

# NUMERICAL SPECTRAL LEGENDRE-GALERKIN ALGORITHM FOR SOLVING TIME FRACTIONAL TELEGRAPH EQUATION

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*Abstract.* This paper is concerned with presenting and analyzing a new technique for solving time fractional telegraph equation. This technique is based on applying the spectral Galerkin method that depends on choosing an appropriate basis functions satisfying the underlying boundary conditions. A double shifted Legendre expansion is proposed as an approximating polynomial. A careful study for the convergence and error analysis of the suggested approximate double expansion is performed. Some numerical results are provided aiming to ensure the efficiency and applicability of the proposed algorithm.

*Key words:* Legendre polynomials; Galerkin method; fractional telegraph equation; Zeilberger's algorithm.

## 1. INTRODUCTION

In the last few decades, it has been pointed out by many authors that the generalizations of the ordinary derivatives of integer orders and integrals are very suitable for the description of various real phenomena. In fact, many problems in science and engineering are modeled by fractional differential equations (FDEs), so many theoretical and practical applications of various FDEs have been performed by a large number of mathematicians and physicians. For some of the important problems of FDEs, one can be referred to [1–5].

It is well-known that no exact solutions for the majority of FDEs can be obtained, so it is very necessary to resort to numerical methods for solving these equations. In this regard, several numerical algorithms have been implemented and analyzed for obtaining approximate solutions of such equations. Some of the techniques used are: Adomian's decomposition method [6], finite difference method [7], tau method [8–11], wavelets methods [12, 13], homotopy analysis transform method [14] and operational matrix methods [15–17]. Some other techniques can be found in [18–22].

Spectral methods have been applied extensively for the numerical solution of various differential equations. The main idea of the application of spectral method is based on choosing the solution of a problem under investigation as a combination of some selected polynomials that are often orthogonal. The different kinds of spectral

methods are applied successfully in a variety of papers. For example, the tau method is applied in [23] and the collocation method is employed in [24–27]. Regarding the Galerkin method that requires to select basis functions satisfying the boundary (initial) value problems, it is employed to solve linear boundary value problems, see, for example [28]. The tau method has the advantage that it avoids some of the problems of the Galerkin methods, since we can choose any set of orthogonal polynomials as basis functions and the boundary (initial) conditions are set as constraints (see, for example [29]). The philosophy of applying the collocation method is to select certain collocation points such that the differential equation is satisfied exactly at these points. This method is convenient for treating nonlinear problems (see, for example [25–27, 30, 31]).

Telegraph equations are commonly used in the study of wave propagation of electric signals in a cable transmission line and also in wave phenomena. In recent years, much attention has been given in the literature to the development of numerical schemes for the telegraph type equations (see, for example [32–38]).

The main objective of this work is to present and analyze a new method for approximating the time fractional telegraph equation based on the application of spectral Legendre-Galerkin method. The content of the paper is as follows. The next Section presents some fundamentals of the fractional calculus theory. Also, some properties and new formulae concerned with Legendre polynomials are given in this Section. A Galerkin approach for treating fractional telegraph type equation is presented in detail in Sec. 3. Discussion of the convergence and error analysis of the suggested double expansion is given in Sec. 4. Numerical tests are displayed in Sec. 5. Finally, some conclusions are reported in Sec. 6.

## 2. PRELIMINARIES

In this Section we present some fundamentals of the fractional calculus theory that will be useful throughout this article. In addition, some properties and formulae concerned with the shifted Legendre polynomials are presented.

### 2.1. SOME ESSENTIALS OF FRACTIONAL CALCULUS

This subsection is devoted to presenting some fundamental definitions and preliminary facts of the fractional calculus theory. For fundamentals of this branch, one can be referred to [39].

**Definition 1.** *The Riemann-Liouville fractional integral operator  $I^\alpha$  of order  $\alpha$  on the usual Lebesgue space  $L_1[0, 1]$  is defined as*

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \quad (1)$$

The following properties of  $I^\alpha$  are valid:

$$\begin{aligned} (i) \quad I^\alpha I^\beta &= I^{\alpha+\beta}, \\ (ii) \quad I^\alpha I^\beta &= I^\beta I^\alpha, \\ (iii) \quad I^\alpha t^\nu &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} t^{\nu+\alpha}, \end{aligned}$$

where  $\alpha, \beta \geq 0$ , and  $\nu > -1$ .

**Definition 2.** The left and right handed Riemann-Liouville fractional-order operators are defined, respectively, as follows:

$$({}^R D_x^\gamma f)(x) = \frac{1}{\Gamma(r-\gamma)} \frac{d^r}{dx^r} \int_a^x (x-\tau)^{r-\gamma-1} f(\tau) d\tau, \quad (2)$$

$$({}^R D_b^\gamma f)(x) = \frac{(-1)^r}{\Gamma(r-\gamma)} \frac{d^r}{dx^r} \int_x^b (\tau-x)^{r-\gamma-1} f(\tau) d\tau, \quad (3)$$

where  $r-1 \leq \gamma < r, r \in \mathbb{N}$ .

**Definition 3.** For a function  $f$  defined on the interval  $I = [a, b]$ , the left and right handed Caputo fractional-order derivatives are defined as:

$$({}^C D_x^\gamma f)(x) = \frac{1}{\Gamma(r-\gamma)} \int_a^x (x-\tau)^{r-\gamma-1} f^{(r)}(\tau) d\tau, \quad \gamma > 0, t > 0, \quad (4)$$

$$({}^C D_b^\gamma f)(x) = \frac{(-1)^r}{\Gamma(r-\gamma)} \int_x^b (\tau-x)^{r-\gamma-1} f^{(r)}(\tau) d\tau, \quad \gamma > 0, t > 0, \quad (5)$$

where  $r-1 \leq \gamma < r, r \in \mathbb{N}$ .

It is worthy to mention here that the operator  ${}^C D_t^\alpha$  satisfies the following fundamental properties, for  $n-1 \leq \alpha < n$ ,

$$\begin{aligned} ({}^C D_t^\alpha I^\alpha f)(t) &= f(t), \\ {}^C D_t^\alpha t^k &= \frac{\Gamma(k+1)}{\Gamma(k+1-\alpha)} t^{k-\alpha}, \quad k \text{ is a positive integer, } k \geq \lceil \alpha \rceil, \end{aligned} \quad (6)$$

where  $\lceil \alpha \rceil$  is the well-known ceiling function.

## 2.2. SOME PROPERTIES AND RELATIONS CONCERNED WITH SHIFTED LEGENDRE POLYNOMIALS

It is well-known that the standard Legendre polynomials  $L_k(z); z \in [-1, 1]$  form a complete orthogonal system for  $L^2(-1, 1)$ . Moreover, these polynomials

satisfy the following orthogonality relation

$$\int_{-1}^1 L_j(z) L_k(z) dz = \frac{2}{2k+1} \delta_{j,k}.$$

where  $\delta_{j,k}$  is the well-known Kronecker delta function.

We denote by  $L_k^*(z)$  the shifted Legendre polynomials defined on  $[0, 1]$  as:

$$L_k^*(z) = L_k(2z - 1).$$

The shifted Legendre polynomials form a complete orthogonal system for  $L^2[0, 1]$ . They have the following orthogonality relation

$$\int_0^1 L_j^*(z) L_k^*(z) dz = \frac{1}{2k+1} \delta_{j,k}. \quad (7)$$

$L_k^*(z)$  has the following analytic form (see, [40])

$$L_k^*(z) = \sum_{i=0}^k \frac{(-1)^{k+i} (k+i)!}{(k-i)! (i!)^2} z^i. \quad (8)$$

Also, the following recurrence relation is satisfied by  $L_k^*(z)$ :

$$(k+1)L_{k+1}^*(z) = (2k+1)(2z-1)L_k^*(z) - kL_{k-1}^*(z), \quad L_0^*(z) = 1, \quad L_1^*(z) = 2z-1, \\ k \geq 1. \quad (9)$$

Now, assume that for any function  $f(z) \in L^2[0, 1]$ ,  $f(z)$  can be expanded as

$$f(z) = \sum_{k=0}^{\infty} c_k L_k^*(z),$$

where

$$c_k = (2k+1) \int_0^1 f(z) L_k^*(z) dz. \quad (10)$$

For more properties of Legendre polynomials see, for example [40].

The following theorem and lemma are needed in the sequel.

**Theorem 1.** If  $I_i^{(r)}(x)$  denotes the  $r$  times repeated integration of  $L_i^*(x)$ :

$$I_i^{(r)}(x) = \underbrace{\int \int \cdots \int}_{r \text{ times}} L_i^*(x) \underbrace{dx dx \cdots dx}_{r \text{ times}}.$$

The following relation holds ([41]):

$$I_i^{(r)}(x) = \frac{(b-a)^r}{2^{2r}} \sum_{j=0}^r \binom{r}{j} (-1)^j \frac{(i+r-2j+\frac{1}{2}) \Gamma(i-j+\frac{1}{2})}{\Gamma(i+r-j+\frac{3}{2})} L_{i+r-2j}^*(x) + \pi_{r-1}(x),$$

where  $\pi_{r-1}(x)$  is a polynomial of degree does not exceed  $(r-1)$ .

**Lemma 1.** Let  $\phi_k(t) = t(1-t)L_k^*(t)$  and let  $\mu$  be any real number. The following integral formula is valid:

$$\int_0^1 t^\mu \phi_k(t) dt = \frac{(2+\mu-k-k^2)(\Gamma(2+\mu))^2}{\Gamma(3+\mu-k)\Gamma(4+\mu+k)}. \quad (11)$$

*Proof.* First, we will show that, for any real number  $\mu$ , one has

$$\int_0^1 t^\mu L_k^*(t) dt = \frac{(\Gamma(\mu+1))^2}{\Gamma(\mu-k+1)\Gamma(\mu+k+2)}. \quad (12)$$

The power form representation (8) for  $L_k^*(t)$  enables one to write

$$\begin{aligned} \int_0^1 t^\mu L_k^*(t) dt &= \sum_{i=0}^k \frac{(-1)^{i+k}(i+k)!}{(i!)^2(k-i)!} \int_0^1 t^{i+\mu} dt \\ &= \sum_{i=0}^k \frac{(-1)^{i+k}(i+k)!}{(i!)^2(i+\mu+1)(k-i)!}. \end{aligned}$$

Now, if we set

$$B_{\mu,k} = \sum_{i=0}^k \frac{(-1)^{i+k}(i+k)!}{(i!)^2(i+\mu+1)(k-i)!},$$

then, it can be shown by means of Zeilberger's algorithm that  $B_{\mu,k}$  satisfies the following difference equation:

$$(k-\mu)B_{\mu,k} + (k+\mu+2)B_{\mu,k+1} = 0, \quad B_{\mu,0} = \frac{1}{\mu+1},$$

which can be immediately solved to give

$$B_{\mu,k} = \frac{(\Gamma(\mu+1))^2}{\Gamma(\mu-k+1)\Gamma(\mu+k+2)}.$$

Now, since

$$\int_0^1 t^\mu \phi_k(t) dt = \int_0^1 (t^{\mu+1} - t^{\mu+2}) L_k^*(t) dt,$$

then making use of (12), formula (11) can be obtained. This proves Lemma 1.  $\square$

### 3. GALERKIN APPROACH FOR TREATING FRACTIONAL TELEGRAPH TYPE EQUATION

This Section focuses on presenting and analyzing in detail how to handle fractional telegraph type equation (see [42]):

$$(D_t^\alpha + D_t^{\alpha-1} - D_x^2 + 1)u(x,t) = f(x,t), \quad (13)$$

with the initial and boundary conditions:

$$u(x, 0) = \xi(x), \quad u(x, 1) = \eta(x), \quad (14)$$

$$u(0, t) = \rho(t), \quad u(1, t) = \sigma(t), \quad (15)$$

where  $\alpha \in (1, 2]$ ,  $f(x, t)$ ,  $\xi(x)$ ,  $\eta(x)$ ,  $\rho(t)$ , and  $\sigma(t)$  are continuous functions.

We will employ the spectral Galerkin method for solving (13)-(15). The philosophy of the application of this method is basically built on choosing suitable basis functions satisfying the underlying conditions governed by the differential equation. Now, we make use of the transformation:

$$v(x, t) = u(x, t) - \mu(x, t),$$

where

$$\begin{aligned} \mu(x, t) = & (1-x)\rho(t) + x\sigma(t) + (1-t)(\xi(x) - (1-x)\xi(0) - x\xi(1)) + t(\eta(x) \\ & - (1-x)\rho(1) - x\sigma(1)), \end{aligned}$$

to turn equation (13) governed by (14) and (15) into the following modified one:

$$(D_t^\alpha + D_t^{\alpha-1} - D_x^2 + 1)v(x, t) = g(x, t), \quad (16)$$

governed by the homogeneous initial and boundary conditions:

$$v(x, 0) = v(x, 1) = v(0, t) = v(1, t) = 0. \quad (17)$$

and

$$g(x, t) = f(x, t) - (D_t^\alpha + D_t^{\alpha-1} - D_x^2 + 1)\mu(x, t).$$

To seek for a numerical solution for (16)-(17), we propose the following approximate solution:

$$v(x, t) \approx v_n(x, t) = \sum_{i=0}^n \sum_{j=0}^n c_{i,j} \phi_i(x) \phi_j(t), \quad (18)$$

where the basis functions  $\phi_i(z)$  are chosen to be

$$\phi_k(z) = z(1-z)L_k^*(z).$$

Note that the polynomials  $\{\phi_k(z)\}_{k \geq 0}$  are orthogonal on  $[0, 1]$  with respect to  $w(z) = \frac{1}{z(1-z)}$ .

The expansion coefficients in (18) are determined by the formula

$$c_{i,j} = (2i+1)(2j+1) \int_0^1 \int_0^1 v(x, t) \phi_i(x) \phi_j(t) w(x, t) dx dt, \quad (19)$$

where  $w(x, t) = \frac{1}{x(1-x)t(1-t)}$ .

Now, if we define the following spaces:

$$\begin{aligned} S_n &= \text{span}\{L_i^*(x)L_j^*(t) : i, j = 0, 1, \dots, n\}, \\ Z_n &= \{z(x, t) \in S_n : z(x, 0) = z(x, 1) = z(0, t) = z(1, t) = 0, \quad 0 < x, t < 1\}, \end{aligned} \quad (20)$$

then the Legendre-Galerkin approximation to (16)-(17) is to find  $v_n(x, t) \in Z_n$  such that  $\forall v(x, t) \in Z_n$

$$((D_t^\alpha + D_t^{\alpha-1} - D_x^2 + 1)v_n(x, t), v(x, t)) = (g(x, t), v(x, t)), \quad (21)$$

where  $(u, v) = \int \int_\Omega uv \, dx \, dt$  is the scalar product in the Sobolev space  $L^2(\Omega)$ ,  $\Omega = (0, 1) \times (0, 1)$ .

The Galerkin variational formulation (21) is equivalent to

$$\begin{aligned} ((D_t^\alpha + D_t^{\alpha-1} - D_x^2 + 1)v_n(x, t), \phi_r(x)\phi_s(t)) &= (g(x, t), \phi_r(x)\phi_s(t)), \\ 0 \leq r, s \leq n. \end{aligned} \quad (22)$$

If we denote

$$v_n(x, t) = \sum_{i=0}^n \sum_{j=0}^n c_{ij} \phi_i(x)\phi_j(t), \quad C = (c_{ij})_{0 \leq i, j \leq n}, \quad (23)$$

$$a_{r,s} = (\phi_r(x), \phi_s(x)), \quad A = (a_{r,s})_{0 \leq r, s \leq n}, \quad (24)$$

$$b_{r,s}^{(\alpha)} = (D_t^\alpha \phi_r(t), \phi_s(t)) \quad B^{(\alpha)} = (b_{r,s}^{(\alpha)})_{0 \leq r, s \leq n}, \quad (25)$$

$$d_{r,s} = (D_x^2 \phi_r(x), \phi_s(x)), \quad D = (d_{r,s})_{0 \leq r, s \leq n}, \quad (26)$$

$$g_{r,s} = (g(x, t), \phi_r(x)\phi_s(t)), \quad G = (g_{r,s})_{0 \leq r, s \leq n}, \quad (27)$$

then equation (22) can be written alternatively in the following matrix form:

$$(B^{(\alpha)} + B^{(\alpha-1)} - D + A)C A^T - G = 0, \quad (28)$$

where the nonzero elements of the matrices  $A, B^{(\alpha)}$ , and  $D$  are given explicitly in the following theorem.

**Theorem 2.** *The nonzero elements of the matrices  $A, B^{(\alpha)}$ , and  $D$  are given explic-*

itly as follows:

$$\begin{aligned}
 b_{r,s}^{(\alpha)} &= \sum_{j=0}^r \frac{(-1)^{j+r} (j+r)! (\kappa_{j,s}^{(\alpha)} - \kappa_{j+1,s}^{(\alpha)})}{(j!)^2 (r-j)!}, \\
 \kappa_{j,s}^{(\alpha)} &= \frac{(j+1)! \Gamma(j-\alpha+3)^2 (-\alpha+j-s(s+1)+3)}{\Gamma(j-\alpha+2) \Gamma(j-s-\alpha+4) \Gamma(j+s-\alpha+5)}, \\
 d_{r,r} &= -\frac{r^4 + 2r^3 + 3r^2 + 2r - 2}{2(2r-1)(2r+1)(2r+3)}, \\
 d_{r,r+2} = d_{r+2,r} &= \frac{(r+1)^2 (r+2)^2}{4(2r+1)(2r+3)(2r+5)}, \\
 a_{r,r} &= \frac{3r^4 + 6r^3 - 11r^2 - 14r + 12}{8(2r-3)(2r-1)(2r+1)(2r+3)(2r+5)}, \\
 a_{r,r+2} = a_{r+2,r} &= -\frac{(r+1)(r+2)(r^2+3r-2)}{4(2r-1)(2r+1)(2r+3)(2r+5)(2r+7)}, \\
 a_{r,r+4} = a_{r+4,r} &= \frac{(r+1)(r+2)(r+3)(r+4)}{16(2r+1)(2r+3)(2r+5)(2r+7)(2r+9)}.
 \end{aligned}$$

*Proof.* First, to compute the elements of the matrix  $A = (a_{r,s})$ , we have

$$a_{r,s} = (x(1-x)L_r^*(x), x(1-x)L_s^*(x)). \quad (29)$$

If we write the recurrence relation (9) as

$$xL_r^*(x) = \gamma_r L_{r-1}^*(x) + \beta_r L_r^*(x) + \alpha_r L_{r+1}^*(x), \quad (30)$$

where

$$\gamma_r = \frac{r}{2(2r+1)}, \quad \beta_r = \frac{1}{2}, \quad \alpha_r = \frac{r+1}{2(2r+1)},$$

then it is easy to see that

$$x(1-x)L_r^*(x) = \sum_{\ell=0}^4 \theta_{\ell,r} L_{r+\ell-2}^*(x), \quad (31)$$

and

$$\begin{aligned}
 \theta_{1,r} &= -\gamma_r \gamma_{r-1}, & \theta_{2,r} &= \gamma_r (1 - \beta_r - \beta_{r-1}), & \theta_{3,r} &= \beta_r - \alpha_r \gamma_{r+1} - (\beta_r)^2 - \gamma_r \alpha_{r-1}, \\
 \theta_{4,r} &= \alpha_r (1 - \beta_{r+1} - \beta_r), & \theta_{5,r} &= -\alpha_r \alpha_{r+1}.
 \end{aligned}$$

Now, in virtue of relation (31), the elements  $a_{r,s}$  are given by the formula

$$a_{r,s} = \left( \sum_{\ell=0}^4 \theta_{\ell,r} L_{r+\ell-2}^*(x), \sum_{\ell=0}^4 \theta_{\ell,s} L_{s+\ell-2}^*(x) \right). \quad (32)$$

With the aid of (32) together with the orthogonality relation (7), the elements  $a_{r,s}$  can be easily obtained.

Now, to compute the nonzero elements of the matrix  $D = (d_{r,s})$ . We have

$$d_{r,s} = (D^2 \phi_r(x), \phi_s(x)) = \int_0^1 D^2 (x(1-x)L_r^*(x)) x(1-x)L_s^*(x) dx,$$

which can be –after performing some computations– written as

$$d_{r,s} = \sum_{i=0}^r \frac{(-1)^{i+r}(i+r)!}{(i!)^2(r-i)!} \{i(i+1)\xi_{s,i-1} - (i+1)(i+2)\xi_{s,i}\},$$

and

$$\xi_{s,i} = \int_0^1 x^i \phi_s(x).$$

Lemma 1 enables one to write the elements  $d_{r,s}$  in the following explicit form

$$a_{r,s} = \sum_{i=0}^r \frac{(-1)^{i+r}(i+r)!}{(r-i)!} \left( \frac{i(i+1)(i-s(s+1)+1)}{(i-s+1)!(i+s+2)!} - \frac{(i+1)^3(i+2)(i-s(s+1)+2)}{(i-s+2)!(i+s+3)!} \right). \quad (33)$$

The summation in right hand side of (33) can be computed in a closed form for all values of  $r$  and  $s$ . Hence all nonzero elements of the matrix  $a_{r,s}$  can be obtained.

For  $s = r + 2$ , it can be shown with the aid of Zeilberger's algorithm that  $\xi_r = a_{r,r+2}$  satisfies the following difference equation:

$$(2r+7)(r+1)^2 \xi_{r+1} - (2r+1)(r+3)^2 \xi_r = 0, \quad \xi_0 = \frac{1}{15}. \quad (34)$$

The exact solution of (34) is

$$d_{r,r+2} = \frac{(r+1)^2(r+2)^2}{4(2r+1)(2r+3)(2r+5)}.$$

Also, for  $s = r$ ,  $\eta_r = d_{r,r}$  satisfies the following difference equation:

$$(1-2r)(r^4+6r^3+15r^2+18r+6)\eta_r + (2r+5)(r^4+2r^3+3r^2+2r-2)\eta_{r+1} = 0, \\ \eta_0 = -\frac{1}{3}. \quad (35)$$

The exact solution of (35) is

$$d_{r,r} = -\frac{r^4+2r^3+3r^2+2r-2}{2(2r-1)(2r+1)(2r+3)}.$$

Moreover, it can be shown that

$$d_{r+2,r} = d_{r,r+2},$$

and all the other elements of the matrix  $D$  are zeros.

Finally, we compute the elements of the matrix  $B^{(\alpha)}$ . The elements  $b_{r,s}^{(\alpha)}$  can be calculated by the formula

$$b_{r,s}^{(\alpha)} = (D^\alpha \phi_r(x), \phi_s(x)).$$

Relation (8) along with Lemma 1 enables one to write

$$b_{r,s}^{(\alpha)} = \sum_{i=0}^r \frac{(-1)^{r+i} (r+i)!}{(r-i)! (i!)^2} \left[ \frac{(i+1)!}{\Gamma(i+2-\alpha)} \zeta_{s,i+1-\alpha} - \frac{(i+2)!}{\Gamma(i+3-\alpha)} \zeta_{s,i+2-\alpha} \right],$$

where

$$\zeta_{k,\mu} = \frac{(2+\mu-k-k^2)(\Gamma(2+\mu))^2}{\Gamma(3+\mu-k)\Gamma(4+\mu+k)},$$

and hence

$$b_{r,s}^{(\alpha)} = \sum_{i=0}^r \frac{(-1)^{r+i} (r+i)!}{(r-i)! (i!)^2} \left[ \kappa_{i,s}^{(\alpha)} - \kappa_{i+1,s}^{(\alpha)} \right].$$

This completes the proof of the Theorem.  $\square$

#### 4. DISCUSSION OF THE CONVERGENCE AND ERROR ANALYSIS OF THE SUGGESTED DOUBLE EXPANSION

This Section is devoted to investigating the convergence and error analysis of the proposed approximate solution. We give *a priori* estimate of the resulting global error. In this respect, we state and prove two theorems. First, the following lemma is needed.

**Lemma 2.** For every positive integer  $p$ , one has

$$\sum_{r=0}^p \frac{\binom{p}{r} (i+p-2r+\frac{1}{2}) \Gamma(i-r+\frac{1}{2})}{\Gamma(i+p-r+\frac{3}{2})} = \frac{\Gamma(\frac{2i-2p+3}{4})}{\Gamma(\frac{2i+2p+3}{4})}. \quad (36)$$

*Proof.* If we set

$$A_p = \sum_{r=0}^p \frac{\binom{p}{r} (i+p-2r+\frac{1}{2}) \Gamma(i-r+\frac{1}{2})}{\Gamma(i+p-r+\frac{3}{2})},$$

and make use of the celebrated algorithm of Zeilbeger, then it can be shown that the following difference equation of order one is satisfied by  $A_p$

$$(2i-2p-1)(2i+2p+3)A_{p+2} - 16A_p = 0, \quad A_0 = 1.$$

The above recurrence relation can be immediately solved to give

$$A_p = \frac{\Gamma\left(\frac{2i-2p+3}{4}\right)}{\Gamma\left(\frac{2i+2p+3}{4}\right)}.$$

This proves the lemma.  $\square$

**Theorem 3.** Let  $v(x, t)$  be the exact solution of Eqs. (16)-(17). Suppose that  $v(x, t) = v_1(x)v_2(t)$ , and assume that there exist positive constants  $M_1, M_2$ , such that,  $\left|\frac{d^p v_1}{dx^p}\right| < M_1$ ,  $\left|\frac{d^q v_2}{dt^q}\right| < M_2$ , for some positive integers  $p, q$ . If  $v(x, t)$  is approximated as:  $v(x, t) \approx \sum_{i=0}^n \sum_{j=0}^n c_{i,j} \phi_i(x) \phi_j(t)$ , then the expansion coefficients  $c_{i,j}$  satisfy the following estimate

$$|c_{i,j}| = \mathcal{O}(i^{\frac{1}{2}-p} j^{\frac{1}{2}-q}), \quad \forall i > p > \frac{3}{2}, j > q > \frac{3}{2}.$$

*Proof.* Starting from Eq. (19), we have

$$c_{i,j} = (2i+1)(2j+1) \int_0^1 \int_0^1 v(x, t) \phi_i(x) \phi_j(t) w(x, t) dx dt.$$

The choice of the polynomials  $\phi_i(x)$  implies that

$$c_{i,j} = (2i+1)(2j+1) \int_0^1 v_1(x) L_i(2x-1) dx \int_0^1 v_2(t) L_j(2t-1) dt. \quad (37)$$

If integration by parts is applied  $p$  times to the first integral in the right hand side of (37) and  $q$  times to the second integral, then after taking Theorem 3 into consideration, the coefficients  $c_{i,j}$  are given by

$$c_{i,j} = (2i+1)(2j+1)(-1)^{p+q} \int_0^1 v_1^{(p)}(x) I_i^{(p)}(x) dx \int_0^1 v_2^{(q)}(t) I_j^{(q)}(t) dt,$$

where

$$I_i^{(p)}(x) = 4^{-p} \sum_{r=0}^p \frac{(-1)^r \binom{p}{r} (i+p-2r+\frac{1}{2}) \Gamma(i-r+\frac{1}{2})}{\Gamma(i+p-r+\frac{3}{2})} L_{i+p-2r}(2x-1).$$

Now, the hypothesis of the theorem along with Bernstein type inequality

$$|L_i(2x-1)| \leq (2\pi i(x-x^2))^{-\frac{1}{2}},$$

yield

$$\begin{aligned}
 |c_{ij}| &< 2^{-(1+2p+2q)} \pi M_1 M_2 (2i+1)(2j+1) \sum_{r=0}^p \frac{\binom{p}{r} (i+p-2r+\frac{1}{2}) \Gamma(i-r+\frac{1}{2})}{\Gamma(i+p-r+\frac{3}{2}) \sqrt{i-p}} \\
 &\times \sum_{r=0}^q \frac{\binom{q}{r} (j+q-2r+\frac{1}{2}) \Gamma(j-r+\frac{1}{2})}{\Gamma(j+q-r+\frac{3}{2}) \sqrt{j-q}} \\
 &< 2^{-(1+2p+2q)} \pi M_1 M_2 (2i+1)(2j+1) \sum_{r=0}^p \frac{\binom{p}{r} (i+p+\frac{1}{2}) \Gamma(i-r+\frac{1}{2})}{\Gamma(i+p-r+\frac{3}{2}) \sqrt{i-p}} \\
 &\times \sum_{r=0}^q \frac{\binom{q}{r} (j+q+\frac{1}{2}) \Gamma(j-r+\frac{1}{2})}{\Gamma(j+q-r+\frac{3}{2}) \sqrt{j-q}}
 \end{aligned}$$

The application of Lemma 2 leads to

$$|c_{ij}| < \frac{2^{-(1+2p+2q)} \pi M_1 M_2 (2i+1)(2j+1) \Gamma(\frac{3}{4} + \frac{i}{2} - \frac{p}{2}) \Gamma(\frac{3}{4} + \frac{j}{2} - \frac{q}{2})}{\sqrt{i-p} \sqrt{j-q} \Gamma(\frac{3}{4} + \frac{i}{2} + \frac{p}{2}) \Gamma(\frac{3}{4} + \frac{j}{2} + \frac{q}{2})}.$$

Finally, if we apply the Stirling asymptotic approximation of Gamma function (see [43])

$$\Gamma(z) = \mathcal{O}(z^{z-\frac{1}{2}} e^{-z}),$$

we get the desired result. □

**Theorem 4.** *Under the hypothesis of Theorem 3, we have the following error estimate*

$$e_n = |u(x, t) - u_n(x, t)| = \mathcal{O}(n^{3-p-q}).$$

*Proof.* We have

$$\begin{aligned}
 e_n &= |u(x, t) - u_n(x, t)| \\
 &= \left| \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} c_{i,j} L_i(2x-1) L_j(2t-1) \right| \\
 &\leq \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} |c_{i,j} L_i(2x-1) L_j(2t-1)|.
 \end{aligned}$$

If we use the well-known identity  $|L_r(2z-1)| \leq 1$  and apply Theorem 3, then we conclude that there exists a generic constant  $\Upsilon$ , such that

$$|c_{ij}| \leq \Upsilon i^{\frac{1}{2}-p} j^{\frac{1}{2}-q},$$

and accordingly, we have

$$e_n \leq \Upsilon \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} i^{1-p} j^{1-q}. \quad (38)$$

Since the remainder of the convergent series  $\sum_{i=0}^{\infty} f(i)$ , satisfies the following estimate (see[44])

$$\sum_{i=n+1}^{\infty} f(i) \leq \int_n^{\infty} f(x) dx - \frac{1}{2} f(n+1) < \int_n^{\infty} f(x) dx,$$

then we have

$$\begin{aligned} e_n &\leq \Upsilon \int_n^{\infty} x^{\frac{1}{2}-p} dx \int_n^{\infty} y^{\frac{1}{2}-q} dy \\ &= \frac{\Upsilon}{(p-\frac{3}{2})(q-\frac{3}{2})} n^{3-p-q}, \end{aligned}$$

which completes the proof of the Theorem.  $\square$

## 5. TEST PROBLEMS

In this Section, in order to illustrate the efficiency of the method described in this paper, which we call Legendre-Galerkin matrix method (LGMM), we present some numerical results to find approximations  $u_n(x, t)$  to the solutions  $u(x, t)$  of some test problems. We consider the following error norm,

$$E = \max_{(x,t) \in (0,1) \times (0,1)} |u(x, t) - u_n(x, t)|.$$

All calculations are carried out using Mathematica 11.

**Example 1.** Consider the following telegraph problem [42]:

$$\begin{aligned} \frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial^{\alpha-1} u}{\partial t^{\alpha-1}} + u - \frac{\partial^2 u}{\partial x^2} &= (2 - 2t + t^2)(x - x^2)e^{-t} + 2t^2 e^{-t}, \\ (x, t) &\in (0, 1) \times (0, 1), \end{aligned} \quad (39)$$

subject to the initial and boundary conditions

$$u(0, t) = u(1, t) = 0, \quad t \in [0, 1],$$

$$u(x, 0) = 0, \quad u(x, 1) = (x - x^2)e^{-1}, \quad x \in [0, 1].$$

The exact solution of Eq. (39) in case  $\alpha = 2$  is given by  $u(x, t) = e^{-t} x^{1+\alpha}$ . We apply LGMM, the maximum pointwise errors are listed in Table 1, and we compare our results with those obtained by the method developed in [42] in Table 2.

Table 1

Maximum pointwise error of Example 1.

$n$	$\alpha$	$E$	$\alpha$	$E$	$\alpha$	$E$
4		$2.54 \cdot 10^{-5}$		$2.65 \cdot 10^{-5}$		$7.38 \cdot 10^{-5}$
8	1.25	$5.61 \cdot 10^{-9}$	1.5	$3.27 \cdot 10^{-9}$	1.5	$7.36 \cdot 10^{-9}$
12		$4.83 \cdot 10^{-13}$		$8.31 \cdot 10^{-13}$		$1.37 \cdot 10^{-13}$

Table 2

Maximum absolute errors of Example 1 for the case  $\alpha = 2$ .

$t$	$n = 3$		$n = 6$		$n = 10$	
	LGMM	[42]	LGMM	[42]	LGMM	[42]
0.6	$9.27 \cdot 10^{-5}$	$9.62 \cdot 10^{-3}$	$3.67 \cdot 10^{-7}$	$4.43 \cdot 10^{-3}$	$2.22 \cdot 10^{-12}$	$2.19 \cdot 10^{-3}$
0.7	$3.84 \cdot 10^{-5}$	$1.998 \cdot 10^{-2}$	$8.51 \cdot 10^{-7}$	$2.67 \cdot 10^{-3}$	$1.27 \cdot 10^{-12}$	$1.45 \cdot 10^{-3}$
0.8	$4.87 \cdot 10^{-5}$	$1.29 \cdot 10^{-2}$	$5.22 \cdot 10^{-7}$	$2.14 \cdot 10^{-3}$	$2.17 \cdot 10^{-12}$	$1.62 \cdot 10^{-3}$
1	0	0	0	0	0	0

**Example 2.** Consider the following telegraph problem:

$$(D_t^\alpha + D_t^{\alpha-1} - D_x^2 + 1)u(x, t) = \frac{t^{-\alpha} \sinh x}{\Gamma(1 - \alpha)}, \quad (x, t) \in (0, 1) \times (0, 1), \quad (40)$$

subject to the initial and boundary conditions

$$u(0, t) = 0 \quad u(1, t) = e^{-1} \sinh 1, \quad t \in [0, 1],$$

$$u(x, 0) = \sinh x \quad u(x, 1) = e^{-1} \sinh x, \quad x \in [0, 1],$$

with the exact solution  $u(x, t) = e^{-t} \sinh x$ . We apply LGMM, the maximum pointwise errors are listed in Table 3.

Table 3

Maximum pointwise error of Example 2.

$n$	$\alpha$	$E$	$\alpha$	$E$	$\alpha$	$E$	$\alpha$	$E$
5		$1.47 \cdot 10^{-5}$		$2.35 \cdot 10^{-6}$		$3.49 \cdot 10^{-6}$		$5.13 \cdot 10^{-7}$
10	1.2	$5.61 \cdot 10^{-10}$	1.6	$3.27 \cdot 10^{-10}$	1.8	$1.57 \cdot 10^{-10}$	2	$2.14 \cdot 10^{-11}$
15		$7.35 \cdot 10^{-15}$		$6.28 \cdot 10^{-15}$		$5.91 \cdot 10^{-15}$		$8.65 \cdot 10^{-16}$

**Example 3.** Consider the following telegraph problem:

$$(D_t^\alpha + D_t^{\alpha-1} - D_x^2 + 1)u(x, t) = g(x, t), \quad (x, t) \in (0, 1) \times (0, 1), \quad (41)$$

subject to the initial and boundary conditions

$$u(0, t) = u(1, t) = 0, \quad t \in [0, 1],$$

$$u(x, 0) = u(x, 1) = 0, \quad x \in [0, 1].$$

The function  $g(x, t)$  is chosen such that the exact solution of (41) is  $u(x, t) = (1 - x)(1 - t) \sinh x \sin t$ . We apply LGMM and list the maximum pointwise errors in Table 4.

Table 4

Maximum pointwise error of Example 3.

$n$	$\alpha$	$E$	$\alpha$	$E$	$\alpha$	$E$	$\alpha$	$E$
6		$2.4 \cdot 10^{-7}$		$3.5 \cdot 10^{-7}$		$4.3 \cdot 10^{-7}$		$8.1 \cdot 10^{-9}$
9	1.25	$5.4 \cdot 10^{-11}$	1.5	$7.2 \cdot 10^{-11}$	1.75	$5.2 \cdot 10^{-11}$	2	$4.7 \cdot 10^{-12}$
12		$3.7 \cdot 10^{-14}$		$2.8 \cdot 10^{-14}$		$4.5 \cdot 10^{-14}$		$2.2 \cdot 10^{-16}$

## 6. CONCLUDING REMARKS

This work provides a new approach to obtain numerical solutions for fractional order telegraph problem. A double basis function in terms of shifted Legendre polynomials is used, and the spectral Galerkin method is applied for obtaining the proposed numerical solutions. To the best of our knowledge, the Galerkin method proposed in this paper to treat the fractional telegraph equation is novel. The presented numerical results show that the suggested algorithm is efficient, applicable, and easy in implementation.

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