GENERALIZATION OF A FRACTIONAL MODEL FOR THE TRANSPORT EQUATION INCLUDING EXTERNAL PERTURBATIONS

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Abstract. A generalized fractional transport equation for particles is proposed and studied. This model includes local effects (through Fokker-Planck equation) and non-local spatial effects (Levy flights modelled using fractional derivatives). External perturbations are introduced in the model as source term in the fractional equation. A specific code based on matrix approach is built in order so study numerically the model.

Key words: Fractional dynamics, transport equation, fusion, tokamak.

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1. INTRODUCTION

The strange transport that is sometimes called anomalous transport, is very known in dynamical systems theory. This issue has been the object of intense study, due to its applications in very different fields, *e.g.* the turbulent fluid and the turbulent plasma theories [1]-[8]. The aim of this paper is to study a generalized fractional transport equation which includes local effects (through Fokker-Planck equation) and non-local spatial effects (Levy flights modeled by fractional derivatives). External perturbations are introduced in the model as source term in the fractional equation.

$$\begin{cases} \left({}_t^C D_0^\alpha T\right)(x,t) - A \cdot \left({}_x^R D_{l_1,r_1}^\beta T\right)(x,t) - B \cdot \left({}_x^R D_{l_2,r_2}^\gamma T\right)(x,t) = S(x,t), \\ T(0,t) = T(L,t) = 0 \quad \forall t \geq 0, \\ T(x,0) = T0(x) \quad \forall x \in [0,L], \end{cases} \tag{1}$$

where $T = T(x,t)$ is the transported scalar quantity (for example temperature or density), $A, B \in \mathbf{R}$ are parameters, $\alpha, \beta, \gamma \in (0,2]$ represent the fractional orders of partial derivatives and $l_1, r_1, l_2, r_2 \in [0,1]$ are related to the symmetry/asymmetry of the spatial derivatives and are parameters.

In this model the time derivative is the Caputo derivative [9] and the spatial derivatives are built using the Riesz derivatives [10].

If $\alpha = 1$, $\beta = 1$, $\gamma = 2$ the fractional partial derivatives coincide with the classical ones and one obtains in (1) the well-known Fokker-Planck equation.

Memory effects, respectively nonlocal spatial effects, are considered in the model (1) if $\alpha \notin \mathbf{N}$, respectively $\beta \notin \mathbf{N}$ or $\gamma \notin \mathbf{N}$.

The equation (1) is obtained from Montroll-Weiss equation by considering specific time and jump distribution functions and the order of the time derivative is determined by the waiting time distribution function, respectively the orders of the fractional spatial derivatives are determined by the algebraic asymptotic scaling of the jump distribution function.

Particular forms of the transport equation (1) were already used in order to describe radial transport in magnetically confined plasmas. The radial displacement of tracers was studied in [11] using a space-symmetric fractional model corresponding to $B = 0$, and $l_1 = r_1 = 1/2$. It was shown that, for some specific fractional orders of derivation, this model reproduces the shape and space-time scaling of the probability density function of the radial displacement, being in quantitative agreement with the turbulence transport calculations. The non-local radial transport was studied using non-symmetric fractional models [12] corresponding to $\gamma = 2$ and $S(x,t) = 0$. As a consequence of the spatial asymmetry some pinch were observed, accompanied by the development of an uphill transport region. The propagation of cold pulses in some experiments conducted in JET were fitted using fractional diffusion models [13]. The non-local transport in the reversed field pinch was described using a fractional diffusion equation: the particles transport across the unperturbed flux surfaces is due to a spectrum of Levy flights which is associated to a fractional equation [14]. There are not yet studies related for example to the extension to fractional equations of symmetry methods as applied in [15], [16]. In this paper we study a generalized version (1) of the one-dimensional fractional transport equation by considering a non-void source term $S(x,t)$. In the homogenous case $S(x,t) = 0$ we give the analytical solution in terms of Mittag-Lefler functions. In the non-homogenous case $S(x,t) \neq 0$ we present a specific code based on matrix approach. The numerical algorithm uses the approximation of Caputo and Riesz derivatives through Grunwald-Letnikov derivatives which are numerically tractable.

Numerical simulations are performed for various source terms and initial conditions. The results are compared with those obtained from classical transport equation in order to observe the influence of non-local spatial effects and of memory effects on the dynamics of the system.

The rest of the paper is organized as follows: Section 2 contains some considerations about the model. In Section 3 numerical method is presented. It is applied for solving specific problems in Section 4. Conclusions and discussions are contained in Section 5.

2. THE GENERALIZED FRACTIONAL TRANSPORT MODEL

In model (1) the time derivative is computed using the Caputo's approach [9]:

$$\left({}_t^C D_0^{\alpha} T\right)(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{T^{(m)}(x,\tau)}{(t-\tau)^{a-m+1}} d\tau \quad \text{if} \quad m-1 < \alpha \leq m. \tag{2}$$

The spatial derivatives are computed using the Riesz derivative on the interval $[a,b]$ [10]:

$$\left({}_x^R D_{l,r}^{\delta} T\right)(x,t) =$$
$$= \frac{1}{\Gamma(p-\delta)} \left(\frac{\partial}{\partial x}\right)^p \left(l \int_a^x \frac{T(\xi,t)}{(x-\xi)^{a-p+1}} d\xi + r(-1)^p \int_x^b \frac{T(\xi,t)}{(x-\xi)^{b-p+1}} d\xi \right), \tag{3}$$

where $p-1 < \delta \leq p$.

Riesz derivative is a weighted sum of left and right Riemann-Liouville (R-L) derivatives. If $\delta \notin \mathbf{N}$, the coefficients l and r measure the influence of the R-L left derivative, respectively R-L right derivative: if $r = 0$ only the right R-L derivative is took into account, hence only right spatial effect are considered; if $l = 0$ only left R-L derivative are used for computing the Riesz derivative, consequently only left spatial effects are involved in the equation. The symmetric Riesz derivative, denoted by $\frac{\partial^{\beta} T}{\partial |x|^{\beta}}$, is obtained for $l = r = 1/2$.

In order to understand the role of fractional derivatives we briefly recall the way to obtain the transport equation.

The starting point is the Montroll-Weiss equation [18] which describes stochastic processes where the position of a particle is influenced by its (random) interaction with other particles and/or sources:

$$P(x,t) = \delta(x) \cdot \int_t^{\infty} \psi(t')dt' + \int_0^t \left[\int_{-\infty}^{\infty} \lambda(x-x')P(x,t')dx' \right] \psi(t-t')dt'. \tag{4}$$

Here $P(x,t)$ is the probability of finding the particle in the position x at the time t, δ is the Dirac function, ψ is the probability density function of waiting times and λ is the probability density function of jumps. In equation (4) the first in the sum is the contribution to P of particles that have not move from x in the time interval $[0,t]$ and the second term denotes the contribution to P of the particles that came in the position x in the same time interval. Introducing in (4) the Laplace transform in time

$$(LP(x,t))(s) = \widetilde{P}(x,s) = \int_0^\infty e^{-st} P(x,t)\,dt$$

and Fourier transform in space

$$(FP(x,t))(s) = \widehat{P}(k,t) = \int_{-\infty}^{\infty} e^{ikx} P(x,t)\,dx$$

one obtains the operational solution

$$\widehat{\widetilde{P}} = \frac{1-\widetilde{\psi}(s)}{s} \cdot \frac{1}{1-\widetilde{\psi}(s)\cdot\widehat{\lambda}(k)}. \tag{5}$$

The solution $P(x,t)$ is obtained through the inversion of the Fourier-Laplace transform. If the distribution of the waiting times is a Poisson distribution $\psi(t) = \mu e^{-\mu t}$ and the jump distribution is a Gaussian one

$$\lambda(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}},$$

it can be proved [11] that $P(x,t)$ is the solution of the classical diffusion equation $\frac{\partial P}{\partial t} = -\frac{\sigma}{2\mu}\frac{\partial^2 P}{\partial x^2}$ *i.e.* $P(x,t) \approx \frac{1}{\sqrt{t}} e^{-\frac{2\mu x^2}{\sigma t}}$.

But if the waiting time and the jump distributions are Levy flights distributions, *i.e.* $\psi(t) \approx \left(\frac{1}{t}\right)^{1+\alpha}$ and $\lambda(x) \approx \left(\frac{1}{|x|}\right)^{1+\gamma}$ then $P(x,t)$ obeys the fractional equation

$${}_0^C D_t^\alpha P = B \cdot D_{|x|}^\gamma P, \tag{6}$$

where B is a constant that depends on α and γ [11].

If we consider also a drive term $A \cdot D_{|x|}^\beta P$ which possibly includes non-local spatial effects and a source term $S(x,t)$, we obtain a particular case of the equation (1), corresponding to $l_1 = r_1 = l_2 = r_2 = 1/2$.

In terms of random walk schemes, the symmetric derivative corresponds to a symmetric jump probability distribution of the transported quantities. The asymmetry of the space derivative accounts a preferable direction of jumps which may occur. Different degrees of asymmetry can be modeled using appropriate values of l, r, in equation (3).

Equation (1) is the model of this general case.

Fractional equations can not be usually solved through analytical methods. Even if this is possible, the solution is expressed in terms of special functions, namely Mittag-Lefler functions: $E_\delta(x) = \sum_{k=0}^{\infty} \frac{x^{k\delta}}{\Gamma(k\delta+1)}$, where $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the Gamma function.

In the simplest case, ${}_0^C D_t^{\alpha} P = D_{|x|}^{\gamma} P$, the solution is [19]:

$$P(x,t) = \int_{-\infty}^{\infty} \left[\int_{0}^{\infty} e^{st} \frac{s^{\alpha-1}}{s^{\alpha} + |k|^{\gamma}} ds \right] e^{-ikx} dk = \int_{-\infty}^{\infty} E_{\alpha}(-|k|^{\gamma} t^{\alpha}) e^{-ikx} dk.$$

Obviously, this analytical form is not very helpful in order to describe the properties of the solution. It can be done only through numerical computations. There is no hope to obtain the analytical form of the solution in the general case (1). For this reason the numerical methods are the unique way to approach the solution.

In the next section we present a numerical method based on the matrix approach.

3. A NUMERICAL METHOD FOR SOLVING TRANSPORT EQUATIONS

Many numerical methods for solving fractional partial differential equations have been proposed (see [18]-[22] and references therein).

Many toolboxes for solving such equations were developed in Matlab and are freely downloadable from Matlab Central File Exchange. We mention the toolbox CRONE, created by CRONE team [23], the Fractional State-Space toolkit [24], the matlab code fde12.m [25], the code flmm.m [26].

In this paper we use the matrix approach presented in [27] for solving the fractional partial differential equations. We prefer to use it because is very accurate and rapid. In usual numerical methods the solution of the equation is obtained step by step by moving from the previous moment to the next one. In the matrix approach the solution is obtained in one step in the whole time interval.

The matrix approach is based on the following observations:

1) Caputo derivative ${}_0^C D_t^{\alpha} T$ and left R-L derivative ${}_t D_{0+}^{\alpha} T$ coincide if we have $T(x,0) = 0$ for all $x \in [0,L]$.

2) the left (respectively right) R-L derivatives and the left (respectively right) Grundwald-Letnikov derivatives (computed on the same interval) coincide. In some initial conditions R-L derivatives can be approximated using discretized G-L derivatives with prescribed step.

3) the discretized Grundwald-Letnikov derivatives with prescribed step can be computed using the matrix approach, which transforms the partial differential equation into a linear system of equations whose unknowns are the values of T in the grid's nodes [17].

We first considered the special case, $T0(x) = 0, \forall x \in [0,L]$.

In order to solve numerically the problem (1) on $[0,L] \times [0,T]$ we consider a grid with p nodes in x direction and n nodes in t direction. The nodes are (ih_x, jh_t) where $h_x = \frac{L}{p-1}$, $h_t = \frac{T}{n-1}$ and $i \in \{1,2,...p\}$, j$\in \{1,2,...,n\}$. We denote $T(ih_x, jh_t)$

by $T_{i,j}$ respectively $S(ih_x, jh_t)$by $S_{i,j}$ and we consider the following matrices $U = (T_{p-i,n-j}) \in \mathcal{M}(p,n)$, respectively $F = (S_{p-i,n-j}) \in \mathcal{M}(p,n)$.

Following [19] we can say that the Caputo derivative in time at these nodes can be approximated using discretized Grundwald-Letnikov operators: for fixed $i \in \{1,2,...p\}$ one has

$$\left[T_{i,n}^{(\alpha)}\ T_{i,n-1}^{(\alpha)}\ ...\ T_{i,2}^{(\alpha)}\ T_{i,1}^{(\alpha)}\right]' = B_n^{(\alpha)} \times [T_{i,n}\ T_{i,n-1}\ ...\ T_{i,2}\ T_{i,1}]',$$

where $w_s^{(\delta)} = (-1)^s \frac{\Gamma(\delta+1)}{\Gamma(\delta-s+1)\cdot\Gamma(s+1)}$ and

$$B_n^{(\alpha)} = \frac{1}{h_t^{\alpha}} \begin{pmatrix} w_1^{(\alpha)} & w_2^{(\alpha)} & ... & ... & w_{n-1}^{(\alpha)} & w_n^{(\alpha)} \\ 0 & w_1^{(\alpha)} & w_2^{(\alpha)} & ... & ... & w_{n-1}^{(\alpha)} \\ ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & 0 & w_1^{(\alpha)} & w_2^{(\alpha)} \\ 0 & ... & ... & 0 & 0 & w_1^{(\alpha)} \end{pmatrix}.$$

Similarly we approximate ${}_x^R D_{l,r}^{\beta} T$ (the derivative in x-direction) by:

$$\left[T_{p,k}^{(\beta)}\ T_{p-1,k}^{(\beta)}\ ...\ T_{2,k}^{(\beta)}\ T_{1,k}^{(\beta)}\right]' = RL_p^{(\beta)} \times [T_{p,k}\ T_{p-1,k}\ ...\ T_{2,k}\ T_{1,k}]',$$

where $RL_p^{(\beta)} = \left(l \cdot B_p^{(\beta)} + r \cdot F_p^{(\beta)}\right)$and $F_p^{(\beta)} = \left(B_p^{(\beta)}\right)'$.

The system (1) is transformed in a matrix equation [19]:

$$\left(B_n^{(\alpha)} \otimes Id_p - A \cdot Id_n \otimes RL_p^{(\beta)} - B \cdot Id_n \otimes RL_p^{(\gamma)}\right) \times U = F, \tag{7}$$

which, in the non-degenerate case, gives the solution

$$U = \left(B_n^{(\alpha)} \otimes Id_p - A \cdot Id_n \otimes RL_p^{(\beta)} - B \cdot Id_n \otimes RL_p^{(\gamma)}\right)^{-1} \times F.$$

If $T0(x) \neq 0$ for some $x \in (0,L)$, one must consider the additional function $V(x,t) = T(x,t) - T0(x)$ and the system (1) reads

$$\begin{cases} \left({}_t^C D_0^{\alpha} V\right)(x,t) - A \cdot \left({}_x^R D_{l_1,r_1}^{\beta} V\right)(x,t) - B \cdot \left({}_x^R D_{l_2,r_2}^{\beta} V\right)(x,t) = S_1(x,t), \\ V(0,t) = V(L,t) = 0 \quad \forall t \geq 0, \\ V(x,0) = 0 \quad \forall x \in [0,L], \end{cases} \tag{8}$$

where

$$S_1(x,t) = S(x,t) - A \cdot \left({}_x^R D_{l_1,r_1}^{\beta} T0\right)(x,t) - B \cdot \left({}_x^R D_{l_2,r_2}^{\beta} T\right)(x,t). \tag{9}$$

The system (8), (9) can be numerically solved using the matrix approach and then one can obtain the solution of (1) as $T(x,t) = V(x,t) + T0(x)$.

The matrix approach can be also used for solving Fokker-Planck equation, because unifies the numerical differentiation of arbitrary (including integer) order. It leads to significant simplification of the numerical solution of partial differential equations.

The matrix approach was extended for the study of transport equations with two spatial directions [28]-[29].

4. NUMERICAL SIMULATIONS AND APPLICATIONS

Using classical methods we previously studied from analytical and numerical point of view the transport equations for ions and electrons or for magnetic field lines [4]-[7], [30], [31] in different cases concerning the diffusion of sheared stochastic magnetic field or the Markovian or non-Markovian transport of ions and/or electrons in combinations of magnetic field and stochastic electrostatic field.

In order to apply transport equations for the description of the radial transport in magnetically confined plasmas one can consider $a = 0$, $b = 1$ in the definition of Riesz derivative, which corresponds to the magnetic axis, respectively to normalized plasma boundary.

In the numerical simulations of (1) we consider the time-independent source term $S(x,t) = \frac{\sigma}{\sqrt{2\pi}} \exp\left(-0.5\left(\frac{x-\mu}{\sigma}\right)^2\right)$ with $\mu = 0.75$, $\sigma = 0.05$ and the initial condition $T0(x) = 0.25x(1-x)$.

In order to observe the memory effects we integrate the classical spatial equation (*i.e.* $\beta = 1$, $\gamma = 2$) for various values of $\alpha \in (0,2]$. The classical value of α is $\alpha = 1$. In this situation the fractional equation is in fact Fokker-Planck equation. In Figure1 the solutions at the moment $t_0 = 0.0099333$ are represented for $x \in [0,1]$ and various values of α. The solution of Fokker-Planck equation is the red curve. One can observe that the memory effects increase, even for short evolution time, when $|\alpha - 1|$ increases. These effects are amplified when time passes.

The effects of the asymmetry of the Riesz spatial derivatives can be observed if at least one spatial derivative has fractional order. In the simulations we considered $\alpha = 1$, $\beta = 1.75$ and $\gamma = 2$. The profile of the solution at $t_0 = 0.0099333$ is drawn in Figure 2. The blue curve corresponds to $l_1 = 1$, $l_2 = 0$ (only left spatial effects are considered), the red one corresponds to $l_1 = r_1 = 1/2$ (symmetric spatial effects), respectively the black one corresponds to $l_1 = 0$, $r_1 = 0$ (only right spatial effects are considered). The modification of the profile (moving to the left or right depending on the spatial effect include in the equation) encourage us to use non-symmetric derivatives in order to describe some transport characteristics that were observed in some experiments, for example the uphill transport.

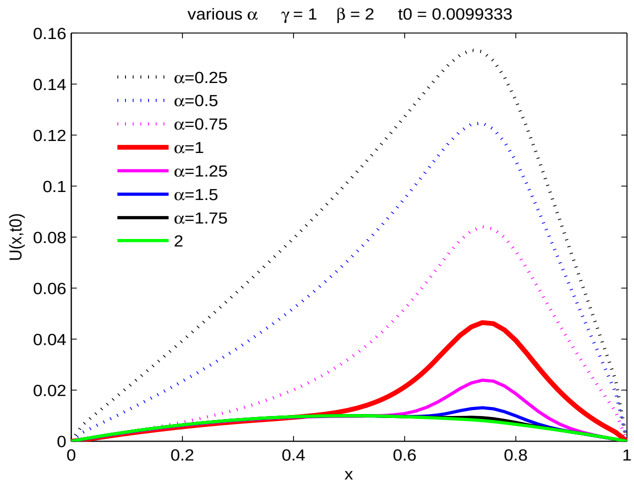

Fig. 1 – Solution of equation (1) with $\beta = 1$, $\gamma = 2$, at $t_0 = 0.0099333$. The solution of Fokker=Planck equation corresponds to $\alpha = 1$.

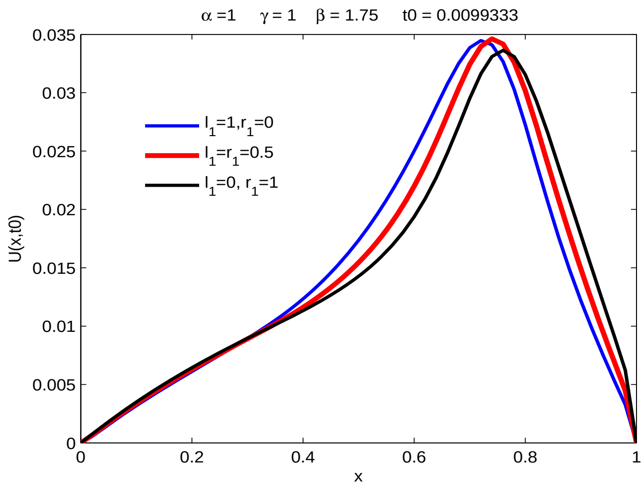

Fig. 2 – Solutions of the fractional equation (1) for $\alpha = 1$, $\beta = 1.75$, $\gamma = 2$ and various values of l_1 and r_1.

5. CONCLUSIONS

A class of non-local models based on fractional derivatives is proposed in order to describe anomalous transport phenomena. The use of fractional derivative enables us to consider in a unified framework asymmetric non-Fickian transport, non-Markovian (memory) effects [28]. Using a numerical method based on matrix approach some numerical simulations were performed. The memory effects and the non-local spatial effects analyzed and a special attention was paid to the asymmetric processes. The general model can be applied for understanding some features of the anomalous transport in fusion plasmas.

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