or whatever, of the operators q and p . Any of those orderings satisfies

$$[p_\kappa, \langle f(q^\mu) \rangle] = -i\partial_\kappa \langle f \rangle \equiv -i \langle \partial_\kappa \langle f \rangle \rangle, \quad (29)$$

where $\langle \dots \rangle$ denotes the operation of taking a given ordering. Eq. (29) and the commutativity of operators under ordering symbols allow one to adapt step by step the derivation (22–26) to the operatorial case. The Schrödinger equation $H(q, p)\Psi = -i\partial_t\Psi$ is a more delicate issue, being ordering dependent. The kinetic part of H gives, in addition to the Laplace-Beltrami operator, an ordering dependent correction. This well-known issue is discussed at length from the usual point of view in [9].

Considering simultaneously nontrivial $[p, p]$ and $[q, p]$ commutators only amounts to a superposition of electromagnetic and gravitational backgrounds. The constraints imposed by the Jacobi identity remain those of a flat background (independence of F from p and Bianchi for $F(q)$). On the other hand, one can allow the metric to depend on both q and p , $g(q, p)$. This leads to the same Jacobi constraints as above but to equations of motion more complicated than (26). More details can be found in [7].

We generated the three known types of gauge backgrounds *via* $[p, p]$ and $[q, p]$ commutators. We proceed with nontrivial $[q, q]$ commutators [7, 10, 11].

3. NONCOMMUTATIVE SPACE

3.1. ELECTROMAGNETISM AND NONCOMMUTATIVITY

Consider first a generalized electromagnetic background $F(q, p)$, living on a space with noncommutativity field $\theta(q, p)$, and flat metric $g_{ij} = \delta_{ij}$:

$$[q^i, q^j] = i\theta^{ij}(q, p) \quad [q^i, p^j] = i\delta^{ij} \quad [p_i, p_j] = iF_{ij}(q, p). \quad (30)$$

We will stay mainly in a nonrelativistic set-up, although the relativistic generalization $\delta^{ij} \rightarrow \eta^{\mu\nu}$ is formally straightforward. To avoid explicit referral to ordering issues, we work with Poisson brackets, *i.e.* $-i[,] \rightarrow \{, \}$. For a generic Hamiltonian $H(p, q)$, the equations of motion read $\dot{q}^m = \{q^m, H(p)\}$, $\dot{p}_m = \{p_m, H(p)\}$, or

$$\dot{q}^m = \frac{\partial H}{\partial p_m} + \theta^{mn}(q, p) \frac{\partial H}{\partial q_n}, \quad \dot{p}_m = F_{mn}(q, p) \frac{\partial H}{\partial p_n} - \frac{\partial H}{\partial q_m}. \quad (31)$$

If the Hamiltonian depends only on the momenta, $H(p)$, the equations of motion are not modified *in form* by a nonzero θ^{ij} . Their content changes however, due to the dependence of F_{mn} on both q and p .

The Jacobi identities on the other hand are completely independent of the Hamiltonian $H(p, q)$; they read

$$\{q^k, F_{ij}\} = \frac{\partial F_{ij}}{\partial p_k} - \frac{\partial F_{ij}}{\partial q^m} \theta^{mk} = 0 \quad (32)$$

$$\{\theta^{ij}, p_k\} = \frac{\partial \theta^{ij}}{\partial q^k} + \frac{\partial \theta^{ij}}{\partial p_m} F_{mk} = 0 \quad (33)$$

$$\{F_{ij}, p_k\} + cyclic = \left(\frac{\partial F_{ij}}{\partial q^k} + \frac{\partial F_{ij}}{\partial p_m} F_{mk} \right) + cyclic = 0 \quad (34)$$

$$\{q^k, \theta^{ij}\} + cyclic = \left(\frac{\partial \theta^{ij}}{\partial p_k} - \frac{\partial \theta^{ij}}{\partial q^m} \theta^{mk} \right) + cyclic = 0. \quad (35)$$

The Jacobi identities ensure the invariance of the commutation relations under time evolution, for *any* Hamiltonian $H(p, q)$. For instance,

$$\{\{q^m, p_n\} - \delta_n^m, H\} = -\partial_{p_s} H \{F_{ns}, q^m\} + \partial_s H \{\theta^{ms}, p_n\} \stackrel{(32,33)}{=} 0, \quad (36)$$

They also restrict the form of F and θ . In particular, (32) forbids an electromagnetic field strength to depend only on the coordinates, $F_{ij}(q)$, if $\theta^{ij} \neq 0$. We proceed to discuss all the possible situations in turn.

θ and F constant, or noncommutative quantum mechanics [12], the simplest case. It is not our main interest here.

F constant and $\theta(q, p)$. Eq. (33) constrains the functional dependence of θ to be of the form $\theta(\bar{p}_m) = \theta(p_m - F_{mn} q^n)$. To satisfy (35) also, one can either block-diagonalize θ , or require $\partial_{\bar{p}_s} \theta^{ij}(\bar{p}) [\delta_s^k + F_{sm} \theta^{sk}(\bar{p})] + cyclic = 0$. The equations of motion do not change if $H(p)$, they remain those of a classical particle in a constant electromagnetic field. The Schrödinger equation in (2+1)-dimensions [13] confirms that: $\psi(q^1, p_2)$ satisfies the same equation as when $\theta \equiv 0$. The only difference is that objects like $\psi(q^1, q^2)$ and $\psi(p_1, p_2)$ are not definable anymore. Two limiting cases are:

1. $F = 0$ and $\theta(p)$, which implies a 'p-Bianchi identity' for θ ,

$$\partial_{p_k} \theta^{ij}(p) + \partial_{p_i} \theta^{jk}(p) + \partial_{p_j} \theta^{ki}(p) = 0. \quad (37)$$

$\theta(p)$ plays the role of a magnetic field in momentum space [13]. To make the duality $q \leftrightarrow -p$ fully manifest one may introduce a harmonic potential, $V \sim q^2$. Again, if $\partial_q H(p, q) = 0$, θ has no effect on the equations of motion, which remain those of a free particle.

2. $F^{-1} \rightarrow 0$ and $\theta \rightarrow \theta(q)$. If one allows a potential term also, $H(q, p) = p^2/2 + V(q)$, given that now $F \rightarrow \infty$, one gets $\ddot{q}_k \simeq F_{km}(\dot{q})(\dot{q}_m - \theta^{ms} \partial_s V)$.

$F(q, p)$ and θ constant.

If θ is constant, Eq. (32) requires that $F = F(\bar{q}^m) = F(q^m + \theta^{mn} p_n)$. To satisfy (34) one can either partially block-diagonalize F and θ such that they couple only pairs of directions, or just look for a solution of (34) for $F(\bar{q}^m)$. (34) is automatically satisfied in two dimensions (2D), where our main interest will be. For small θ one obtains, for a particle of mass m ,

$$m\ddot{q}^m = F_{mn}(q^s + \theta^{st} p_t) \dot{q}^n \simeq F_{mn}(q^s) \dot{q}^n + m \partial_s F_{mn}(q) \theta^{st} \dot{q}^t \dot{q}^n. \quad (38)$$

We have thus the superposition of a usual electromagnetic background, linear in velocities, and of a gravitational-like force quadratic in the velocities, $\partial_s F(q)_{mn} \theta^{st} \dot{q}^t \dot{q}_n$. The term $\gamma_{tn}^s \equiv \partial_s F(q)_{mn} \theta^{st}$ simulates a gravitational connection, although it does not behave like a Christoffel symbol under generalized coordinate transformations.

It is important that a nonzero noncommutativity, $\theta \neq 0$, requires $F(q, p)$. Noncommutativity might consequently be detected through the effects of the additional term in (38). A simple classical, low-energy set-up actually provides a good bound for θ , with $\theta^{-\frac{1}{2}}$ going easily into the *TeV* region [10].

Finally, two limiting cases are worth mentioning:

1. $\theta = 0$ and $F(q)$, which is usual electromagnetism.
2. $\theta^{-1} \rightarrow 0$ and $F \rightarrow F(p)$, which has the following dynamical content: If $H(p) = p^2/2$, get $\ddot{q}_k = F_{km}(\dot{q}) \dot{q}_m$, *i.e.* first order differential equation in \dot{q} . If $H(q, p) = p^2/2 + V(q)$, due to $\theta \rightarrow \infty$, the potential term will dominate the dynamics; in a first approximation $\ddot{q}^k \simeq \theta^{ks} \theta^{lm} \partial_l \partial_s V \partial_m V$.

$F(q, p)$ and $\theta(q, p)$ – Dimensional reduction.

We proceed with the case in which both θ and F are nonconstant. The Jacobi identities imply that they should depend on both q 's and p 's. In the 2D case, with $F_{12} \equiv F$, $\theta_{12} \equiv \theta$, Eqs. (34,35) are identically satisfied, whereas (32,33) read:

$$F \partial_{p_2} \theta = \partial_1 \theta, \quad F \partial_{p_1} \theta = -\partial_2 \theta, \quad (39)$$

$$\partial_{p_2} F = \theta \partial_1 F, \quad \partial_{p_1} F = -\theta \partial_2 F. \quad (40)$$

A variety of situations suggest [11] that solutions of Eqs. (39,40) obeying

$$F\theta = 1 \quad (41)$$

are an important subset of all the solutions. Due to the $F \longleftrightarrow \theta^{-1}$ symmetry of the Jacobi identities (39,40), F and $1/\theta$ actually satisfy the same equations. The additional $q_1, p_2 \longleftrightarrow -q_2, p_1$ symmetry of the system is also evident.

We thus begin by assuming Eq. (41). It leaves us with a single pair of equations

$$\partial_1 F = F \partial_{p_2} F, \quad \partial_2 F = -F \partial_{p_1} F, \quad (42)$$

which can be analyzed independently without loss of generality. Their general solution (consider the first one for instance) is well-known under the implicit form

$$F(q_1, p_2) = f(q_1 + F(q_1, p_2)p_2), \quad (43)$$

where f is defined by the (in this case) phase-space boundary condition

$$F(q_1, p_2 = 0) = f(q_1). \quad (44)$$

An crucial point is that $F = \frac{1}{\theta}$ arises if and only if the q 's and p 's are not independent, *i.e.* if dimensional reduction takes place. In an arbitrary number of dimensions.

assuming a relation between the q^i 's and the p_j 's means

$$q^n = g^n(p_m) \quad p_m = f_m(q^s) = (g^{-1})_m(q^s). \quad (45)$$

Asking consistency of the commutation relations, one obtains [11]

$$\frac{\partial q^m}{\partial p_s} = -\theta^{ms} \quad \frac{\partial p_k}{\partial q^n} = F_{kn}. \quad (46)$$

This follows, for instance, from $\{f(q(p)), q^s\} = \frac{\delta f}{\delta q^m} \theta^{ms} = -\frac{\delta f}{\delta q^m} \frac{\partial q^m}{\partial p_s}$. Then

$$\theta^{mk} F_{kn} = -\delta_n^m, \quad (47)$$

or $F_{12} \equiv F = \theta^{-1} \equiv \theta_{12}^{-1}$ in 2D parlance. Conversely, (47) implies (45), with the relationship between the q 's and the p 's actually taking the more precise form [11]

$$q^m = -\theta^{mn}(q, p)p_n + cst. \quad \forall H(p, q). \quad (48)$$

One may also enquire which conditions render (48) and (46) compatible. Those conditions turn out to be precisely the Jacobi identities [11].

For θ and F constant in (47), the dimensional reduction occurring when $F = \theta^{-1}$ has long been known in a slightly different form [4, 5], cf. also [13, 14]. Here we generalized it to the case of nonconstant backgrounds.

$F(q, p)$ and $\theta(q, p)$ – General case.

Consider now the case in which dimensional reduction does not take place, hence $F\theta \neq 1$. It is convenient to use throughout this paragraph the notation $q_1 \equiv x$, $p_2 \equiv y$, $G \equiv \theta^{-1}$ [the second pair of variables - (q_2, p_1) - in Eqs. (39,40) can be treated similarly]. The ensuing equations are

$$\partial_x F = G \partial_y F, \quad \partial_x G = F \partial_y G, \quad (49)$$

with obvious symmetry. To avoid dimensional reduction, we must search for solutions with $F \neq G$. The Ansatz (43) can be generalized in various ways. One able to generate in the end non-reduced solutions is [11]

$$f(F, G) = xG + y, \quad g(F, G) = xF + G. \quad (50)$$

If $F \neq G$, the following conditions ensure the compatibility of (50) and (49):

$$\frac{\partial f}{\partial G} = \frac{\partial g}{\partial F} = x. \quad (51)$$

In order to use (51) into (50) it is convenient to see x and y as functions of F and G , $x(F, G), y(F, G)$, a kind of hodograph method [11]. Eqs. (51) and (50) imply then the *linear* partial differential equations

$$G \frac{\partial x}{\partial G} + \frac{\partial y}{\partial G} = 0, \quad F \frac{\partial x}{\partial F} + \frac{\partial y}{\partial F} = 0, \quad (52)$$

easily solved once boundary conditions are given. We present here a particular solution which has two desirable properties: one can explicitly invert it to recover F and G as functions of x and y , and the solution reaches continuously the $F = G$ case. It thus allows one to 'interpolate' in between the general and reduced regimes of the system. The solution is

$$x = a(F - G), \quad y = -\frac{a}{2}(F^2 - G^2), \quad (53)$$

with a having dimensionality $[length]^3$. Then

$$F = -\frac{y}{x} + \frac{x}{2a}, \quad G = -\frac{y}{x} - \frac{x}{2a}. \quad (54)$$

If in appropriate units $a \rightarrow \infty$, the system reduces, $F \rightarrow G = 1/\theta$.

3.2. GRAVITY AND NONCOMMUTATIVITY

Consider for simplicity the case in which the metric depends only on the coordinates, $g(q)$, and the electromagnetic-like background vanishes, $F = 0$:

$$\{q^i, p_j\} = \delta_j^i \quad \{q^i, q^j\} = \theta^{ij} \quad \{p_i, p_j\} = 0, \quad (55)$$

and $p^i = g^{ij}(q^s)p_j$, $2H = p_i p^i$ [going to Minkowskian signature is formally straightforward: $\delta_{ij} \rightarrow \eta_{\mu\nu}$, $q^i \rightarrow q^\mu = (q^0, q^i)$, $p_i \rightarrow p_\mu = (p_0, p_i)$, $t \rightarrow \tau$]. If $\theta \neq 0$, the bracket $\{q^i, p^j\}$ is not symmetric anymore, and one gets also a nonzero $\{p^i, p^j\}$ bracket. The Jacobi identities read the same as in flat space, but the equations of motion change. First, we have

$$\dot{q}^k = p^k + \frac{1}{2}p_i p_j \partial_s g^{ij} \theta^{ks}, \quad (56)$$

and

$$\dot{q}^k = \{g^{kj} p_j + \frac{1}{2}p_i p_j \partial_s g^{ij} \theta^{ks}, \frac{1}{2}p_m p_n g^{mn}\}. \quad (57)$$

For small θ , one obtains

$$p^k \simeq \dot{q}^k + \frac{1}{2}p^m p^n \theta^{ks} \partial_s g_{mn} \quad (58)$$

and in the end one gets the following equations of motion [7]

$$\ddot{q}^k = -\Gamma_{mn}^k \dot{q}^m \dot{q}^n + \Delta_{mnp}^k \dot{q}^m \dot{q}^n \dot{q}^p, \quad (59)$$

with the correction to the commutative motion given by

$$\Delta_{mnp}^k = \Gamma_{np}^l \theta^{ks} \partial_s g_{lm} - \Gamma_{pl}^k \theta^{ls} \partial_s g_{mn} + \frac{1}{2}g^{lk} \{g_{lm}, g_{np}\} - \theta^{kl} \frac{1}{2} \partial_l \partial_p g_{mn}. \quad (60)$$

It is understood that $\{g_{lm}, g_{np}\} = \partial_s g_{lm} \theta^{st} \partial_t g_{np}$. The correction is proportional to the first power of θ only for small θ . It then happens that while F and g introduce in

the equations of motion terms linear, respectively quadratic, in the velocities \dot{q}^m , θ brings in terms *cubic* in the velocities. This happens once gravity is already in; otherwise, θ -corrections amount to gravitationally-like terms quadratic in the velocities, as already seen in (38).

The equation of motion for a test particle (59) can be used to put a stringent bound on θ , *e.g.* by studying the additional precession it induces for the Laplace-Runge-Lenz vector of planetary orbits. However, negligible improvement occurs over the extremely strong bound ($\sqrt{\theta} \sim 10^3 l_{Planck}$) already put in [15] using strictly Newtonian NC-deformed gravity. This bound suggests that noncommutativity is unlikely to be found at accessible, say LHC, scales.

3.3. SIGNIFICANCE OF θ

We saw that F is related to the field-strength of a gauge field, and g to the metric of a gravitational theory. One may then ask: what does θ correspond to?

To provide an answer, one can first go to a higher dimensional space, by promoting the phase-space $\{q, p\}$ to an ordinary (commutative) configuration space $\{x\} = \{q, p\}$. The fields $F(q, p)$ and $\theta(q, p)$ are then united into one field $\mathcal{F}(x)$, living in the enlarged space x , on which the Maxwell equations are imposed. θ is thus interpreted as a part of an electromagnetic background living in a higher dimensional space. The relation with noncommutative dynamics is established in the following way. Consider a particle with mass M_0 and 'higher dimensional trajectory' $x(t)$, interacting with the \mathcal{F} field. The Lagrangian is

$$L' = \int dt \left(\frac{M_0}{2} \dot{x}_a^2 - \mathcal{A}_b \dot{x}_b \right) \quad (61)$$

where $\mathcal{F}_{ab}(x) = \partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a$. x_a, x_b denote both q_i and p_j . To distinguish them, use also the notation $q_i \equiv x_i, q_j \equiv x_j, p_i \equiv x_I, p_j \equiv x_J$, etc. Consider now a background field \mathcal{F} of the form

$$\mathcal{F}_{ij} = F_{ij}^{-1}(x) \quad \mathcal{F}_{IJ} = \theta_{IJ}^{-1}(x) \quad \mathcal{F}_{iI} = 1 \quad \mathcal{F}_{iJ} = 0. \quad (62)$$

By a simple extension of a classic argument [4] it can be shown that in the limit

$$M_0 \rightarrow 0 \quad (63)$$

the until now ordinary point particle will transform into one evolving in a phase-space $\{q_i, p_j\}$ having the 'noncommutative' symplectic structure (30), and driven by a quadratic Hamiltonian [unless additional terms appear in (61)].

In conclusion, one can interpret the noncommutativity form θ as a part of a higher dimensional electromagnetic field \mathcal{F} , with the noncommutative dynamics (30) of test particles arising upon taking the limit $\frac{M_0}{\mathcal{F}} \rightarrow 0$. This interpretation also sug-

gests that one can give dynamics to θ simply by starting at the level of the higher-dimensional electromagnetic-like field \mathcal{F} .

4. DARBOUX TRANSFORMATIONS

To further understand how nontrivial commutators can generate gauge-type backgrounds, we provide explicit Darboux transformations for the symplectic structures of noncommutative mechanics with arbitrary nonconstant noncommutativity parameter and magnetic field [16]. Consider the following nonconstant Poisson brackets:

$$\{q_i, p_i\} = \delta_{ij}, \quad \{q_i, q_j\} = \theta_{ij}(q, p), \quad \{p_i, p_j\} = F_{ij}(q, p), \quad (64)$$

Usually the Poisson brackets are taken to be constant and space is taken to be two-dimensional ($i, j = 1, 2$; $\theta_{12} \equiv \theta$; $F_{12} \equiv F$ above). In this case Darboux coordinates Q_i, P_j (in terms of which the Poisson brackets are canonical: $\{Q_i, P_j\} = \delta_{ij}$, $\{Q_i, Q_j\} = 0$, $\{P_i, P_j\} = 0$) are easily found [12], *e.g.*

$$Q_1 = \frac{q_1 + \theta p_2}{1 - F\theta}, \quad Q_2 = q_2, \quad P_1 = p_1, \quad P_2 = \frac{p_2 + Fq_1}{1 - F\theta}. \quad (65)$$

We wish to generalize the above to nonconstant Poisson brackets, a technically more difficult case. For brevity, we will stay in (2+1) dimensions.

Given a symplectic form $\omega(q, p)$ and a Hamiltonian $H_0(q, p)$, one may in principle render the symplectic form canonical, ω_0 , by going to Darboux coordinates $Q(q, p), P(q, p)$. Explicit expressions are rarely available; we will find them for the generic structure (64). To find the new Hamiltonian $H(Q, P)$ one must invert the Darboux transformation, *i.e.* express the old phase-space coordinates q, p in terms of the Darboux ones Q, P , in $H_0(q, p)$. The new $H(Q, P)$ and the canonical ω_0 are completely equivalent dynamically to the initial noncanonical system (ω, H_0) .

4.1. GENERAL VIEW

Start with the simpler case in which $\theta F = 0$. Take first $\theta = 0$. By the Jacobi identities implied by (64), F cannot depend on the momenta. Then $\{p_i, p_j\} = F_{ij}(q)$ simply mimics a magnetic background. More precisely,

$$Q_i = q_i, \quad P_i = p_i + A_i(q), \quad (66)$$

are canonical space coordinates, provided

$$\partial_i A_j - \partial_j A_i = F_{ij}. \quad (67)$$

If one substitutes (66) in the Hamiltonian $H(p, q) = \frac{p^2}{2m} + V(q)$ one obtains

$$H(P, Q) = \frac{(P_i - A_i)^2}{2m} + V(Q), \quad (68)$$

the Hamiltonian in presence of a magnetic background (67), expressed now in terms of canonical coordinates.

If on the other hand $F = 0$ in (64), the Jacobi identities imply that θ cannot depend on the coordinates, $\theta(p)$. The Darboux transformation is

$$Q_i = q_i - \mathcal{A}_{i+2}(p), \quad P_i = p_i, \quad (69)$$

$$\partial_{p_i} \mathcal{A}_{j+2} - \partial_{p_j} \mathcal{A}_{i+2} = \theta_{ij}(p). \quad (70)$$

Upon replacement in $2H = p^2 + q^2$ a 'P-space' magnetic field results, which upon the subsequent transformation (canonical this time) $P' = Q, Q' = -P$ is mapped into a real one. If $V(q)$ contains higher than quadratic terms a similar picture applies, but with nonquadratic kinetic term in the end.

Consider now the simplest form of the $F\theta \neq 0$ case, namely F and θ constant. Denote by x_a , $a = 1, 2, 3, 4$ the phase space coordinates, $x_{1,2,3,4} = q_1, q_2, p_1, p_2$. Eqs. (64) can then be rewritten as $\{x_a, x_b\} = \Theta_{ab}$, where

$$\Theta = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & F \\ 0 & -1 & -F & 0 \end{pmatrix} \text{ i.e., } \omega = \frac{1}{1-\theta F} \begin{pmatrix} 0 & F & -1 & 0 \\ -F & 0 & 0 & -1 \\ 1 & 0 & 0 & \theta \\ 0 & 1 & -\theta & 0 \end{pmatrix}. \quad (71)$$

Above, $\Theta_{ab} = (\omega^{-1})_{ab}$, and ω is the symplectic form, which enters the action

$$S = \int dt \left(\frac{1}{2} \omega_{ab} x_a \dot{x}_b - H(x) \right). \quad (72)$$

out of which one variationally derives [22] the equations of motion

$$\dot{x}_a = \{x_a, H\} = \Theta_{ab} \frac{\partial H}{\partial x_b}. \quad (73)$$

If $F\theta \neq 0$ two different possibilities appear: We can put the $\det \Theta^{-1} = 1/(1-\theta F)$ factor either in the symplectic form - like in Eq. (71), or into the commutation relations - by transferring it from ω to Θ in (71). Both cases present interest and lead to important differences for nonconstant $\Theta(x)$. One can start either from the general Eqs. (64), or from the generalization of Eq. (72),

$$\tilde{S} = \int dt [\mathcal{A}_a(x) \dot{x}_a - H(x)], \quad (74)$$

where

$$\partial_a \mathcal{A}_b - \partial_b \mathcal{A}_a = \omega_{ab}(x) \quad (75)$$

and $\omega_{ab}(x)$ has the form (71), but without the $(1-\theta F)^{-1}$ factor in front; it now multiplies $\Theta \equiv \omega^{-1}$.

4.2. STARTING FROM THE COMMUTATION RELATIONS

Consider a generalized electromagnetic background $F(q, p)$, living on a space with noncommutativity field $\theta(q, p)$, and flat metric $g_{ij} = \delta_{ij}$, with Poisson brackets (64). The Jacobi identities are given in Eqs. (32,33,34,35). They imply that F 's and θ 's should depend on both q 's and p 's (unless they are constant) and consequently mix the q - and p -dependence. In the two-dimensional case Eqs. (34,35) are identically satisfied, whereas (32,33) read become Eqs. (39,40). q_1 and p_2 , respectively q_2 and p_1 , appear in pairs in θ and F .

Let us search for canonical coordinates Q_i, P_j . The simplest possible Ansatz is

$$Q_1 = q_1 + f(q, p), \quad Q_2 = q_2, \quad P_1 = p_1, \quad P_2 = p_2 + g(q, p). \quad (76)$$

Imposing $\{Q_1, Q_2\} = 0$ and $\{Q_1, P_1\} = 1$ leads to

$$\frac{\partial f}{\partial q_1} = \frac{F\theta}{1-\theta F}, \quad \frac{\partial f}{\partial p_2} = \frac{\theta}{1-\theta F}. \quad (77)$$

The integrability condition $\partial_{p_2} \frac{F\theta}{1-\theta F} = \partial_{q_1} \frac{\theta}{1-\theta F}$ is satisfied thanks to the Jacobi identities, and one obtains

$$f = \int dp_2 \frac{\theta}{1-F\theta} + f_1(q_1; q_2, p_1) = \int dq_1 \frac{F\theta}{1-\theta F} + f_2(p_2; q_2, p_1). \quad (78)$$

Similarly the integrability condition for

$$\frac{\partial g}{\partial q_1} = \frac{F}{1-\theta F}, \quad \frac{\partial g}{\partial p_2} = \frac{F\theta}{1-\theta F}. \quad (79)$$

is automatically satisfied due to the Jacobi identities and one has

$$g = \int dp_2 \frac{\theta F}{1-F\theta} + g_1(q_1; q_2, p_1) = \int dq_1 \frac{F}{1-\theta F} + g_2(p_2; q_2, p_1). \quad (80)$$

Above, f_1, f_2, g_1, g_2 are for the time being arbitrary functions of the variables indicated in Eqs. (78,80). Since $\{Q_2, P_1\} = 0$ is trivially satisfied, the last condition to impose is $\{Q_1, P_2\} = 0$. The resulting nonlinear partial differential equation *linearizes* when Eqs. (77,79) are used. It simplifies then to $\frac{\partial f}{\partial q_2} + \frac{\partial g}{\partial p_1} = 0$, which is automatically obeyed by the already fixed part of the solutions (78,80) above, again due to the Jacobi identities! The functions f_1, f_2, g_1, g_2 must consequently also obey

$$\frac{\partial f_1}{\partial q_2} + \frac{\partial g_1}{\partial p_1} = 0. \quad (81)$$

This is automatically satisfied if we take the initial Poisson brackets to depend only on the pair q_1, p_2 . The Darboux coordinates are then found to be given by Eqs. (76, 78, 80).

When θ and F are constant, Eqs. (77,79) are immediately solved by

$$f = q_1 \frac{\theta F}{1 - F\theta} + p_2 \frac{\theta}{1 - F\theta}, \quad g = q_1 \frac{F}{1 - F\theta} + p_2 \frac{F\theta}{1 - F\theta}. \quad (82)$$

One recovers (65) and the inverse transformation $q_1 = Q_1 - \theta P_2$, $p_2 = P_2 - FQ_1$.

In order for the case θ, F constant to follow smoothly from the above solutions (78,80), we should also request

$$f_1 \xrightarrow{F, \theta \text{ cst}} q_1 \frac{F\theta}{1 - F\theta}; \quad f_2 \xrightarrow{F, \theta \text{ cst}} p_2 \frac{\theta}{1 - F\theta}; \quad (83)$$

One solution is for instance [we consider first $\theta, F(q_1, p_2)$ only]

$$f_1 = q_1 \frac{F(q_1, p_2^0)\theta(q_1, p_2^0)}{1 - F(q_1, p_2^0)\theta(q_1, p_2^0)}, \quad f_2 = p_2 \frac{\theta(q_1^0, p_2)}{1 - F(q_1^0, p_2)\theta(q_1^0, p_2)} \quad (84)$$

for some conveniently chosen p_2^0 and q_1^0 .

Remark: If $\theta F = 1$ then one gets dimensional reduction and *constant* brackets in the reduced space - cf. Section 3 and [11].

4.3. STARTING FROM THE SYMPLECTIC STRUCTURE

The Poisson brackets are now

$$\{q_1, q_2\} = \frac{\theta}{1 - \theta F}, \quad \{p_1, p_2\} = \frac{F}{1 - \theta F}, \quad \{q_i, p_j\} = \frac{\delta_{ij}}{1 - \theta F}. \quad (85)$$

The main difference is that the Jacobi identities now imply that $F(q)$ and $\theta(p)$ (with then usual Jacobi), and thus decouple the q - and p -dependence. To find a Darboux transformation for arbitrary $F_{12}(q)$ and $\theta_{12}(p)$, define

$$P_1 \equiv p_1 + A_1(q), \quad P_2 \equiv p_2 + A_2(q), \quad (86)$$

$$Q_1 \equiv q_1 - \mathcal{A}_3(p), \quad Q_2 \equiv q_2 - \mathcal{A}_4(p), \quad (87)$$

where we are guided by the Ansatz

$$\begin{aligned} \frac{\partial A_2(q)}{\partial q_1} - \frac{\partial A_1(q)}{\partial q_2} + \{A_1, A_2\} &= F(q), \\ \frac{\partial \mathcal{A}_4(p)}{\partial p_1} - \frac{\partial \mathcal{A}_3(p)}{\partial p_2} + \{\mathcal{A}_3, \mathcal{A}_4\} &= \theta(p), \end{aligned} \quad (88)$$

remnescent of the Non-Abelian definition of field strenghts. It is simplest to achieve canonical Poisson brackets between the Q 's and the P 's through, for instance,

$$A_1 = 0, \quad \mathcal{A}_4 = 0, \quad \frac{\partial A_2(q)}{\partial q_1} = F(q), \quad \frac{\partial \mathcal{A}_3(p)}{\partial p_2} = -\theta(p). \quad (89)$$

It is easy to check that the new phase-space coordinates $Q_{1,2}, P_{1,2}$ are canonical. In consequence one possible Darboux transformation is

$$Q_1 = q_1 + \int_{p_1} \theta(p), \quad Q_2 = q_2, \quad P_1 = p_2, \quad P_2 = p_2 + \int_{q_1} F(q). \quad (90)$$

Remark: Due to the dependence of F on q and of θ on p , no more dimensional reduction (enforced iff $\theta F = 1$) can take place, unless F, θ are both constant.

5. FEYNMAN APPROACH

Feynman assumed a set of commutation relations plus the Jacobi property of the commutator, plus the Leibniz (derivation) property for time derivatives, together with second order equations of motion. He obtained a Lorentz-like force law plus the two sourceless Maxwell equations (no magnetic monopoles and Faraday law). Let us review Feynman's reasoning, according to Dyson [6]. Assume

$$[x_i, x_j] = 0 \quad (91)$$

$$m[x_i, \dot{x}_j] = \hbar \delta_{ij} \quad (92)$$

$$m\ddot{x}_j = F_J(x, \dot{x}, t) \quad (93)$$

Taking the time derivative of (92)

$$[x_i, F_j] = -m[\dot{x}_i, \dot{x}_j] \equiv f_{ij} = -f_{ji} \quad (94)$$

We got an antisymmetric tensor. Using now the Jacobi identity

$$[x_k, [\dot{x}_i, \dot{x}_j]] + [\dot{x}_i, [\dot{x}_j, x_k]] + [\dot{x}_j, [x_k, \dot{x}_i]] = 0 \quad (95)$$

we see that f_{ij} does not depend on velocities

$$f_{ij} = f_{ij}(x, t). \quad (96)$$

Introducing the pseudovector \vec{B} via

$$f_{kj} = \epsilon_{kji} B_i, \quad B_i = \frac{1}{2} \epsilon_{kji} f_{kl} \quad (97)$$

and using once more Jacobi

$$[\dot{x}_k, [\dot{x}_i, \dot{x}_j]] + \text{cyclic} = 0 \quad (98)$$

we obtain

$$\nabla \cdot \vec{B} = \partial_k B_k = 0., \quad (99)$$

i.e. the divergenceless character of the magnetic field.

Given that (up to orderings)

$$[x_i, F_j] = \frac{\partial}{\partial \dot{x}_i} F_j = f_{ij}(x, t) \quad (100)$$

we can decompose the force F_j into a term independent of the velocities and a term linear in the velocities:

$$F_j = E_j(x) + \dot{x}_k f_{kj}(x) = E_j + \epsilon_{kji} \dot{x}_k B_i. \quad (101)$$

Finally, taking the time derivative of $f_{ij} = -m[\dot{x}_i, \dot{x}_j]$ we obtain after a little bit of massaging the Faraday law part of Maxwell's equations:

$$\partial_t \vec{B} = \nabla \times \vec{E}. \quad (102)$$

Beyond the “how is that possible?” question that we will quickly discuss, two precise issues appear right away: with Dyson, one wonders what happens to the other pair of Maxwell equations, and how relativistic results follow from nonrelativistic assumptions. Dyson's statement that the other two Maxwell equations “merely define” the sources was quickly corrected in subsequent comments [17] stressing their essential and independent character. As for “Galilean mechanics and Maxwell equations coexisting peacefully” [6] in the first pair, one recalls that those are kinematical equations ensuring the existence of vector potentials, and indeed are invariant under both Galilei and Lorentz transformations. The other pair however is not invariant under Galilei transformations [17], see also [23, 25]. A Galilean electrodynamics was actually constructed [18] (see also [19]) and - in one of the two possible versions - Maxwell's displacement current is absent from Ampere's law, rendering that equation nonrelativistically invariant. Hence there is no real coexistence of relativistic and nonrelativistic symmetry groups.

Concerning the structure of the proof, one first notices that the use of commutators automatically implies the Leibniz rule and the Jacobi identity. On the other hand, at the classical level a Hamiltonian formalism would be needed to implement the Leibniz derivation rule. Yet, such a formalism is guaranteed by the Feynman assumptions, cf. [20, 21]. This is why he failed to go beyond it. Therefore one can parallel the proof working exclusively with Poisson brackets, although assuming a Hamiltonian formalism from the beginning makes things easier [20]. Feynman probably chose to work with velocities because they can be immediately related to coordinates by taking a time derivative, allowing to interconnect the assumptions. On the other hand, starting with generalized momenta would have required an explicit Hamiltonian to tell the relationship between the q 's and the p 's.

Ref. [21] showed that the commutativity assumption (91) is essential for the existence of a Lagrangian variational formalism (see also [22]). Starting with [20] - which stressed that Feynman's arguments restrict *any* kind of force (not just electromagnetic), examples from different areas appeared [23, 24]. A relativistic formula-

tion was discussed in [26, 27]. The dynamics of Yang-Mills particles and its relationship to the Wong effective equations appeared in [26, 28]. Gravitational backgrounds were discussed in [26].

6. NC FEYNMAN PROBLEM

Allowing coordinates not to commute, Feynman's hypotheses get modified to

$$\{x_i, x_j\} = \theta_{ij} \quad (103)$$

$$m\{x_i, \dot{x}_j\} = \hbar\delta_{ij} \quad (104)$$

$$m\ddot{x}_j = F_j(x, \dot{x}, t) \quad (105)$$

Taking the time derivative of (104) we obtain again an antisymmetric tensor f_{ij} :

$$\{x_i, F_j\} = -m\{\dot{x}_i, \dot{x}_j\} \equiv f_{ij} \quad (106)$$

In three dimensions the connection with a magnetic field appears of course through $f_{ij} = \epsilon_{ijk}B_k$, $B_k = \frac{1}{2}\epsilon_{ijk}f_{ij}$. The situation differs from the commutative case through the dependence of f_{ij} on both coordinates and velocities, as we will see below.

Considering Jacobi for a triplet of velocities already shows the differences, as

$$\{\dot{x}_i, \{\dot{x}_j, \dot{x}_k\}\} + \text{cyclic} \sim \{\dot{x}_i, f_{jk}\} + \text{cyclic} = 0, \quad (107)$$

therefore

$$\frac{\partial f_{jk}}{\partial x_i} + f_{im} \frac{\partial f_{jk}}{\partial \dot{x}_m} + \text{cyclic}\{i; j; k\} = 0 \quad (108)$$

which does not ensure anymore the divergenceless character of \vec{B} found previously in (99).

The novelty clearly appears through the Jacobi identity for two velocities and one coordinate. The *constancy* of the coordinate-velocity commutator/bracket (104),

$$\{x_k, \{\dot{x}_i, \dot{x}_j\}\} = -\{\dot{x}_i, \{\dot{x}_j, x_k\}\} - \{\dot{x}_j, \{x_k, \dot{x}_i\}\} \equiv 0, \quad (109)$$

leads to

$$\{x_k, f_{ij}\} = \theta_{kl} \frac{\partial f_{ij}}{\partial x_l} + \frac{\partial f_{ij}}{\partial \dot{x}_k} = 0. \quad (110)$$

One notes the parallel with (32) in Section 3. Again, explicitating this constraint will be the key to further progress. Although one can also handle more complicated higher dimensional cases, consider the simplest one, in which only $\theta_{12} = -\theta_{21}$ is nonzero. Working further in two dimensions and denoting

$$\theta_{12} = -\theta_{21} \equiv \theta, \quad f_{12} = -f_{21} \equiv f \quad (111)$$

Eq. (110) reduces to

$$\{x_1, f\} = \frac{\partial f}{\partial \dot{x}_1} + \theta \frac{\partial f}{\partial x_2} = 0, \quad \{x_2, f\} = \frac{\partial f}{\partial \dot{x}_2} - \theta \frac{\partial f}{\partial x_1} = 0; \quad (112)$$

which amounts to saying that f is actually a function of only two variables instead of four [compare to Eqs. (38) and (40)]

$$f(x_1, x_2; \dot{x}_1, \dot{x}_2) = f(x_1 + \theta \dot{x}_2; x_2 - \theta \dot{x}_1) \quad (113)$$

On the other hand, switching to the natural variables

$$\begin{aligned} u_1 &= x_1 + \theta \dot{x}_2, & v_1 &= \dot{x}_1, \\ u_2 &= x_2 - \theta \dot{x}_1, & v_2 &= \dot{x}_2, \end{aligned} \quad (114)$$

(through a unity-Jacobian transformation) leads to

$$\{x_1, \cdot\} = \frac{\partial}{\partial \dot{x}_1} + \theta \frac{\partial}{\partial x_2} = \frac{\partial}{\partial v_1}, \quad \{x_2, \cdot\} = \frac{\partial}{\partial \dot{x}_2} - \theta \frac{\partial}{\partial x_1} = \frac{\partial}{\partial v_2}. \quad (115)$$

Therefore (112) simply says that f is a function only of u_1 and u_2 . Furthermore, noting that (106) now becomes

$$f(u_1, u_2) = -\{x_2, F_1\} = -\frac{\partial F_1}{\partial v_2}, \quad \frac{\partial F_1}{\partial v_1} = 0, \quad (116)$$

$$f(u_1, u_2) = +\{x_1, F_2\} = +\frac{\partial F_2}{\partial v_1}, \quad \frac{\partial F_2}{\partial v_2} = 0, \quad (117)$$

immediately leads to a generalized Lorentz force law

$$F_1 = -v_2 f(u_1, u_2) + E_1(u_1, u_2), \quad F_2 = +v_1 f(u_1, u_2) + E_2(u_1, u_2). \quad (118)$$

Again, the force depends linearly on the (now generalized) velocities v_1 and v_2 , and electric fields E_1 , E_2 and a magnetic field f_{12} depending exclusively on the (generalized) coordinates u_1, u_2 can be identified.

To derive the generalization of Faraday's law (102) we take the time derivative of $f_{12} \equiv f$ in two different ways. First,

$$\begin{aligned} \frac{df}{dt} &= \frac{d}{dt} (-m\{\dot{x}_1, \dot{x}_2\}) \\ &= -\{F_1, \dot{x}_2\} - \{\dot{x}_1, F_2\} \\ &= -\{E_1(u) - v_2 f(u), v_2\} - \{v_1, E_2(u) + v_1 f(u)\} \\ &= (1 + \theta f) [(\partial_1 E_2 - \partial_2 E_1) + v_1 \partial_1 f + v_2 \partial_2 f]. \end{aligned} \quad (119)$$

In the last equality we used the notation $\partial_1 f = \frac{\partial f}{\partial u_1}$, $\partial_2 f = \frac{\partial f}{\partial u_2}$ and the Poisson brackets of the u, v variables:

$$\begin{aligned} \{v_1, v_2\} &\equiv -\frac{1}{m} f, & \{u_1, u_2\} &= -\theta(1 - \theta f), \\ \{u_1, v_1\} &= 1 + \theta f, & \{u_1, v_2\} &= 0, \\ \{u_2, v_2\} &= 1 + \theta f, & \{u_2, v_1\} &= 0. \end{aligned} \quad (120)$$

Second,

$$\begin{aligned}
\frac{df}{dt} &= \partial_t f + \dot{u}_1 \partial_1 f + \dot{u}_2 \partial_2 f \\
&= \partial_t f + (\partial_1 f)(\dot{x}_1 + \theta \ddot{x}_2) + (\partial_2 f)(\dot{x}_2 - \theta \ddot{x}_1) \\
&= \partial_t f + (\partial_1 f)(v_1 + \frac{\theta}{m} F_2) + (\partial_2 f)(v_2 - \frac{\theta}{m} F_1) \\
&= \partial_t f + (v_1 \partial_1 f + v_2 \partial_2 f)(1 + \theta f) + \frac{\theta}{m} (E_2 \partial_1 f - E_1 \partial_2 f).
\end{aligned} \tag{121}$$

Equating (119) and (121) one obtains

$$(\partial_1 E_2 - \partial_2 E_1) - \partial_t f = \theta [(E_2 \partial_1 f - E_1 \partial_2 f) - (\partial_1 E_2 - \partial_2 E_1) f] \tag{122}$$

which is the NC generalization of the Faraday law obtained by pushing to its limits Feynman's procedure. The right hand side in (122) represents the NC correction *in the form* of the law. In content, one should also notice that in the left hand side as well the partial derivatives are taken with respect to the u_1, u_2 variables, not with respect to the initial coordinates x_1 and x_2 .

The NC Feynman problem was first considered in [29], although with little success. They incorrectly concluded that noncommutativity enforces a constant magnetic field and proceeded to some artificial assumptions in order to bypass that apparent constraint. The problem was taken further in [30], which obtained the first modified law, Eq. (108). They however did not solve the Jacobi identity constraints (110) either, and as such were unable to *deduce* the correct dependence of the force law on velocities. This of course affected their derivation of the modified Faraday law as well. The derivations given in this section were taken from [31].

7. DISCUSSION AND FURTHER TOPICS

7.1. EARLY REFERENCES

It is difficult to trace back with certainty the first reference which transferred dynamics from the Hamiltonian to the symplectic structure. The idea was apparently rediscovered independently by several workers, and this reviewer did not attempt to ascribe priorities. Possibly the earliest reference using brackets and especially symplectic forms to introduce interactions is [32].

On the other hand, the reasoning of Souriau somehow parallels (in a more sophisticated set-up) the ideas of Feynman (cca. 1948, cf. F. Dyson). If one sticks to commutation relations and either their representation involving gauge potentials or their form involving field strengths, one might take a serious look at the early contributions of Dirac and Landau. Further on, Souriau seems to acknowledge a debt towards ideas of Lagrange and Maxwell...

7.2. WIGNER PROBLEM

One should also mention a kind of inverse problem: *Given* the equations of motion, can one find - uniquely or not - the commutation relations?

The question was asked for the first time by Wigner [33], and answered in the negative for the harmonic oscillator and the free particle. On the other hand, the linear and cubic potentials led uniquely to the standard Heisenberg commutation relations. The paper of Wigner immediately gave rise to an interesting discussion [34]. For instance, Putnam extended Wigner's positive answer to odd integer power potentials, whereas Yang and Vachaspati tried to disprove Wigner's negative conclusion by introducing additional criteria. Field theory discussions were attempted/produced by Schweber and by Arnowitt and Deser.

More recently, the problem came back into attention, for instance in the context of the inverse problem of the calculus of variations [35, 36]. More sophisticated approaches were also introduced, see for instance [37]. Wigner systems were also discussed in the context of the Feynman problem [38] or were considered as an autonomous mini-field of research [39]. An interesting reference replacing a canonical structure with an equivalent noncanonical one, in the context of vortex dynamics, is [40].

7.3. APPLICATIONS

The generation of magnetic fields through commutators was put to use to bypass problems with gauge invariance in a NC context [41, 42]. The role played in obtaining simple experimental bounds [10] and in dimensional reduction [11] was already mentioned.

Fields generated *via* commutators enter naturally in the path integral approach to NC mechanics [13, 43] and the equivalence of NC dynamics to a higher order system [43, 44]. An alternative approach to NC mechanics "à la Souriau" and its applications was pursued in Refs. [14, 45–47]. A nice review trying to strike a balance between the commutator and symplectic views is [48], see also [49].

A formalism, subsequently called "the covariant formalism" in which conserved quantities were investigated starting from bracket introduced interactions was proposed in [50]. Examples of its use can be found in [51–53].

Acknowledgements

Financial support from the ANCS, Contract PN 16 42 01 01/2016, and from UEFISCDI (Contract Idei-PCE No 121/2011) is acknowledged.

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