NUMERICAL SOLUTIONS OF THE QUADRATIC FORM OF THE SPINLESS SALPETER-TYPE EQUATION

JIAO-KAI CHEN

School of Physics and Information Science,
Shanxi Normal University, Linfen 041004, China
E-mail: chenjk@sxnu.edu.cn, chenjkphy@yahoo.com

Received March 12, 2017

Abstract. In this paper, numerical solutions of the quadratic form of the spinless Salpeter-type equation (QSSE) are given by employing the Nyström method with the TANH rule. The upper bounds on the ground-state energy are presented by employing the variational method.

Key words: QSSE, numerical solutions, upper bounds on eigenvalues.

PACS: 03.65.Pm, 03.65.Ge, 11.10.St, 12.39.Pn.

1. INTRODUCTION

The well-known spinless Salpeter equation (SSE) [1, 2] is a relativistic extension of the nonrelativistic Schrödinger equation and a well-defined standard approximation to the Bethe-Salpeter equation [3, 4] which is the appropriate tool to describe the bound states within the relativistic quantum field theory. The SSE takes the linear mass form and the quadratic form of the reduction of the Bethe-Salpeter equation is presented in Refs. [5, 6].

In momentum space, the spinless Salpeter equation is written as

$$M\Psi(\mathbf{p}) = (\omega_1+\omega_2)\Psi(\mathbf{p}) + \int \frac{d^3\mathbf{p}'}{(2\pi)^3} V(\mathbf{p},\mathbf{p}')\Psi(\mathbf{p}'), \tag{1}$$

where M is the mass of the bound state, $\omega_i = \sqrt{\mathbf{p}^2+m_i^2}$, m_i is the mass of the constituent i, and $V(\mathbf{p},\mathbf{p}')$ is the potential. The quadratic mass form of the spinless Salpeter-type equation (QSSE) reads [5, 6]

$$M^2\Psi(\mathbf{p}) = (\omega_1+\omega_2)^2\Psi(\mathbf{p}) + \int \frac{d^3\mathbf{p}'}{(2\pi)^3} V'(\mathbf{p},\mathbf{p}')\Psi(\mathbf{p}'), \tag{2}$$

where $V' = 2(m_1+m_2)V$ in the nonrelativistic limit. The mass operator of SSE (1) is linear

$$M = \omega_1+\omega_2+V, \tag{3}$$

while the mass operator of QSSE (2) takes the quadratic form

$$M^2 = (\omega_1+\omega_2)^2 + V'. \tag{4}$$

These two semirelativistic equations (1) and (2) can be obtained from the Salpeter equation for the bound state composed of two scalar constituents [6, 7]

$$\left[M^2-(\omega_1+\omega_2)^2\right]\Psi(\mathbf{p})=\frac{\omega_1+\omega_2}{2\omega_1\omega_2}\eta(\mathbf{p}), \tag{5}$$

where $\eta(\mathbf{p})$ reads

$$\eta(\mathbf{p})=\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\mathcal{V}(\mathbf{p},\mathbf{p}')\Psi(\mathbf{p}'). \tag{6}$$

Equations (1) and (2) can also be derived from the Salpeter equation for a fermion-antifermion system [3] which is written in the center-of-momentum frame of the bound state

$$\Psi(\mathbf{p})=\frac{\Lambda_1^+(\mathbf{p})\gamma^0\eta'(\mathbf{p})\gamma^0\Lambda_2^-(-\mathbf{p})}{M-\omega_1-\omega_2}-\frac{\Lambda_1^-(\mathbf{p})\gamma^0\eta'(\mathbf{p})\gamma^0\Lambda_2^+(-\mathbf{p})}{M+\omega_1+\omega_2}, \tag{7}$$

by neglecting the small term and then neglecting all the spin degrees of freedom of constituents. In Eq. (7), $\Lambda_i^{\pm}(\mathbf{p})$ is the energy projection operator for particle i,

$$\eta'(\mathbf{p})=\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\mathcal{V}'(\mathbf{p},\mathbf{p}')\Psi(\mathbf{p}'). \tag{8}$$

In Ref. [5], QSSE is obtained by using the assumption that $\ln W$ (W being the Wilson loop correlator) can be written in QCD as the sum of its perturbative expression and an area term.

Although Eq. (1) is a semirelativistic equation, Eq. (1) is applied not only to the nonrelativistic case but also to the relativistic case, even to the ultrarelativistic case [2, 8]. In Ref. [6], we discussed SSE and QSSE by the generalized virial theorem. In this paper, we will discuss them numerically.

This paper is organized as follows: In Sec. 2, the numerical solutions of the quadratic form of the spinless Salpeter-type equation are presented and we conclude in Sec. 3.

2. NUMERICAL SOLUTIONS OF QSSE

In this section, the Nyström method with the TANH rule is employed to solve the quadratic form of the spinless Salpeter-type equation. And the upper bounds on the eigenvalue of the ground state of the QSSE is given by applying the variational method.

2.1. ANALYTICAL TREATMENT

The Coulomb potential is taken as an example in this paper, which is the most prominent potential among the central potentials which are of particular importance

in all physics,

$$V = -\frac{4\pi\alpha}{(\mathbf{p}-\mathbf{p}')^2}. \tag{9}$$

Applying the formula

$$V^l(p,p') = \int d\Omega Y_{lm}(\Omega) \int d\Omega' Y_{l'm'}(\Omega') V(\mathbf{p}-\mathbf{p}'), \tag{10}$$

the partial wave expansion of the momentum-space spinless Salpeter equation is obtained from Eq. (1)

$$M_{nl}\psi_{nl}(p) = (\omega_1+\omega_2)\psi_{nl}(\mathbf{p}) + \frac{1}{(2\pi)^3}\int p'^2 dp' V^l(p,p')\psi_{nl}(p'), \tag{11}$$

where n is the principal quantum number, l is the orbital quantum number. The Coulomb potential (9) is expanded in partial wave as

$$V^l(p,p') = -8\pi^2\alpha\frac{Q_l(z)}{pp'}, \quad z = \frac{p^2+p'^2}{2p'p}, \tag{12}$$

where $Q_l(z)$ is the Legendre polynomial of the second kind,

$$\begin{aligned} Q_l(z) &= P_l(z)Q_0(z) - w_{l-1}(z), \quad Q_0(z) = \frac{1}{2}\ln\frac{z+1}{z-1}, \\ w_{l-1}(z) &= \sum_{m=1}^{l}\frac{1}{m}P_{l-m}(z)P_{m-1}(z). \end{aligned} \tag{13}$$

$Q_0(z)$ has a logarithmic singularity as $z \to 1$.

Applying the Landé subtraction method [9–11] to handle the logarithmic singularity, Eq. (11) is rewritten as

$$\begin{aligned} M_{nl}\psi_{nl}(p) =& (\omega_1+\omega_2)\psi_{nl}(p) - \frac{\alpha}{\pi}I_0(z)P_l(1)p\phi_{nl}(p) \\ &- \frac{\alpha}{\pi p}\int_0^\infty \frac{Q_0(z)}{p'}\left[p'^2P_l(z)\phi_{nl}(p') - p^2P_l(1)\phi_{nl}(p)\right]dp' \\ &+ \frac{\alpha}{\pi p}\int_0^\infty w_{l-1}(z_\beta)p'\phi_{nl}(p')\,dp'. \end{aligned} \tag{14}$$

In calculations, we have used the useful identity

$$I_0(z_\eta) = \int_0^\infty \frac{1}{p'}Q_0(z_\eta)\,dp' = \frac{\pi^2}{2} - \pi\arctan\frac{\eta}{p}, \quad z_\eta \equiv \frac{p^2+p'^2+\eta^2}{2pp'}. \tag{15}$$

If the infinite range is cut at a finite point Λ in numerical calculations, then the previ-

ous identity becomes

$$I_0'(z_\eta) = \int_0^\Lambda \frac{1}{p'} Q_0(z_\eta)\, dp' \\ = \frac{1}{2}\left[Li_2(-\frac{i\Lambda}{\eta - ip}) - Li_2(\frac{i\Lambda}{\eta - ip}) + Li_2(\frac{i\Lambda}{\eta + ip}) - Li_2(-\frac{i\Lambda}{\eta + ip}) \right], \tag{16}$$

where $Li_2(z)$ is the Spence's function. The improved identities (15) and (16) are adoptable for $\eta \geq 0$. The partial wave expansion of the QSSE (2) can be obtained by the similar way.

2.2. NUMERICAL RESULTS AND DISCUSSIONS

By using the method proposed in Refs. [9, 10], the eigenvalue integral equations are resolved. The Nyström method with the extended trapezoidal rule is applied to approximate the integral. By employing the variation of the TANH rule, that is to say, the variable transformation [12], $x = e^t$, the numerical results can be calculated efficiently, and the accuracy of the obtained results will be extremely high even as the simple extended trapezoidal rule is implemented.

For simplicity, the special case of equal masses of the two involved bound-state constituents is considered, $m_1 = m_2 = m$. The obtained eigenvalues are listed in Table 1. The input parameters are $h = 0.05$, $p_0/(m\alpha) = 10^{-5}$, $N = 320$, $l = 0$.

Table 1

The binding energies of SSE and QSSE. $\epsilon_{n_r} = (M_{n_r} - 2m)/(m\alpha^2)$ is the binding energy of SSE, and $\epsilon'_{n_r} = (\sqrt{M_{n_r}^2} - 2m)/(m\alpha^2)$ is the bind energy of QSSE. $\alpha_0 = 7.2973525698 \times 10^{-3}$. ϵ_0'' is the upper bound on the ground-state energy, $\epsilon_0'' = \sqrt{4-\alpha^2}/\alpha^2 - 2/\alpha^2$ which is from Eqs. (31) and (35). n_r is equal to the number of nodes in the radial wave function. A negative sign before the energy has been omitted everywhere.

		$\alpha = \alpha_0$	$\alpha = 0.1$	$\alpha = 0.5$
ϵ_{n_r}	$n_r = 0$	0.250011	0.250747	0.267618
	$n_r = 1$	0.062508	0.062629	0.065492
	$n_r = 2$	0.027785	0.027824	0.028739
ϵ'_{n_r}	$n_r = 0$	0.250008	0.250163	0.254041
	$n_r = 1$	0.062507	0.062517	0.062753
	$n_r = 2$	0.027785	0.027487	0.027833
ϵ_0''	$n_r = 0$	0.2500008	0.250156	0.254033

In calculations, potential in QSSE takes the simple form $V' = 2(m_1 + m_2)V$ where V is the potential in SSE equation because V' is complicated especially in

the configuration space. As expected, the yielded eigenvalues from different equations are in agreement well in the nonrelativistic limit when α is small, see Table 1. As the coupling strength α increases, relativistic corrections will be more and more important. Out of expectation, the numerical ground-state eigenvalue of QSSE with $V' = 2(m_1 + m_2)V$ is closer to the upper bounds obtained in subsection 2.3 by the variational method than the calculated ground-state eigenvalue of SSE.

2.3. VARIATIONAL BOUNDS ON THE GROUND-STATE EIGENVALUE

It is well-known that the min-max principle [13–15] provides the theoretical foundation to derive the rigorous upper bounds on the eigenvalues of some self-adjoint operators which are bounded from below. As a consequence of the min-max principle, the Rayleigh-Ritz method provides a straightforward and efficient means of computing non-increasing upper bounds on eigenvalues.

The Rayleigh quotient [16] is defined as

$$F_H(\psi) = \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle}. \tag{17}$$

There is the Rayleigh principle [14]

$$E_0 = \min_{\psi\in D(H)} F_H(\psi). \tag{18}$$

In consequence, there is the inequality given by Rayleigh

$$E_0 \leq F_H(\psi) \quad (\psi \in D(H)), \tag{19}$$

which is a particular case of the min-max principle.

By the Fourier transform

$$\begin{aligned} \psi(\mathbf{r}) &= \frac{1}{(2\pi)^3}\int \Psi(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}} d\mathbf{q}, \\ \Psi(\mathbf{q}) &= \int \psi(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} d\mathbf{r}, \end{aligned} \tag{20}$$

the spinless Salpeter equation (1) and the quadratic form of the spinless Salpeter-type equation (2) are written in the configuration space as, respectively,

$$M\psi(\mathbf{r}) = H_1\psi(\mathbf{r}), \quad H_1 = (\omega_1 + \omega_2) + V(\mathbf{r}), \tag{21}$$

and

$$M^2\psi(\mathbf{r}) = H_2\psi(\mathbf{r}), \quad H_2 = (\omega_1 + \omega_2)^2 + V'(\mathbf{r}), \tag{22}$$

where ω_i are the square-root operators of the relativistic kinetic energy of particles with mass m_i and momentum $\boldsymbol{p}$,

$$\omega_i = \sqrt{\mathbf{p}^2 + m_i^2} = \sqrt{-\Delta + m_i^2}, \tag{23}$$

which are the nonlocal square-root differential operators.

We use the eigenfunction of the ground state of the hydrogen-like atom as a trial function

$$\psi(\mathbf{r}) = \sqrt{\frac{\beta^3}{\pi}} e^{-\beta r}, \quad \langle\psi|\psi\rangle = 1, \tag{24}$$

to evaluate the energy expectation value of the Hamiltonians in Eqs. (21) and (22). Here β is the variational parameter to minimize the expectation value. Using Eqs. (17), (21) and (24), the expectation value of H reads

$$\begin{aligned} F_{H_1}(\psi) &= \langle\psi|H_1|\psi\rangle \\ &\leq \sqrt{\beta^2 + m_1^2} + \sqrt{\beta^2 + m_2^2} - \alpha\beta. \end{aligned} \tag{25}$$

In the above calculation, we have used the relations

$$\langle\psi|\mathbf{p}^2|\psi\rangle = \beta^2, \quad \left\langle\psi\middle|\frac{\alpha}{r}\middle|\psi\right\rangle = \alpha\beta, \tag{26}$$

and the inequality

$$\left\langle\psi\middle|\sqrt{\mathbf{p}^2 + m_i^2}\middle|\psi\right\rangle \leq \sqrt{\langle\psi|\mathbf{p}^2 + m_i^2|\psi\rangle}. \tag{27}$$

In the nonrelativistic limit, we obtain the ground-state mass of Eq. (21) by minimizing Eq. (25)

$$M_0 \leq m_1 + m_2 - \frac{\mu\alpha^2}{2}, \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \tag{28}$$

when

$$\beta_{min} = \alpha\mu. \tag{29}$$

From Eq. (25), we can see that in the ultrarelativistic limit, $m_1 \to 0$, $m_2 \to 0$ or $\beta \to \infty$, the coupling strength should be $\alpha \leq 2$ for the semirelativistic Hamiltonian H (21) which is bounded from below. The mass of the bound state in the ultrarelativistic limit becomes zero, *i.e.*, the bound state is massless, $F_H(\psi) = 0$. As $m_1 = m_2 = m$, minimizing Eq. (25) gives [17]

$$\beta_{min} = \frac{m}{2} \frac{\alpha}{\sqrt{1 - \frac{\alpha^2}{4}}} \tag{30}$$

and

$$M_0 \leq 2m\sqrt{1 - \frac{\alpha^2}{4}}. \tag{31}$$

In case of QSSE, using Eqs. (17), (22) and (24), the expectation value of H_2

reads

$$F_{H_2}(\psi) = \langle\psi|H_2|\psi\rangle \leq \left[\sqrt{\beta^2+m_1^2}+\sqrt{\beta^2+m_2^2}\right]^2 - 2(m_1+m_2)\alpha\beta. \quad (32)$$

In the above calculation, $\langle \mathbf{p}^4\rangle$ is replaced by $\langle \mathbf{p}^2\rangle^2$ for consistence. In the nonrelativistic limit, the upper bound on the ground-state mass of the bound state is obtained by minimizing Eq. (32),

$$M_0 \leq \sqrt{(m_1+m_2)^2 - m_1 m_2 \alpha^2}. \quad (33)$$

Eq. (33) is consistent with Eq. (28). In the ultrarelativistic limit, the expectation value approaches massless, too. As $m_1 = m_2 = m$, minimizing Eq. (32) gives

$$\beta_{min} = \frac{m}{2}\alpha \quad (34)$$

and

$$M_0 \leq 2m\sqrt{1-\frac{\alpha^2}{4}}. \quad (35)$$

The obtained upper bound (35) on the ground state of QSSE is the same as that of SSE (31), which gives the critical value of the coupling strength, $\alpha \leq 2$. As shown in Table 1, the numerical eigenvalues obtained by employing the Nyström method are consistent with the upper bounds on the ground-state energy, see Eqs. (31) and (35).

3. CONCLUSIONS

In this paper, we solve numerically the quadratic form of the spinless Salpeter-type equation by employing the Nyström method with the TANH rule. The numerical eigenvalues of QSSE are in consistence with those of SSE as the coupling strength is small while the differences between them become large as α increases. The upper bounds on the ground-state energy of QSSE by the variational method are presented.

REFERENCES

1. D. P. Stanley and D. Robson, Phys. Rev. D **21**, 3180 (1980); W. Lucha and F. F. Schöberl, Fizika B **8**, 193 (1999).
2. A. Gara, B. Durand, L. Durand and L. J. Nickisch, Phys. Rev. D **40**, 843 (1989).
3. E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951).
4. E. E. Salpeter, Phys. Rev. **87**, 328 (1952).
5. M. Baldicchi, A. V. Nesterenko, G. M. Prosperi, D. V. Shirkov and C. Simolo, Phys. Rev. Lett. **99**, 242001 (2007); M. Baldicchi and G. M. Prosperi, Phys. Rev. D **62**, 114024 (2000); M. Baldicchi, arXiv: hep-ph/9911268; N. Brambilla, E. Montaldi and G. M. Prosperi, Phys. Rev. D **54**, 3506 (1996).

6. J. K. Chen, Acta Phys. Pol. B **47**, 1155 (2016).
7. J. K. Chen, Z. X. Tang and Q. D. Chen, Mod. Phys. Lett. A **22**, 2979 (2007); J. K. Chen and Y. Liu, IL Nuovo Cimento B **122**, 1 (2007).
8. W. Lucha and F. F. Schöberl, Phys. Rev. Lett. **64**, 2733 (1990).
9. J. K. Chen, Phys. Rev. D **86**, 036013 (2012); Phys. Rev. D **89**, 099904(E) (2014); J. K. Chen, Phys. Rev. D **88**, 076006 (2013).
10. J. K. Chen, Few-Body Syst. **54**, 2081 (2013).
11. J. W. Norbury, K. M. Maung, and D. E. Kahana, Phys. Rev. A **50**, 3609 (1994); R. H. Landau, Phys. Rev. C **28**, 1324 (1983); R. H. Landau, Phys. Rev. C **27**, 2191 (1983); Y. R. Kwon and F. Tabakin, Phys. Rev. C **18**, 932 (1978); J. W. Norbury, K. M. Maung, and D. E. Kahana, Phys. Rev. A **50**, 2075 (1994); K. M. Maung, D. E. Kahana, and J. W. Norbury, Phys. Rev. D **47**, 1182 (1993); A. Tang and J. W. Norbury, Phys. Rev. E **63**, 066703 (2001).
12. William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery, *Numerical Recipes in C: the art of scientific computing* (Cambridge University Press, New York, 1992).
13. M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators* (Academic Press, New York, 1978).
14. A. Weinstein and W. Stenger, *Methods of Intermediate Problems for Eigenvalues – Theory and Ramifications* (Academic Press, New York, 1972).
15. G. Teschl, *Mathematical Methods in Quantum Mechanics: with applications to Schrödinger Operators* (http://www.mat.univie.ac.at/∼gerald/ftp/book-schroe/schroe.pdf).
16. C. T. H. Baker, *The Numerical Treatment of Integral Equations* (Oxford University Press, New York, 1977); L. M. Delves and J. L. Mohamed, *Computational Methods for Integral Equations* (Cambridge Univeristy Press, New York, 1992).
17. W. Lucha and F. F. Schöberl, Phys. Rev. D **50**, 5443 (1994).