SECOND-ORDER LAGRANGIAN FORMULATION OF LINEAR FIRST-ORDER FIELD EQUATIONS

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A second-order Lagrangian formulation with respect to a set of linear first-order field equations (either relativistic or not) is proposed. The general formalism is illustrated in the case of first-order field equations containing a single spatial derivative.

Key words: field equations, Lagrangian approach, second-class constraints.

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The relationship between the Lagrangian [1] and Hamiltonian [2] formalisms stands for a subject of interest in theoretical physics in the context of both degenerate and non-degenerate dynamical systems [3–5]. It is well-known that the first-order Hamilton equations usually originate from a second-order Lagrangian formulation. For instance, the Klein–Gordon [6, 7], Maxwell [8–12] (written in terms of potentials), or Einstein [13] actions can be expressed in both second- and first-order form. Recently it has been shown that Schrödinger's equation [14] also allows for a second-order Lagrangian formulation [15]. On the other hand, neither of Dirac [16], chiral bosons in two dimensions [17], or chiral (self-dual) p-forms [18] actions allows an equivalent (local) second-order form. Thus, there appears to be an antisymmetry between the first- and second-order formulations of dynamics.

In this paper we investigate the possibility to derive a second-order Lagrangian formulation of independent, linear first-order field equations (relativistic or not), not necessarily in a Hamiltonian form.

Let $Q^A\left(t, x^1, \cdots, x^n\right)$ be a set of fields that parameterize the time-evolution of a dynamical system. We denote by ε_A the Grassmann parity of Q^A. Assume that the considered system is described by the system of independent, linear first-order equations

$$\mathcal{H}^A \equiv \dot{Q}^A - \sum_{s=1}^{k} \Gamma^{A\ j_1 \cdots j_s}_{\ B} \partial_{j_1} \cdots \partial_{j_s} Q^B - m^A{}_B Q^B = 0, \tag{1}$$

where the objects $\Gamma^{A\ j_1 \cdots j_s}_{\ B}$ and $m^A{}_B$ may be functions of $\mathbf{x} = \left(x^1, \cdots, x^n\right)$.

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Let us try some solutions of the form

$$Q^A = \dot{\Phi}^A + \hat{O}^A{}_B \Phi^B, \tag{2}$$

of equations (1), where we used the notations

$$\hat{O}^A{}_B = \sum_{s=1}^{k} \Gamma^{A\ j_1 \cdots j_s}_{\ B} \partial_{j_1} \cdots \partial_{j_s} + m^A{}_B. \tag{3}$$

Under these considerations, the following result can be shown to hold: Q^A given by (2) are solutions to the first-order equations given in (1) if and only if Φ^A are solutions to the second-order equations

$$\mathcal{E}^A \equiv \ddot{\Phi}^A - \hat{O}^A{}_C \hat{O}^C{}_B \Phi^B = 0. \tag{4}$$

Moreover, it is easy to show that formulas (2) express the general form of the solutions to equations (1). The above result emphasizes that the Φ^A's play the role of "potentials" associated to the "observables" Q^A.

Now, we investigate the following problem: can the second-order equations (4) originate in some Euler–Lagrange equations? In view of this, we consider a constant and invertible matrix ρ_{AB} with the symmetry properties

$$\rho_{AB} = -\left(-\right)^{(\varepsilon_A + 1)(\varepsilon_B + 1)} \rho_{BA}. \tag{5}$$

Then, we easily find the relations

$$\rho_{AB} \mathcal{E}^B = \frac{\partial}{\partial t} \frac{\partial^L \left(\frac{1}{2} \rho_{AB} \dot{\Phi}^A \dot{\Phi}^B \right)}{\partial \dot{\Phi}^A} - \rho_{AB} \hat{O}^B{}_C \hat{O}^C{}_D \Phi^D, \tag{6}$$

which suggest us to try a Lagrangian of the type

$$\bar{\mathcal{L}}_0 = \frac{1}{2} \rho_{AB} \dot{\Phi}^A \dot{\Phi}^B - \bar{\mathcal{V}} \left(\Phi, \partial_i \Phi, \partial_i \partial_j \Phi, \cdots \right), \tag{7}$$

and to require that the second-order equations of the type (4) to be expressed in the form

$$\rho_{AB} \mathcal{E}^B = -\frac{\delta^L \bar{\mathcal{L}}_0}{\delta \Phi^A}. \tag{8}$$

From (6) and (8) we obtain the equations

$$\frac{\delta^L \bar{\mathcal{V}}}{\delta \Phi^A} = -\rho_{AB} \hat{O}^B{}_C \hat{O}^C{}_D \Phi^D. \tag{9}$$

In consequence, we find the following answer to above mentioned problem: the second-order equations as in (4) originate in some Euler–Lagrange equations of the form (8) if and only if there exists a constant and invertible matrix ρ_{AB} (that satisfies (5)) such that equations (9) possess solutions.

On the one hand, from (9) we get that

$$\Phi^A \frac{\delta^L \bar{\mathcal{V}}}{\delta \Phi^A} = -\rho_{AB} \Phi^A \hat{O}^B{}_C \hat{O}^C{}_D \Phi^D. \tag{10}$$

Meanwhile, for any non-integrated density the standard formula holds

$$\Phi^A \frac{\delta^L \bar{\mathcal{V}}}{\delta \Phi^A} = \hat{N} \bar{\mathcal{V}} + \partial_i K^i, \tag{11}$$

for some K^i, where $\hat{N}$ denotes the counting operator

$$\hat{N} = \sum_{n \geqslant 0} \left(\partial_{j_1} \cdots \partial_{j_n} \Phi^A\right) \frac{\partial^L}{\partial \left(\partial_{j_1} \cdots \partial_{j_n} \Phi^A\right)}. \tag{12}$$

Then, from (10)–(11) we deduce that

$$\hat{N} \bar{\mathcal{V}} + \partial_i K^i = -\rho_{AB} \Phi^A \hat{O}^B{}_C \hat{O}^C{}_D \Phi^D. \tag{13}$$

Assume that equations (9) posses solutions. In consequence, we have that $\bar{\mathcal{V}}$ has to be quadratic in the fields and their derivatives, such that

$$\hat{N} \bar{\mathcal{V}} = 2 \bar{\mathcal{V}}. \tag{14}$$

Substituting (14) in (13) and taking into account that $\bar{\mathcal{V}}$ is defined up to a total derivative, we arrive at

$$\bar{\mathcal{V}} = -\frac{1}{2} \rho_{AB} \Phi^A \hat{O}^B{}_C \hat{O}^C{}_D \Phi^D. \tag{15}$$

The above considerations lead to the following conclusion: there exists a second-order Lagrangian formulation of the first-order equations (1) in terms of the of the "potentials" Φ^A if and only if there exists a constant and invertible matrix ρ_{AB} (that satisfies (5)) such that $\bar{\mathcal{V}}$ given by (15) is solution to equations (9).

In the sequel we focus on the particular case of first-order equations with a single spatial derivative, *i.e.*, on equations of the type

$$\mathcal{H}^A \equiv \dot{Q}^A - \Gamma^{A}{}_{B}^{j} \partial_j Q^B - m^A{}_B Q^B = 0. \tag{16}$$

In this situation equations (9) take the concrete form

$$\frac{\delta^L \bar{\mathcal{V}}}{\delta \Phi^A} = \lambda^{ij}_{AB} \partial_i \partial_j \Phi^B + \nu^j_{AB} \partial_j \Phi^B + \mu_{AB} \Phi^B, \tag{17}$$

where we used the notations

$$\lambda^{ij}_{AB} = -\frac{1}{2} \rho_{AC} \left(\Gamma^{C}{}_{D}^{i} \Gamma^{D}{}_{B}^{j} + \Gamma^{C}{}_{D}^{j} \Gamma^{D}{}_{B}^{i} \right), \tag{18}$$

$$\nu^j_{AB} = -\rho_{AC} \left(\Gamma^{C}{}_{D}^{i} \partial_i \Gamma^{D}{}_{B}^{j} + \Gamma^{C}{}_{D}^{j} m^D{}_B + m^C{}_D \Gamma^{D}{}_{B}^{j} \right), \tag{19}$$

$$\mu_{AB} = -\rho_{AC} \left(\Gamma^{C}{}_{D}^{j} \partial_j m^D{}_B + m^C{}_D m^D{}_B \right), \tag{20}$$

while formula (15) reads as

$$\bar{\mathcal{V}} = \frac{1}{2}\Phi^A \left(\lambda^{ij}_{AB}\partial_i\partial_j\Phi^B + \nu^j_{AB}\partial_j\Phi^B + \mu_{AB}\Phi^B\right). \tag{21}$$

After some computations we find that (21) is solution to equations (17) if and only if the relations

$$\lambda^{ij}_{AB} = (-)^{\varepsilon_A\varepsilon_B}\lambda^{ij}_{BA}, \tag{22}$$

$$\frac{1}{2}\left(\nu^i_{AB} + (-)^{\varepsilon_A\varepsilon_B}\nu^i_{BA}\right) = (-)^{\varepsilon_A\varepsilon_B}\partial_j\lambda^{ij}_{BA}, \tag{23}$$

$$\mu_{AB} - (-)^{\varepsilon_A\varepsilon_B}\mu_{BA} = \frac{1}{2}\partial_i\left(\nu^i_{AB} - (-)^{\varepsilon_A\varepsilon_B}\nu^i_{BA}\right). \tag{24}$$

are fulfilled. For instance, in the case of Dirac equations

$$\mathcal{H} \equiv \partial_0\psi + \mathrm{i}\gamma^0\left(m\psi - \mathrm{i}\gamma^j\partial_j\psi\right) = 0, \tag{25}$$

$$\bar{\mathcal{H}} \equiv \partial_0\bar{\psi} - \mathrm{i}\left(m\bar{\psi} + \mathrm{i}\partial_j\bar{\psi}\gamma^j\right)\gamma^0 = 0, \tag{26}$$

relations (22)–(24) are satisfied with the choice

$$\rho_{AB} = \begin{pmatrix} \mathbf{0}_{4\times4} & -\mathbf{1}_{4\times4} \\ \mathbf{1}_{4\times4} & \mathbf{0}_{4\times4} \end{pmatrix}. \tag{27}$$

For this example the "potentials" Φ^A take the form

$$\Phi^A = \begin{pmatrix} \chi \\ \bar{\chi} \end{pmatrix}, \tag{28}$$

with χ a Dirac spinor, such that $\bar{\mathcal{V}}$ is given by

$$\bar{\mathcal{V}} = -\left(\partial_j\bar{\chi}\partial^j\chi - m^2\bar{\chi}\chi\right). \tag{29}$$

Using (27)–(29) in (7) we derive the Klein–Gordon Lagrangian

$$\bar{\mathcal{L}}_0 = \partial_\mu\bar{\chi}\partial^\mu\chi - m^2\bar{\chi}\chi, \tag{30}$$

from which we obtain the second-order equations

$$\mathcal{E} \equiv \left(\Box + m^2\right)\chi = 0, \tag{31}$$

$$\bar{\mathcal{E}} \equiv \left(\Box + m^2\right)\bar{\chi} = 0. \tag{32}$$

Based on formulas (2) we infer that in the context of the example under study the following relations hold

$$\psi = \partial_0\chi - \mathrm{i}\gamma^0\left(m\chi - \mathrm{i}\gamma^j\partial_j\chi\right), \tag{33}$$

$$\bar{\psi} = \partial_0\bar{\chi} + \mathrm{i}\left(m\bar{\chi} + \mathrm{i}\partial_j\bar{\chi}\gamma^j\right)\gamma^0. \tag{34}$$

We remark that if the spinors $(\chi,\bar{\chi})$ satisfy equations (31)–(32), then the spinors $\left(\mathrm{i}\gamma^0\chi, -\mathrm{i}\bar{\chi}\gamma^0\right)$ obey exactly the same equations. Consequently, if we perform the transformation

$$\chi \to \mathrm{i}\gamma^0\chi,\ \bar{\chi} \to -\mathrm{i}\bar{\chi}\gamma^0, \tag{35}$$

which leaves invariant also the Lagrangian (30), formulas (33)–(34) lead to the expressions

$$\psi = \mathrm{i}\gamma^\mu\partial_\mu\chi + m\chi, \tag{36}$$
$$\bar{\psi} = -\mathrm{i}\partial_\mu\bar{\chi}\gamma^\mu + m\bar{\chi}, \tag{37}$$

which emphasize the next general relationship between Dirac and Klein–Gordon equations: $(\psi, \bar{\psi})$ given by (36)–(37) are solutions to the Dirac equations (25)–(26) if and only if $(\chi, \bar{\chi})$ are solutions to the Klein–Gordon equations (31)–(32).

In the final part of this paper we investigate the first-order equations expressed like in (16) in the particular case

$$\Gamma^{A\,j}_{\ B} = \begin{pmatrix} 0 & \Gamma^{a\,j}_{\ b} \\ -\Gamma^{a\,j}_{\ b} & 0 \end{pmatrix},\ m^A{}_B = \begin{pmatrix} 0 & m^a{}_b \\ -m^a{}_b & 0 \end{pmatrix}. \tag{38}$$

In this situation the fields Q^A split into two subsets of the type

$$Q^A = (Q^a, P^a), \tag{39}$$

such that equations (16) read as

$$\mathcal{H}^a \equiv \dot{Q}^a - \Gamma^{a\,j}_{\ b}\partial_j P^b - m^a{}_b P^b = 0, \tag{40}$$
$$\bar{\mathcal{H}}^a \equiv \dot{P}^a + \Gamma^{a\,j}_{\ b}\partial_j Q^b + m^a{}_b Q^b = 0. \tag{41}$$

Similarly, the “potentials” Φ^A split into two subsets

$$\Phi^A = (\varphi^a, \phi^a), \tag{42}$$

while equations (4) are expressed by

$$\begin{aligned} \mathcal{E}^a \equiv \ddot{\varphi}^a &+ \frac{1}{2}\left(\Gamma^{a\,i}_{\ c}\Gamma^{c\,j}_{\ b} + \Gamma^{a\,j}_{\ c}\Gamma^{c\,i}_{\ b}\right)\partial_i\partial_j\varphi^b + \Big(\Gamma^{a\,i}_{\ c}\partial_i\Gamma^{c\,j}_{\ b} + \\ &+ \Gamma^{a\,j}_{\ c}m^c{}_b + m^a{}_c\Gamma^{c\,j}_{\ b}\Big)\partial_j\varphi^b + \\ &+ \left(\Gamma^{a\,j}_{\ c}\partial_j m^c{}_b + m^a{}_c m^c{}_b\right)\varphi^b = 0, \end{aligned} \tag{43}$$

$$\begin{aligned} \bar{\mathcal{E}}^a \equiv \ddot{\phi}^a &+ \frac{1}{2}\left(\Gamma^{a\,i}_{\ c}\Gamma^{c\,j}_{\ b} + \Gamma^{a\,j}_{\ c}\Gamma^{c\,i}_{\ b}\right)\partial_i\partial_j\phi^b + \Big(\Gamma^{a\,i}_{\ c}\partial_i\Gamma^{c\,j}_{\ b} + \\ &+ \Gamma^{a\,j}_{\ c}m^c{}_b + m^a{}_c\Gamma^{c\,j}_{\ b}\Big)\partial_j\phi^b + \\ &+ \left(\Gamma^{a\,j}_{\ c}\partial_j m^c{}_b + m^a{}_c m^c{}_b\right)\phi^b = 0. \end{aligned} \tag{44}$$

In this setting formula (2) yields the relations

$$Q^a = \dot{\varphi}^a + \Gamma^{a}{}_{b}^{j}\partial_j\phi^b + m^a{}_b\phi^b, \tag{45}$$

$$P^a = \dot{\phi}^a - \Gamma^{a}{}_{b}^{j}\partial_j\varphi^b - m^a{}_b\varphi^b. \tag{46}$$

Due to the fact that equations (43)–(44) have exactly the same form, we can set

$$\varphi^a = \phi^a, \tag{47}$$

such that formulas (45)–(46) yield

$$Q^a = \dot{\varphi}^a + \Gamma^{a}{}_{b}^{j}\partial_j\varphi^b + m^a{}_b\varphi^b, \tag{48}$$

$$P^a = \dot{\varphi}^a - \Gamma^{a}{}_{b}^{j}\partial_j\varphi^b - m^a{}_b\varphi^b. \tag{49}$$

In consequence, in this particular case we are able to describe the solutions to equations (40)–(41) only in terms of the subset of "potentials" denoted by φ^a. By means of the notations

$$\lambda_{ab}^{ij} = -\frac{1}{2}\rho_{ac}\left(\Gamma^{c}{}_{d}^{i}\Gamma^{d}{}_{b}^{j} + \Gamma^{c}{}_{d}^{j}\Gamma^{d}{}_{b}^{i}\right), \tag{50}$$

$$\nu_{ab}^{j} = -\rho_{ac}\left(\Gamma^{c}{}_{d}^{i}\partial_i\Gamma^{d}{}_{b}^{j} + \Gamma^{c}{}_{d}^{j}m^d{}_b + m^c{}_d\Gamma^{d}{}_{b}^{j}\right), \tag{51}$$

$$\mu_{ab} = -\rho_{ac}\left(\Gamma^{c}{}_{d}^{j}\partial_j m^d{}_b + m^c{}_d m^d{}_b\right), \tag{52}$$

and applying a line similar to that employed with respect to equations (16), we get that the second-order equations of the type (43) may be expressed as Euler–Lagrange equations if and only if the following relations hold

$$\lambda_{ab}^{ij} = (-)^{\varepsilon_a\varepsilon_b}\,\lambda_{ba}^{ij}, \tag{53}$$

$$\frac{1}{2}\left(\nu_{ab}^{i} + (-)^{\varepsilon_a\varepsilon_b}\,\nu_{ba}^{i}\right) = (-)^{\varepsilon_a\varepsilon_b}\,\partial_j\lambda_{ba}^{ij}, \tag{54}$$

$$\mu_{ab} - (-)^{\varepsilon_a\varepsilon_b}\,\mu_{ba} = \frac{1}{2}\partial_i\left(\nu_{ab}^{i} - (-)^{\varepsilon_a\varepsilon_b}\,\nu_{ba}^{i}\right). \tag{55}$$

Equations (53)–(55) are automatically verified in the case where

$$\Gamma^{a}{}_{b}^{i} = \rho^{ac}\Gamma^{i}{}_{cb},\ m^a{}_b = \rho^{ac}m_{cb},\ \rho_{ab}\rho^{bc} = \delta_a{}^c, \tag{56}$$

with $\Gamma^{i}{}_{cb}$ and m_{cb} some constants subject to relations

$$\Gamma^{i}{}_{(ac)}\rho^{cd}\Gamma^{j}{}_{[db]} + \Gamma^{i}{}_{[ac]}\rho^{cd}\Gamma^{j}{}_{(db)} + \Gamma^{j}{}_{(ac)}\rho^{cd}\Gamma^{i}{}_{[db]} + \Gamma^{j}{}_{[ac]}\rho^{cd}\Gamma^{i}{}_{(db)} = 0, \tag{57}$$

$$\Gamma^{j}{}_{(ac)}\rho^{cd}m_{(db)} + \Gamma^{j}{}_{[ac]}\rho^{cd}m_{[db]} + m_{(ac)}\rho^{cd}\Gamma^{j}{}_{(db)} + m_{[ac]}\rho^{cd}\Gamma^{j}{}_{[db]} = 0, \tag{58}$$

$$m_{[ac]}\rho^{cd}m_{(db)} + m_{(ac)}\rho^{cd}m_{[db]} = 0, \tag{59}$$

where we employed the notations

$$\Gamma^i{}_{(ab)} = \Gamma^i{}_{ab} + \Gamma^i{}_{ba},\ \Gamma^i{}_{[ab]} = \Gamma^i{}_{ab} - \Gamma^i{}_{ba}, \tag{60}$$

$$m_{(ab)} = m_{ab} + m_{ba},\ m_{[ab]} = m_{ab} - m_{ba}. \tag{61}$$

We emphasize that formulas (56)–(59) are valid for both boson and fermionic systems. Under these considerations, equations (40)–(41) become

$$\mathcal{H}^a \equiv \dot{Q}^a - \rho^{ac}\left(\Gamma^j{}_{cb}\partial_j P^b + m_{cb}P^b\right) = 0, \tag{62}$$

$$\bar{\mathcal{H}}^a \equiv \dot{P}^a + \rho^{ac}\left(\Gamma^j{}_{cb}\partial_j Q^b + m_{cb}Q^b\right) = 0. \tag{63}$$

It is easy to see that equations (57)–(59) are fulfilled for

$$\Gamma^i{}_{ab} = \Gamma^i{}_{ba},\ m_{ab} = -m_{ba}. \tag{64}$$

For purely bosonic systems in the context of relations (56) and (64), the Lagrangian that generates equations (43) takes the expression

$$\begin{aligned}\bar{\mathcal{L}}_0 = {}& \frac{1}{2}\rho_{ab}\dot{\varphi}^a\dot{\varphi}^b + \frac{1}{4}\left(\Gamma^i{}_{ac}\rho^{cd}\Gamma^j{}_{db} + \Gamma^j{}_{ac}\rho^{cd}\Gamma^i{}_{db}\right)\partial_i\varphi^a\partial_j\varphi^b - \\ & -\frac{1}{2}\left(\Gamma^j{}_{ac}\rho^{cd}m_{db} + m_{ac}\rho^{cd}\Gamma^j{}_{db}\right)\varphi^a\partial_j\varphi^b - \frac{1}{2}m_{ac}\rho^{cd}m_{db}\varphi^a\varphi^b.\end{aligned} \tag{65}$$

In this manner Lagrangian (65) precisely provides a second-order description of the first-order equations as in (62)–(63) in terms of the “potentials” involved with relations (48)–(49).

Consider now a Lagrangian of the form

$$\bar{\mathcal{L}}'_0 = \frac{1}{2}\rho_{ab}\dot{\varphi}^a\dot{\varphi}^b - \frac{1}{2}\rho_{ab}Q^aQ^b + Q^a\left(\Gamma^j{}_{ab}\partial_j\varphi^b + m_{ab}\varphi^b\right). \tag{66}$$

Defining the canonical momenta in the standard manner by

$$\pi_a = \frac{\partial\bar{\mathcal{L}}'_0}{\partial\dot{\varphi}^a},\ \Pi_a = \frac{\partial\bar{\mathcal{L}}'_0}{\partial\dot{Q}^a},$$

from the canonical analysis of Lagrangian (66) we infer the constraints

$$\Delta_a \equiv \Pi_a \approx 0,\ \Theta_a \equiv -\rho_{ab}Q^b + \Gamma^j{}_{ab}\partial_j\varphi^b + m_{ab}\varphi^b \approx 0, \tag{67}$$

as well as the canonical Hamiltonian

$$\bar{H}'_0 = \int d^{D-1}x\left(\frac{1}{2}\rho^{ab}\pi_a\pi_b + \frac{1}{2}\rho_{ab}Q^aQ^b - Q^a\left(\Gamma^j{}_{ab}\partial_j\varphi^b + m_{ab}\varphi^b\right)\right). \tag{68}$$

It is easy to see that constraints (67) are second-class. Using the Dirac procedure we find that the only nonvanishing Dirac brackets are given by

$$[\varphi^a(t,\mathbf{x}),\pi_b(t,\mathbf{y})]^* = \delta^a{}_b\delta^{D-1}(\mathbf{x}-\mathbf{y}), \tag{69}$$

$$[Q^a(t,\mathbf{x}),\pi_b(t,\mathbf{y})]^* = \rho^{ac}\left(\Gamma^j{}_{cb}\partial_j^{(x)}+m_{cb}\right)\delta^{D-1}(\mathbf{x}-\mathbf{y}), \tag{70}$$

such that the equations of motion read as

$$\dot{\varphi}^a \approx \rho^{ab}\pi_b, \tag{71}$$

$$\dot{\pi}_a \approx -\frac{1}{2}\left(\Gamma^i{}_{ac}\rho^{cd}\Gamma^j{}_{db}+\Gamma^j{}_{ac}\rho^{cd}\Gamma^i{}_{db}\right)\partial_i\partial_j\varphi^b - \\ -\left(\Gamma^j{}_{ac}\rho^{cd}m_{db}+m_{ac}\rho^{cd}\Gamma^j{}_{db}\right)\partial_j\varphi^b - m_{ac}\rho^{cd}m_{db}\varphi^b, \tag{72}$$

$$\dot{Q}^a \approx \rho^{ac}\rho^{bd}\left(\Gamma^j{}_{cb}\partial_j\pi_d+m_{cb}\pi_d\right), \tag{73}$$

$$\dot{\Pi}_a \approx 0. \tag{74}$$

The concrete expressions (69)–(70) of the Dirac brackets imply that we may take any of the pairs (φ^a,π_a) or respectively (Q^a,π_a) as independent variables.

If we consider the pairs (φ^a,π_a) as independent variables, then the equations of motion are expressed by relations (71)–(72) viewed as strong equalities, while the canonical Hamiltonian from (68) becomes

$$\bar{H}_0' = \int d^{D-1}x\left(\frac{1}{2}\rho^{ab}\pi_a\pi_b - \frac{1}{4}\left(\Gamma^i{}_{ac}\rho^{cd}\Gamma^j{}_{db}+\Gamma^j{}_{ac}\rho^{cd}\Gamma^i{}_{db}\right)\partial_i\varphi^a\partial_j\varphi^b + \right. \\ \left. +\frac{1}{2}\left(\Gamma^j{}_{ac}\rho^{cd}m_{db}+m_{ac}\rho^{cd}\Gamma^j{}_{db}\right)\varphi^a\partial_j\varphi^b+\frac{1}{2}m_{ac}\rho^{cd}m_{db}\varphi^a\varphi^b\right). \tag{75}$$

It is rather obvious that this situation signifies nothing but the Hamiltonian version of the theory with the Lagrangian (65).

If we take now the pairs (Q^a,π_a) as independent variables, then the equations of motion read as

$$\dot{Q}^a = \rho^{ac}\rho^{bd}\left(\Gamma^j{}_{cb}\partial_j\pi_d+m_{cb}\pi_d\right), \tag{76}$$

$$\dot{\pi}_a = -\Gamma^j{}_{ac}\partial_j Q^c - m_{ac}Q^c, \tag{77}$$

while the canonical Hamiltonian is expressed by

$$\bar{H}_0' = \int d^{D-1}x\frac{1}{2}\left(\rho^{ab}\pi_a\pi_b-\rho_{ab}Q^aQ^b\right). \tag{78}$$

Equations (76)–(77) are nothing but the first-order equations (62)–(63) modulo the identification

$$P^b = \rho^{bd}\pi_d. \tag{79}$$

In consequence, the second-order Lagrangian formulation (65) in terms of the "potentials" and equations (62)–(63) may be respectively identified with the parameterisations (φ^a, π_a) and (Q^a, π_a) of the phase-space associated with the degenerate Lagrangian (66).

To conclude with, in this paper we emphasized the conditions that grant that a set of independent, first-order field equations (not necessarily in a Hamiltonian form) allows for a second-order Lagrangian formulation. In this respect we implemented two main steps: i) we expressed the general solution to the first-order equations in terms of the general solution to some second-order equations; ii) we inferred the general conditions that must be fulfilled in order to establish that the second-order equations originate in some Euler–Lagrange equations. Next, we applied this two-step procedure to first-order equations containing a single spatial derivative. In this context we determined the general relationship between Klein–Gordon and Dirac equations. Moreover, we showed that for a particular form of the first-order equations with a single spatial derivative there exists a second-order Lagrangian formulation subject to purely second-class constraints. In this special case the first-order equations can be expressed in Hamiltonian form in terms of the Dirac bracket built with respect to this second-class constraint set. In a future work we hope to extend the previous results to first-order systems of the type recently investigated in [19–28].

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