NONCOMMUTATIVE GRAVITY VIA $SO(2,3)$ NONCOMMUTATIVE GAUGE THEORY

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In this paper the noncommutative gravity is treated as a gauge theory of the noncommutative $SO(2,3)_\star$ group on the noncommutative space with the constant noncommutativity. The enveloping algebra approach and the Seiberg-Witten map are used to relate noncommutative and the commutative gauge theory. By combining different actions a noncommutative gravity model is constructed in such a way that the cosmological constant term is not present in the commutative limit, but it is generated by the noncommutativity and it appears in the higher order expansion. We calculate the second order correction to this model and obtain terms that are zero-th, first, . . . and fourth power of the curvature tensor. Finally, we discuss physical consequences of those correction terms in the low energy limit.

Key words: gauge theory of gravity, Seiberg-Witten map, 2nd order expansion, x-dependent cosmological constant.

1. INTRODUCTION

Noncommutative (NC) gauge theories have been extensively studied and well understood. Different approaches include the enveloping algebra approach and the Seiberg-Witten map [1]. General Relativity (GR) on the other hand is difficult to generalize to the NC setting. One of the main difficulty is the underlying diffeomorphism symmetry. The concept of NC diffeomorphisms is still not well understood. However, GR can also be formulated as a gauge theory of the Poincaré group. Using this approach different models of the NC gravity were studied [2].

Instead of the Poincaré symmetry, one can also use de Sitter $SO(1,4)$ or anti de Sitter $SO(2,3)$ gauge groups. GR is then obtained after the symmetry breaking down to local $SO(1,3)$ [3]. In our previous papers [4], we begun the study of NC gravity based on the AdS gauge group. Here we briefly summarize the main results and discuss the deformed Einstein equation and some possible solutions in different limits of the model.

In the next section we review the commutative model and point to different $SO(2,3)$ gauge invariant action that after the symmetry breaking reduce to the Einstein-Hilbert action with additional terms. In Section 3 we generalize the model to the NC space-time. We work in the simplest NC space-time with the constant noncommutativity. The NC $SO(2,3)_\star$ gauge theory is introduced *via* the SW map. We expand the NC action to the second order in the deformation parameter $\theta^{\alpha\beta}$ and calculate the correction terms to the commutative action. The first order correction vanishes and we confirm the results already present in the literature. Namely, it was shown that if reality of the NC gravity action is imposed, all odd order corrections (in the NC parameter) have to vanish [2]. The first non-vanishing correction is then the second order correction. The correction terms we obtain are of the zeroth, first, ...and fourth power in the curvature tensor and are written in a manifestly covariant way. The term that is the zero-th power in the curvature tensor renormalizes the cosmological constant, that is we obtain a x-dependent cosmological constant. Our model contains three parameters: noncommutativity scale $\theta^{\alpha\beta}$, AdS radius l and the usual energy (renormalization) scale. Combining these parameters we obtain different limits of the model. In the last section we discuss the action and the equations of motion in a specific limit and comment on the possible solutions.

2. COMMUTATIVE ADS GAUGE THEORY

We assume that the space-time has the structure of the 4 dimensional Minkowski space M_4 and follow the usual steps for constructing a gauge theory on M_4 taking the $SO(2,3)$ group as the gauge group. The gauge field takes values in the $SO(2,3)$ algebra, $\omega_\mu = \frac{1}{2}\omega_\mu^{AB}M_{AB}$. Here M_{AB} are the generators of the $SO(2,3)$ group and they fulfil

$$[M_{AB},M_{CD}] = i(\eta_{AD}M_{BC}+\eta_{BC}M_{AD}-\eta_{AC}M_{BD}-\eta_{BD}M_{AC})\,. \tag{1}$$

The $5D$ metric is $\eta_{AB} = \mathrm{diag}(+,-,-,-,+)$. Indices $A,B,\dots$ take the values $0,1,2,3,5$, while indices $a,b,\dots$ take values $0,1,2,3$. A representation of this algebra is given by

$$M_{ab} = \frac{i}{4}[\gamma_a,\gamma_b] = \frac{1}{2}\sigma_{ab}, \quad M_{5a} = \frac{1}{2}\gamma_a\,, \tag{2}$$

where γ_a are four dimensional Dirac gamma matrices. Then the gauge potential ω_μ^{AB} decomposes into ω_μ^{ab} and ω_μ^{a5}

$$\omega_\mu = \frac{1}{2}\omega_\mu^{AB}M_{AB} = \frac{1}{4}\omega_\mu^{ab}\sigma_{ab} - \frac{1}{2}\omega_\mu^{a5}\gamma_a. \tag{3}$$

The field strength tensor is defined in the usual way by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu - i[\omega_\mu,\omega_\nu] = \frac{1}{2}F_{\mu\nu}^{AB}M_{AB} \\ &= \Big(R_{\mu\nu}^{\ ab} - \frac{1}{l^2}(e_\mu^a e_\nu^b - e_\mu^b e_\nu^a)\Big)\frac{\sigma_{ab}}{4} - F_{\mu\nu}^{\ a5}\frac{\gamma_a}{2}\,, \end{aligned} \tag{4}$$

where

$$R_{\mu\nu}^{\ ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac}\omega_\nu^{cb} - \omega_\mu^{bc}\omega_\nu^{ca}\,, \tag{5}$$

$$lF_{\mu\nu}^{\ a5} = D_\mu e_\nu^a - D_\nu e_\mu^a = T_{\mu\nu}^a. \tag{6}$$

Equations (3), (4), (5) and (6) suggest that one can identify ω_μ^{ab} with the spin connection of the Poincaré gauge theory, ω_μ^{a5} with the vielbeins, $R_{\mu\nu}^{ab}$ with the curvature tensor and $F_{\mu\nu}^{a5}$ with the torsion. It was shown in the seventies that one can really do such an identification and relate AdS gauge theory with GR. Different ways were discussed in the literature, see [3]. One way is to start from the following action

$$S_1 = \frac{il}{64\pi G_N}\mathrm{Tr}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\phi\,, \tag{7}$$

where G_N is the Newton gravitational constant. An additional auxiliary field $\phi = \phi^A\Gamma_A$, $\Gamma_A = (i\gamma_a\gamma_5, \gamma_5)$, transforming in the adjoint representation of $SO(2,3)$ is introduced*. One can show that the action (7) is invariant under the $SO(2,3)$ gauge symmetry. However, if we restrict the field ϕ to be $\phi^a = 0,\ \phi^5 = l$, with an arbitrary constant l then the symmetry of the action is reduced to the $SO(1,3)$ gauge symmetry and diffeomorphisms. The constraint on the field ϕ can be introduced *via* a Lagrange multiplier or dynamically [3]. We are not concerned with that problem here. The action obtained after symmetry breaking is given by

$$\begin{aligned} S_1 &= \frac{il^2}{64\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int \mathrm{d}^4x\mathrm{Tr}(F_{\mu\nu}F_{\rho\sigma}\gamma_5) \\ &= -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\Big[\frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R_{\mu\nu}^{\ ab}R_{\rho\sigma}^{\ cd} + eR - e\frac{6}{l^2}\Big]\,, \end{aligned} \tag{8}$$

where $e = \det(e_\mu^a)$. This action is written in the first order formalism: the spin connection ω_μ^{ab} and the vielbeins e_μ^a are independent fields. Varying the action with respect to the spin connection we obtain an equation that relates the spin connection and the vielbeins. After the analysis of the equations of motion, we see that after the symmetry breaking the action (8) describes GR with the negative cosmological constant and the topological Gauss-Bonnet term.

Another model we can use is defined by the action

$$S_2 = \frac{1}{64\pi G_N l}\int d^4x\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(F_{\mu\nu}D_\rho\phi D_\sigma\phi\phi), \tag{9}$$

where $D_\rho\phi = \partial_\rho\phi - i[\omega_\rho, \phi]$ is the $SO(2,3)$ covariant derivative of the auxiliary scalar field ϕ. The $SO(2,3)$ symmetry can be broken in the same way as in the action (7), by choosing $\phi^a = 0,\ \phi^5 = l$. As before, l is an arbitrary constant and the symmetry is reduced to the $SO(1,3)$ and diffeomorphisms. After the symmetry

*Note that $M_{AB} = \frac{i}{4}[\Gamma_A, \Gamma_B]$.

breaking the action reduces to

$$S_2 = -\frac{1}{16\pi G_N}\int d^4x(eR - e\frac{12}{l^2}). \tag{10}$$

The action contains the Einstein-Hilbert and the cosmological constant term, while the Gauss-Bonnet term is absent. Notice that the action (9) is not hermitian (real) and one has to add to it a hermitian conjugate term to impose hermiticity.

Finally, one can also consider a $SO(2,3)$ invariant action of the form

$$S_3 = \frac{i}{64\pi G_N l^3}\int d^4x\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(D_\mu\phi D_\nu\phi D_\rho\phi D_\sigma\phi\phi). \tag{11}$$

After the symmetry breaking (in the same way as in the first two examples) this action reduces to the cosmological constant term

$$S_3 = -\frac{1}{16\pi G_N}\int d^4xe\frac{12}{l^2}. \tag{12}$$

When doing the NC generalization, we will only consider actions (7) and (9). In general, one can consider different combinations of these models depending on the expected/desired results.

3. NC $SO(2,3)_\star$ GAUGE THEORY

The NC space-time we consider is the canonically deformed space-time with the Moyal-Weyl $\star$-product given by

$$\begin{aligned} f\star g\,(x) =& \sum_{n=0}^{\infty}\Big(\frac{i}{2}\Big)^n\frac{1}{n!}\theta^{\rho_1\sigma_1}\dots\theta^{\rho_n\sigma_n} \\ & \Big(\partial_{\rho_1}\dots\partial_{\rho_n}f(x)\Big)\Big(\partial_{\sigma_1}\dots\partial_{\sigma_n}g(x)\Big) \\ =& f\cdot g + \frac{i}{2}\theta^{\rho\sigma}(\partial_\rho f)\cdot(\partial_\sigma g) - \frac{1}{8}\theta^{\rho\sigma}\theta^{\alpha\beta}(\partial_\rho\partial_\alpha f)\cdot(\partial_\sigma\partial_\beta g) \\ & + \mathcal{O}(\theta^3). \end{aligned} \tag{13}$$

Here $\theta^{\rho\sigma}$ is a constant antisymmetric matrix and is considered to be a small deformation parameter. Indices ρ, σ take values $0,1,2,3$ and the four dimensional Minkowski metric is $\eta_{\mu\nu} = diag(1,-1,-1,-1)$.

In order to construct the NC $SO(2,3)_\star$ gauge theory we use the enveloping algebra approach and the Seiberg-Witten (SW) map [1]. This map enables to express the NC variables (NC gauge parameter, NC gauge field) as functions of the corresponding commutative variables. Its direct consequence is the equality of degrees of freedom in the commutative and the corresponding NC gauge theory. That is, NC gauge theory does not introduce new degrees of freedom. However, the algebra of the NC gauge transformations only closes in the enveloping algebra of the original gauge group algebra and all the NC variables are enveloping algebra-valued.

Through this paper we will only use the SW map solutions for the NC field strength tensor $\hat{F}_{\mu\nu}$ and the NC scalar field $\hat{\phi}$. They are given by the following recursive relations [5]

$$\hat{F}^{(n+1)}_{\mu\nu} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{\omega}_\kappa \stackrel{\star}{,} \partial_\lambda \hat{F}_{\mu\nu} + D_\lambda \hat{F}_{\mu\nu}\}\Big)^{(n)} + \frac{1}{2(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{F}_{\mu\kappa}, \stackrel{\star}{,} \hat{F}_{\nu\lambda}\}\Big)^{(n)}, \tag{14}$$

$$\hat{\phi}^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\Big(\{\hat{\omega}_\kappa \stackrel{\star}{,} \partial_\lambda \hat{\phi} + D_\lambda \hat{\phi}\}\Big)^{(n)}. \tag{15}$$

The zeroth order is just given by the commutative field strength tensor $F_{\mu\nu}$ and the scalar field ϕ. Note that NC fields we label with a ”hat”.

Now let us discuss the NC generalization of the actions (7) and (9). The NC generalization of the action (7) is given by

$$S_{NC1} = -\frac{il}{16\pi G_N}\mathrm{Tr}\int \mathrm{d}^4x \epsilon^{\mu\nu\rho\sigma}\hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma} \star \hat{\phi}. \tag{16}$$

The $\star$-product is the Moyal-Weyl $\star$-product (13). The NC fields we expand in terms of the corresponding commutative fields by using the SW map. One can show that the action (16) is invariant under the NC $SO(2,3)$ gauge transformations given by

$$\delta^\star_\epsilon \hat{F_{\mu\nu}} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{F}_{\mu\nu}],$$
$$\delta^\star_\epsilon \hat{\phi} = i[\hat{\Lambda}_\epsilon \stackrel{\star}{,} \hat{\phi}], \tag{17}$$

with the NC gauge parameter Λ. The commutative limit of Λ is the commutative gauge parameter ϵ. In the limit $\theta^{\alpha\beta} \to 0$ the action (16) reduces to the commutative action (7).

The first order correction vanishes, as expected [2]. The second order correction is given by

$$
\begin{aligned}
S^{(2)}_{NC1} = & \frac{il}{64\pi G_N}\frac{1}{8}\theta^{\alpha\beta}\theta^{\kappa\lambda}\mathrm{Tr}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\Big\{ -\frac{1}{8}\{F_{\alpha\beta},\{F_{\mu\nu},F_{\rho\sigma}\}\}\{\phi,F_{\kappa\lambda}\} \\
& +\frac{1}{2}\{F_{\alpha\beta},\{F_{\rho\sigma},\{F_{\kappa\mu},F_{\lambda\nu}\}\}\}\phi+\frac{1}{4}\{\{F_{\mu\nu},F_{\rho\sigma}\},\{F_{\kappa\alpha},F_{\lambda\beta}\}\}\phi \\
& +\frac{i}{4}\{F_{\alpha\beta},[D_\kappa F_{\mu\nu},D_\lambda F_{\rho\sigma}]\}\phi+\frac{i}{2}[\{D_\kappa F_{\mu\nu},F_{\rho\sigma},D_\lambda F_{\alpha\beta}\}]\phi \\
& -\frac{1}{2}\{F_{\rho\sigma},\{F_{\alpha\mu},F_{\beta\nu}\}\}\{\phi,F_{\kappa\lambda}\}+\{F_{\alpha\mu},F_{\beta\nu},\{F_{\kappa\rho},F_{\lambda\sigma}\}\}\phi \\
& +2\{F_{\rho\sigma},\{F_{\beta\nu},\{F_{\kappa\alpha},F_{\lambda\mu}\}\}\}\phi+i\{F_{\rho\sigma},[D_\kappa F_{\alpha\mu},D_\lambda F_{\beta\nu}]\}\phi \\
& +2i[\{F_{\beta\nu},D_\kappa F_{\alpha\mu}\},D_\lambda F_{\rho\sigma}]\phi-\frac{1}{4}\{\phi,F_{\kappa\lambda}\}[D_\alpha F_{\mu\nu},D_\beta F_{\rho\sigma}] \\
& +\frac{i}{2}\{D_\kappa D_\alpha F_{\mu\nu},D_\lambda D_\beta F_{\rho\sigma}\}\phi+[\{F_{\kappa\alpha},D_\lambda F_{\mu\nu}\},D_\beta F_{\rho\sigma}]\phi \\
& +[\{F_{\lambda\nu},D_\alpha F_{\kappa\mu}\},D_\beta F_{\rho\sigma}]\phi+[\{F_{\kappa\mu},D_\alpha F_{\lambda\nu}\},D_\beta F_{\rho\sigma}]\phi\Big\}.
\end{aligned}
$$

This expanded action is manifestly invariant under the commutative $SO(2,3)$ gauge transformations. This result is guaranteed by the SW map. After the symmetry breaking, which is obtained by taking the field ϕ to be $\phi^a=0$ and $\phi^5=l$, we obtain

$$
\begin{aligned}
S^{(2)}_{NC1} = & -\frac{l^2}{64\pi G_N}\theta^{\alpha\beta}\theta^{\kappa\lambda}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\int \mathrm{d}^4x\Big\{\frac{1}{256}\Big(F_{\mu\nu}{}^{cd}F_{\rho\sigma}{}^{ab}F_{\alpha\beta}{}^{mn}F_{\kappa\lambda mn} \\
& -8F_{\mu\nu}{}^{ab}F_{\rho\sigma}{}^{c5}F_{\kappa\lambda}{}^{de}F_{\alpha\beta e}{}^{5}+F_{\alpha\beta}{}^{ab}F_{\kappa\lambda}{}^{cd}(F_{\mu\nu}{}^{mn}F_{\rho\sigma mn}+2F_{\mu\nu}{}^{m5}F_{\rho\sigma m}{}^{5b})\Big) \\
& -\frac{1}{32}\Big(F_{\kappa\lambda}{}^{ab}F_{\mu\nu}{}^{cd}F_{\alpha\rho}{}^{mn}F_{\beta\sigma mn}+2F_{\alpha\beta}{}^{ab}F_{\rho\sigma}{}^{cd}F_{\kappa\mu}{}^{m5}F_{\lambda\nu m}{}^{5}+F_{\kappa\mu}{}^{ab}F_{\lambda\nu}{}^{cd}F_{\alpha\beta}{}^{mn}F_{\rho\sigma mn}\Big) \\
& -\frac{1}{128}\Big(F_{\kappa\alpha}{}^{ab}F_{\lambda\beta}{}^{cd}(F_{\mu\nu}{}^{mn}F_{\rho\sigma mn} \\
& +2F_{\mu\nu}{}^{m5}F_{\rho\sigma m}{}^{5})+F_{\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}(F_{\kappa\alpha}{}^{mn}F_{\lambda\beta mn}+2F_{\kappa\alpha}{}^{m5}F_{\lambda\beta m}{}^{5})\Big) \\
& +\frac{1}{16}F_{\alpha\beta}{}^{ab}\Big((D_\kappa F_{\mu\nu})^{cm})(D_\lambda F_{\rho\sigma})_m{}^{d}+(D_\kappa F_{\mu\nu})^{c5}(D_\lambda F_{\rho\sigma})_5{}^{d}\Big) \\
& -\frac{1}{16}\Big((D_\kappa F_{\mu\nu})^{ab}(D_\lambda F_{\alpha\beta})^{d5}F_{\rho\sigma}^{c5}+(D_\kappa F_{\mu\nu})^{a5}(D_\lambda F_{\alpha\beta})^{b5}F_{\rho\sigma}{}^{cd}\Big) \\
& +\frac{1}{16}F_{\alpha\mu}^{ab}F_{\beta\nu}^{cd}\Big(F_{\kappa\rho}{}^{mn}F_{\lambda\sigma mn}+2F_{\kappa\rho}{}^{m5}F_{\lambda\sigma m5}\Big) \\
& +\frac{1}{16}\Big(F_{\rho\sigma}{}^{ab}F_{\beta\nu}{}^{cd}(F_{\kappa\alpha}{}^{mn}F_{\lambda\mu mn}+2F_{\kappa\alpha}{}^{m5}F_{\lambda\mu m5})+F_{\kappa\alpha}{}^{ab}F_{\lambda\mu}{}^{cd}F_{\rho\sigma}{}^{mn}F_{\beta\nu mn} \\
& -4(F_{\kappa\alpha}{}^{ab}F_{\lambda\mu}{}^{c5}+F_{\kappa\alpha}{}^{a5}F_{\lambda\mu}{}^{bc})F_{\rho\sigma}{}^{de}F_{\beta\nu e5}\Big) \\
& -\frac{1}{8}F_{\rho\sigma}{}^{ab}\Big((D_\kappa F_{\alpha\mu})^{cm}(D_\lambda F_{\beta\nu})_m{}^{d}+(D_\kappa F_{\alpha\mu})^{c5}(D_\lambda F_{\beta\nu})_5{}^{d}\Big) \\
& +\frac{1}{2}\Big(F_{\kappa\mu}{}^{ab}(D_\alpha F_{\lambda\nu})^{c5}(D_\beta F_{\rho\sigma})^{d5}+F_{\kappa\mu}{}^{a5}(D_\alpha F_{\lambda\nu})^{bc}\Big)(D_\beta F_{\rho\sigma})^{d5} \\
& +\frac{1}{8}\Big(F_{\kappa\alpha}{}^{ab}(D_\lambda F_{\mu\nu})^{c5}+F_{\kappa\alpha}{}^{a5}(D_\lambda F_{\mu\nu})^{bc}\Big)(D_\beta F_{\rho\sigma})^{d5} \\
& -\frac{1}{32}(D_\kappa D_\alpha F_{\mu\nu})^{ab}(D_\lambda D_\beta F_{\rho\sigma})^{cd}\Big\}.
\end{aligned} \tag{18}
$$

Here $D_\alpha F_{\mu\nu}$ is the $SO(2,3)$ covariant derivative and its components are

$$(D_\alpha F_{\mu\nu})^{ab} = \nabla_\alpha F_{\mu\nu}^{\ \ ab} - \frac{1}{l^2}(e_\alpha^a T_{\mu\nu}^{\ \ b} - e_\alpha^b T_{\mu\nu}^{\ \ a})$$

$$(D_\alpha F_{\mu\nu})^{a5} = \frac{1}{l}(\nabla_\alpha T_{\mu\nu}^a + e_\alpha^m F_{\mu\nu m}^{\ \ \ \ a}).$$

The result (18) is very complicated. Before we comment on it, let us write the results for the NC generalization of the action (9). The NC action is given by

$$S_{NC2} = \frac{1}{2}(S + S^\dagger),$$

with

$$S = \frac{1}{64\pi G_N l}\int d^4x \epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}(\hat{F}_{\mu\nu} \star D_\rho\hat{\phi} \star D_\sigma\hat{\phi} \star \hat{\phi}), \tag{19}$$

where $D_\rho\hat{\phi} = \partial_\rho\hat{\phi} - i[\hat{\omega}_\rho \stackrel{\star}{,} \hat{\phi}]$. The first order correction vanishes. The second order correction is again very cumbersome and we will not write it here. More details can be found in [6]. Instead we comment on the obtained results in the following section.

4. DISCUSSION

After the expansion of the NC action(s) we obtain very complicated results and it is difficult to give a physical meaning to them. In order to see some of the physical consequences of these model we have to make additional requirements. We will first assume that the torsion in the zero-th order vanishes, $F_{\mu\nu}^{\ \ a5} = 0$. Since we have no fermionic matter, this assumption is valid. Then we will expand $F_{\mu\nu}^{\ \ bc}$ in terms of the curvature tensor and the vielbeins, using (4). Since the torsion vanishes, the curvature tensor will have the usual symmetry properties: $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} = R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}$. We will also write $\sqrt{-g}$ instead of e.

We have already mentioned that there are three parameters in our models: the cosmological constant $\frac{const.}{l^2}$ (the length parameter l is related with the radius of the AdS space), the NC parameter $\theta^{\alpha\beta}$ and the powers of the curvature tensor (powers of derivatives, energy scale). Depending on the values of these three parameters we can analyse different limits of the model: big cosmological constant and low energies (lower powers of curvature dominate), or big cosmological constant and high energies (higher powers of curvature dominate) and so on.

Before deciding which limit we discuss we have to stress one more thing. In the commutative limit of the actions (16) and (19) the cosmological constant term is present. That implies that the flat (Minkowski) space-time cannot be a zero-th order solution of the equations of motion. Instead, one possible zero-th order solution is the AdS space-time. Since the Moyal-Weyl $\star$-product is adopted to the Cartesian coordinates, it would be simpler† to start from a flat zero-th order solution. This we can achieve by combining the actions (16) and (19) in such a way that the cosmological constant term vanishes in the commutative limit. On the other hand, it is interesting to check if the noncommutativity induces the non-zero cosmological constant.

Let us write the expansions of S_{NC1} and S_{NC2} up to the first power in the curvature. They are given by

$$\begin{aligned} S_{NC1} = & -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\Big[\frac{l^2}{16}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R^{ab}_{\mu\nu}R^{cd}_{\rho\sigma} + \sqrt{-g}R - \sqrt{-g}\frac{6}{l^2}\Big] \\ & + \frac{3\theta^{\alpha\beta}\theta^{\kappa\lambda}}{64\pi G_N l^6}\int \mathrm{d}^4x\sqrt{-g}g_{\alpha\kappa}g_{\beta\lambda} \\ & - \frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{64\pi G_N l^4}\int \mathrm{d}^4x\sqrt{-g}\Big(3g_{\alpha\kappa}R_{\beta\lambda} + 3R_{\alpha\beta\kappa\lambda} - 2R_{\alpha\kappa\beta\lambda}\Big) \\ & + \frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{256\pi G_N l^2}\int \mathrm{d}^4x\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\Big\{\frac{2}{l^2}\nabla_\kappa(e_\mu^{\ c}e_\nu^{\ m})\nabla_\lambda(e_{\rho m}e_\sigma^{\ d}) \\ & - \frac{1}{l^2}\nabla_\kappa(e_\alpha^{\ c}e_\mu^{\ m} - e_\alpha^{\ m}e_\mu^{\ c})\nabla_\lambda(e_{\beta m}e_\nu^{\ d} - e_\beta^{\ d}e_{\nu m})\Big]\Big\}. \end{aligned} \tag{20}$$

$$\begin{aligned} S_{NC2} = & -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\Big[\sqrt{-g}R - \sqrt{-g}\frac{12}{l^2}\Big], \\ & + \frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{64\pi G_N l^4}\int \mathrm{d}^4x\sqrt{-g}\Big[-16R_{\alpha\kappa}g_{\beta\lambda} - 7R_{\alpha\kappa\beta\lambda} - 3\nabla_\alpha e^a_\kappa\nabla_\beta e^b_\rho e_{\lambda a}e^\rho_b \\ & - 9\nabla_\alpha e^a_\kappa e^\rho_a\partial_\beta g_{\rho\lambda} - 2g_{\alpha\kappa}\nabla_\beta e^a_\rho\nabla_\lambda e^b_\sigma e^\rho_a e^\sigma_b - 2g_{\alpha\kappa}\nabla_\beta e^a_\rho\nabla_\lambda e^\rho_a \\ & + g_{\beta\lambda}\nabla_\kappa\nabla_\alpha e^a_\rho e^\rho_a + \frac{17}{l^2}g_{\alpha\kappa}g_{\beta\lambda}\Big] \end{aligned} \tag{21}$$

The action $S_{NC} = 2S_{NC1} - S_{NC2}$ has no cosmological constant term in the

†It is non necessary, one can also write a non-flat metric in Cartesian coordinates.

commutative limit and its expansion is given by

$$
\begin{aligned}
S_{NC} = &-\frac{1}{16\pi G_N}\int \mathrm{d}^4x\Big[\frac{l^2}{8}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R^{ab}_{\mu\nu}R^{cd}_{\rho\sigma}+\sqrt{-g}R\Big],\\
&-\frac{11\theta^{\alpha\beta}\theta^{\kappa\lambda}}{16\pi G_N l^6}\int \mathrm{d}^4x\sqrt{-g}g_{\alpha\kappa}g_{\beta\lambda}\\
&-\frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{64\pi G_N l^4}\int \mathrm{d}^4x\sqrt{-g}\Big(3R_{\alpha\beta\kappa\lambda}-10g_{\alpha\kappa}R_{\beta\lambda}-11R_{\alpha\kappa\beta\lambda}\Big)\\
&+\frac{\theta^{\alpha\beta}\theta^{\kappa\lambda}}{256\pi G_N l^2}\int \mathrm{d}^4x\Big\{\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\frac{2}{l^2}\nabla_\kappa(e_\mu^{\ c}e_\nu^{\ m})\nabla_\lambda(e_{\rho m}e_\sigma^{\ d}))\\
&-\frac{1}{l^2}\nabla_\kappa(e_\alpha^{\ c}e_\mu^{\ m}-e_\alpha^{\ m}e_\mu^{\ c})\nabla_\lambda(e_{\beta m}e_\nu^{\ d}-e_\beta^{\ d}e_{\nu m})\Big]\Big\}\\
&+\sqrt{-g}\Big\{-12\nabla_\alpha e^a_\kappa\nabla_\beta e^b_\rho e_{\lambda a}e^\rho_b-36\nabla_\alpha e^a_\kappa e^\rho_a\partial_\beta g_{\rho\lambda}-8g_{\alpha\kappa}\nabla_\beta e^a_\rho\nabla_\lambda e^b_\sigma e^\rho_a e^\sigma_b\\
&-8g_{\alpha\kappa}\nabla_\beta e^a_\rho\nabla_\lambda e^\rho_a+4g_{\beta\lambda}\nabla_\kappa\nabla_\alpha e^a_\rho e^\rho_a\Big\}. \qquad (22)
\end{aligned}
$$

This action is invariant under the local $SO(1,3)$ symmetry, but it is not invariant under the diffeomorphisms because of the terms with $\nabla_\kappa e_\alpha^{\ c}$. Namely, using the metricity condition

$$\nabla^{tot}_\mu e_\rho^{\ a}=\partial_\mu e_\rho^{\ a}+\omega_\mu^{ab}e_{\rho b}-\Gamma^\sigma_{\mu\rho}e_\sigma^{\ a}=0,$$

this term can be rewritten as

$$\nabla_\mu e_\rho^{\ a}=\partial_\mu e_\rho^{\ a}+\omega_\mu^{ab}e_{\rho b}=\Gamma^\sigma_{\mu\rho}e_\sigma^{\ a}.$$

We see that the Cristoffel symbols $\Gamma^\sigma_{\mu\rho}$ appear explicitly in the action and they obviously break the diffeomorphism invariance.

Let us now assume that we are interested in the limit of low energies with all the other scales fixed. In that case, from (22) we include only the term that is zeroth order in the curvature. The resulting action is given by

$$
\begin{aligned}
S_{NC\Lambda} = &-\frac{1}{16\pi G_N}\int \mathrm{d}^4x\Big[\frac{l^2}{8}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}R^{ab}_{\mu\nu}R^{cd}_{\rho\sigma}+\sqrt{-g}R\Big]\\
&-\frac{11\theta^{\alpha\beta}\theta^{\kappa\lambda}}{16\pi G_N l^6}\int \mathrm{d}^4x\sqrt{-g}g_{\alpha\kappa}g_{\beta\lambda}. \qquad (23)
\end{aligned}
$$

To obtain the equations of motion, we vary the action with respect to the metric $g_{\rho\sigma}$. The result is

$$R_{\rho\sigma}-\frac{1}{2}g_{\rho\sigma}R-\frac{11}{l^6}\theta^{\alpha\beta}\theta^{\kappa\lambda}(\frac{1}{2}g_{\rho\sigma}g_{\alpha\kappa}g_{\beta\lambda}+2g_{\beta\lambda}g_{\alpha\rho}g_{\kappa\sigma})=0. \qquad (24)$$

We immediately see that the cosmological constant appears as the x-dependent term $\theta^{\alpha\beta}\theta^{\kappa\lambda}g_{\alpha\kappa}g_{\beta\lambda}$. Therefore we can conclude that the noncommutativity can induce the cosmological constant. If this effect can be used to explain dark energy or inflation remains to be discussed in future.

In addition, one can start from different solutions of the commutative vacuum Einstein equations and analyse how the noncommutativity influences them. A good candidate is the Kasner solution

$$\mathrm{d}s^2 = \mathrm{d}t^2 - t^{2p_1}\mathrm{d}x^2 - t^{2p_2}\mathrm{d}y^2 - t^{2p_3}\mathrm{d}z^2, \tag{25}$$

with $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$. This is anisotropic vacuum solution and is sometimes used in anisotropic cosmological models. Since we expect that the noncommutativity, similarly as a magnetic field, introduces a preferred direction, an anisotropic solution is a good place to start from. One can check if NC corrections can change the relations between the parameters p_i and in which way. This we also leave for future work.

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