ON THE QUANTIZATION OF THE MASSIVE MAXWELL–CHERN–SIMONS MODEL

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Received December 12, 2014

The massive Maxwell–Chern–Simons model and a higher order derivative extension of it are analyzed from the point of view of the Hamiltonian path integral quantization in the framework of the gauge-unfixing approach.

Key words: quantization methods, second-class constraints, gauge theories.

PACS: 11.10.Ef.

In this paper the massive Maxwell–Chern–Simons (MCS) model [1–4] and a higher order derivative extension [5] of it [6] are analyzed from the point of view of the Hamiltonian path integral quantization. The quantization procedure relies on the construction of an equivalent first-class system and then quantizing the resulting first-class system. The construction of the equivalent first-class system can be achieved using the gauge-unfixing (GU) approach [7–15]. The associated first-class system with the original second-class theory satisfies the following requirements: its number of physical degrees of freedom coincides with that of the original second-class theory, the algebras of classical observables are isomorphic, the first-class Hamiltonian (governing the dynamics of the first-class system) restricted to the original constraint surface reduces to the original canonical Hamiltonian of the second-class theory.

1. GAUGE-UNFIXING METHOD

The starting point is a bosonic dynamic system endowed with the canonical Hamiltonian H_c and subject to $2M_0$ second-class constraints

$$\chi_{\alpha_0}(z) \approx 0, \qquad \alpha_0 = \overline{1, 2M_0}. \tag{1}$$

Assume that one can split the second-class constraint set (1) into two subsets

$$\chi_{\alpha_0}(z) \equiv \left(G_{\bar{\alpha}_0}(z), C^{\bar{\beta}_0}(z)\right) \approx 0, \qquad \bar{\alpha}_0, \bar{\beta}_0 = \overline{1, M_0}. \tag{2}$$

such that

$$\left[G_{\bar{\alpha}_0}, G_{\bar{\beta}_0}\right] = D^{\bar{\gamma}_0}_{\bar{\alpha}_0\bar{\beta}_0} G_{\bar{\gamma}_0}. \tag{3}$$

We introduce an operator $\hat{X}$

$$\hat{X}F = F - C^{\bar{\alpha}_0}\left[G_{\bar{\alpha}_0}, F\right] + \tfrac{1}{2} C^{\bar{\alpha}_0} C^{\bar{\beta}_0}\left[G_{\bar{\alpha}_0}, \left[G_{\bar{\beta}_0}, F\right]\right] - \cdots, \tag{4}$$

such that

$$\left[\hat{X}F, G_{\bar{\alpha}_0}\right] = 0. \tag{5}$$

With the help of this operator we construct a first-class Hamiltonian

$$H_{GU} = \hat{X}H_c \tag{6}$$

with respect to the first-class constraints subset.

The original second-class theory and respectively the GU system are classically equivalent since they possess the same number of physical degrees of freedom

$$\mathcal{N}_O = \tfrac{1}{2}\left(2n - 2M_0\right) = \mathcal{N}_{GU}, \tag{7}$$

and the corresponding algebras of classical observables are isomorphic.

$$Phys\left(\mathcal{S}_O\right) = Phys\left(\mathcal{S}_{GU}\right). \tag{8}$$

Consequently, the two systems become also equivalent at the level of the path integral quantization and we can replace the Hamiltonian path integral of the original second-class theory

$$Z_O = \int \mathcal{D}\left(z^a, \lambda^{\alpha_0}\right) \det\left(\left[G_{\bar{\alpha}_0}, C^{\bar{\beta}_0}\right]\right) \exp\left[\mathrm{i}\int dt \left(\dot{q}^i p_i - H_c - \lambda^{\alpha_0}\chi_{\alpha_0}\right)\right] \tag{9}$$

with that associated with the GU first-class system.

$$\begin{aligned} Z_{GU} = \int \mathcal{D}\left(z^a, \lambda^{\bar{\alpha}_0}\right) \left(\prod_{\bar{\alpha}_0} \delta\left(C^{\bar{\alpha}_0}\right)\right) \left(\det\left(\left[G_{\bar{\alpha}_0}, C^{\bar{\beta}_0}\right]\right)\right) \\ \times \exp\left[\mathrm{i}\int dt \left(\dot{q}^i p_i - \hat{X}H_c - \lambda^{\bar{\alpha}_0} G_{\bar{\alpha}_0}\right)\right]. \end{aligned} \tag{10}$$

2. THE MASSIVE MCS MODEL

The massive MCS model is described by the Lagrangian action

$$S = \int d^3x \left(-\tfrac{a}{4}\partial_{[\mu}A_{\nu]}\partial^{[\mu}A^{\nu]} - b\varepsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho - \tfrac{m^2}{2}A_\mu A^\mu\right). \tag{11}$$

By performing the canonical analysis of this model, we find the irreducible second-class constraints

$$\chi^{(1)} \equiv \pi^0 \approx 0, \tag{12}$$

$$\chi^{(2)} \equiv -\tfrac{1}{m^2}\left(\partial^i\pi_i - b\varepsilon_{0ij}\partial^i A^j - m^2 A_0\right) \approx 0, \tag{13}$$

along with the canonical Hamiltonian

$$H_c = \int d^2x \left(-\tfrac{1}{2a}\pi_i\pi^i - A_0\partial_i\pi^i + \tfrac{a}{4}\partial_{[i}A_{j]}\partial^{[i}A^{j]} \right. \\ \left. + b\varepsilon_{0ij}A^0\partial^iA^j + \tfrac{b}{a}\varepsilon_{0ij}A^i\pi^j - \tfrac{b^2}{2a}A_iA^i + \tfrac{m^2}{2}A_\mu A^\mu \right), \tag{14}$$

According to the GU method we consider (13) as the first-class constraint set and the remaining constraints (12) as the corresponding canonical gauge conditions. The first-class Hamiltonian with respect to (13) follows from relation (4)

$$H_{GU} = \int d^2x \left[-\tfrac{1}{2a}\pi_i\pi^i - A_0\partial_i\pi^i + \tfrac{a}{4}\partial_{[i}A_{j]}\partial^{[i}A^{j]} + b\varepsilon_{0ij}A^0\partial^iA^j \right. \\ \left. + \tfrac{b}{a}\varepsilon_{0ij}A^i\pi^j - \tfrac{b^2}{2a}A_iA^i + \tfrac{m^2}{2}A_\mu A^\mu - \pi^0\partial^iA_i + \tfrac{1}{m^2}\left(\partial_i\pi_0\right)\left(\partial^i\pi^0\right) \right]. \tag{15}$$

Performing the canonical transformation

$$A_0 \longrightarrow \tfrac{1}{m}p, \qquad p^0 \longrightarrow -m\varphi, \tag{16}$$

the constraint (13) becomes

$$G \equiv -\tfrac{1}{m^2}\left(\partial^ip_i - b\varepsilon_{0ij}\partial^iA^j - mp\right) \approx 0, \tag{17}$$

and the first-class Hamiltonian (15) takes the form

$$H_{GU} = \int d^2x \left[-\tfrac{1}{2a}p_ip^i + \tfrac{a}{4}\partial_{[i}A_{j]}\partial^{[i}A^{j]} + \tfrac{b}{a}\varepsilon_{0ij}A^ip^j - \tfrac{b^2}{2a}A_iA^i \right. \\ \left. + \tfrac{1}{2}\left(\partial_i\varphi - mA_i\right)\left(\partial^i\varphi - mA^i\right) - \tfrac{1}{m}p\left(\partial_ip^i - b\varepsilon_{0ij}\partial^iA^j - mp\right) - \tfrac{1}{2}p^2 \right]. \tag{18}$$

The original second-class theory and respectively the gauge-unfixed system are classically equivalent since they possess the same number of physical and, moreover, the corresponding algebras of classical observables are isomorphic. Consequently, the two systems become equivalent at the level of the path integral quantization, which allows us to replace the Hamiltonian path integral of the massive MCS model with that of the gauge-unfixed first-class system

$$Z_{GU} = \int \mathcal{D}\left(A_i,\varphi,p^i,p,\lambda\right)\mu\left([A_i],[\varphi]\right)\exp\left\{ \mathrm{i}\int d^3x \left[\left(\partial_0A_i\right)p^i \right.\right. \\ \left.\left. + \left(\partial_0\varphi\right)p - \mathcal{H}_{GU} + \tfrac{1}{m^2}\lambda\left(\partial^ip_i - b\varepsilon_{0ij}\partial^iA^j - mp\right)\right]\right\}. \tag{19}$$

After some field redefinitions and performing some partial integrations over the auxiliary fields we find out that the argument of the exponential from the Hamiltonian path integral of the first-class system reads as

$$S_{GU} = \int d^3x \left[-\tfrac{a}{4}\partial_{[\mu}\bar{A}_{\nu]}\partial^{[\mu}\bar{A}^{\nu]} - b\varepsilon_{\mu\nu\rho}\bar{A}^\mu\partial^\nu\bar{A}^\rho \right. \\ \left. - \tfrac{1}{2}\left(\partial_\mu\varphi - m\bar{A}_\mu\right)\left(\partial^\mu\varphi - m\bar{A}^\mu\right) \right]. \tag{20}$$

The functional (20) associated with the first-class system takes a manifestly Lorentz covariant form and describes a Stückelberg-like coupling between the 1-form $\bar{A}_\mu$ and scalar field φ [16, 17].

In [6], we show that for an appropriate extension of the phase space the massive MCS model is related to another first-class theory. The Hamiltonian path integral of the first-class theory takes a manifestly Lorentz-covariant form

$$S'_{GU} = \int d^3x \left(-\tfrac{a}{4}\partial_{[\mu}\bar{A}_{\nu]}\partial^{[\mu}\bar{A}^{\nu]} - b\varepsilon_{\mu\nu\rho}\bar{A}^\mu\partial^\nu\bar{A}^\rho + \tfrac{1}{4}\partial_{[\mu}\bar{V}_{\nu]}\partial^{[\mu}\bar{V}^{\nu]} + m\varepsilon_{\mu\nu\rho}\bar{A}^\mu\partial^\nu\bar{V}^\rho \right). \tag{21}$$

and describes a Chern–Simons coupling between the two 1-forms $\bar{A}_\mu$ and $\bar{V}_\mu$ [18].

3. THE EXTENDED MASSIVE MCS MODEL

The extended massive MCS model are described by the Lagrangian action

$$S = \int d^3x \left[-\tfrac{1}{4}\partial_{[\mu}A_{\nu]}\partial^{[\mu}A^{\nu]} + \tfrac{1}{2}\varepsilon_{\mu\nu\rho}\left(\partial_\lambda\partial^\lambda A^\mu\right)\partial^\nu A^\rho - \tfrac{1}{2}A_\mu A^\mu \right]. \tag{22}$$

The canonical analysis will be done by a variant of Ostrogradsky method [19–23]. The method consists in embedding the higher order derivative theory to an effective first order one [24–28]. We define the variables $B_\mu = \partial_0 A_\mu$ and enforced the Lagrangian constrains

$$B_\mu - \partial_0 A_\mu = 0, \tag{23}$$

by the Lagrange multiplier ξ^μ. The equivalent first order Lagrangian takes the form

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}\partial_{[i}A_{j]}\partial^{[i}A^{j]} - \tfrac{1}{2}\left(B_i - \partial_i A_0\right)\left(B^i - \partial^i A^0\right)\\ &+ \tfrac{1}{2}\varepsilon_{0ij}\left(\partial_0 B^0 + \partial_k\partial^k A^0\right)\partial^i A^j + \tfrac{1}{2}\varepsilon_{i0j}\left(\partial_0 B^i + \partial_k\partial^k A^i\right)B^j\\ &+ \tfrac{1}{2}\varepsilon_{ij0}\left(\partial_0 B^i + \partial_k\partial^k A^i\right)\partial^j A^0 - \tfrac{1}{2}A_\mu A^\mu + \xi^\mu\left(B_\mu - \partial_0 A_\mu\right).\end{aligned} \tag{24}$$

The canonical analysis of this model displays the irreducible second-class constraints

$$\Phi^{(\xi)}_\mu \equiv \Pi_\mu \approx 0, \tag{25}$$

$$\Phi^{(A)\mu} \equiv p^\mu + \xi^\mu \approx 0, \tag{26}$$

$$\Phi^{(B)} \equiv \pi_0 - \tfrac{1}{2}\varepsilon_{0ij}\partial^i A^j - \tfrac{1}{2}\varepsilon_{0ij}\partial^i \Pi^j \approx 0, \tag{27}$$

$$\Phi^{(B)}_i \equiv \pi_i + \tfrac{1}{2}\varepsilon_{0ij}B^j - \tfrac{1}{2}\varepsilon_{0ij}\partial^j A^0 - \tfrac{1}{2}\varepsilon_{0ij}\partial^i \Pi^0 \approx 0, \tag{28}$$

$$\Phi_{II}^{(B)} \equiv \xi_0 - \tfrac{1}{2}\varepsilon_{0ij}\partial^i B^j \approx 0, \tag{29}$$

$$\Phi_{III}^{(B)} \equiv \partial_i \xi^i + A_0 - \tfrac{1}{2}\varepsilon_{0ij}\partial_k \partial^k \left(\partial^i A^j\right) \approx 0, \tag{30}$$

$$\Phi_{IV}^{(B)} \equiv \partial_i A^i + B_0 \approx 0, \tag{31}$$

and the canonical Hamiltonian

$$\begin{aligned} H_c = \int d^2x \Big[& \tfrac{1}{4}\partial_{[i}A_{j]}\partial^{[i}A^{j]} + \tfrac{1}{2}\left(B_i - \partial_i A_0\right)\left(B^i - \partial^i A^0\right) \\ & - \tfrac{1}{2}\varepsilon_{0ij}\left(\partial_k \partial^k A^0\right)\partial^i A^j - \tfrac{1}{2}\varepsilon_{i0j}\left(\partial_k \partial^k A^i\right) B^j \\ & - \tfrac{1}{2}\varepsilon_{ij0}\left(\partial_k \partial^k A^i\right)\partial^j A^0 - \xi^\mu B_\mu + \tfrac{1}{2}A_\mu A^\mu \Big]. \end{aligned} \tag{32}$$

Eliminating the second-class constraints $\Phi_\mu^{(\xi)} \approx 0$ and $\Phi^{(A)\mu} \approx 0$ (the coordinates of the reduced phase space are $\{A_\mu, B_\mu, p^\mu, \pi^\mu\}$) we are left with a system subject to the second-class constraints

$$\chi_i^{(1)} \equiv \pi_i + \tfrac{1}{2}\varepsilon_{0ij}B^j - \tfrac{1}{2}\varepsilon_{0ij}\partial^j A^0 \approx 0, \tag{33}$$

$$\chi^{(1)} \equiv \partial_i p^i - A_0 + \partial_k \partial^k \pi_0 \approx 0, \tag{34}$$

$$\chi^{(2)} \equiv -p_0 + \partial_i \pi^i \approx 0, \tag{35}$$

$$\chi^{(3)} \equiv -\pi_0 + \tfrac{1}{2}\varepsilon_{0ij}\partial^i A^j \approx 0, \tag{36}$$

$$\chi^{(4)} \equiv \partial_i A^i + B_0 \approx 0, \tag{37}$$

while the canonical Hamiltonian (32) takes the form

$$\begin{aligned} H_c = \int d^2x \Big[& \tfrac{1}{4}\partial_{[i}A_{j]}\partial^{[i}A^{j]} + \tfrac{1}{2}\left(B_i - \partial_i A_0\right)\left(B^i - \partial^i A^0\right) \\ & - \tfrac{1}{2}\varepsilon_{0ij}\left(\partial_k \partial^k A^0\right)\partial^i A^j - \tfrac{1}{2}\varepsilon_{i0j}\left(\partial_k \partial^k A^i\right) B^j \\ & - \tfrac{1}{2}\varepsilon_{ij0}\left(\partial_k \partial^k A^i\right)\partial^j A^0 + p^\mu B_\mu + \tfrac{1}{2}A_\mu A^\mu \Big]. \end{aligned} \tag{38}$$

The matrix of the Poisson bracket between the constraint functions becomes

$$C_{\alpha_0\beta_0} = \begin{pmatrix} \varepsilon_{0ij} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 1 & 0 & 0 \\ \mathbf{0} & -1 & 0 & 0 & 0 \\ \mathbf{0} & 0 & 0 & 0 & 1 \\ \mathbf{0} & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{39}$$

The constraints $\chi_i^{(1)} \approx 0$ generate a submatrix (of the matrix of the Poisson brackets among the constraint functions) of maximum rank, therefore they form an independent subset of second-class constraints. Thus in the sequel we examine from the point of view of the GU method only the constraints $\chi_A \equiv \left\{\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \chi^{(4)}\right\} \approx 0$. According to the GU method we consider $G_a \equiv \left\{\chi^{(1)}, \chi^{(3)}\right\}$ as the first-class constraint set and the remaining constraints $C_a \equiv \left\{\chi^{(2)}, \chi^{(4)}\right\}$ as the corresponding canonical gauge conditions. The concrete form of the first-class Hamiltonian H_{GU} with respect to (34) and (36) is given by

$$\begin{aligned} H_{GU} = H_c + \int d^2x \Big[&- \left(p_0 - \partial_i \pi^i\right)\left(\partial_k A^k + B_0\right) \\ &- \left(p_0 - \partial_i \pi^i\right)\partial_k \partial^k \left(p_0 + \tfrac{1}{2}\varepsilon_{0lm}\partial^l B^m\right) \\ &+ \tfrac{1}{2}\left(p_0 - \partial_i \pi^i\right)\partial_k \partial^k \left(p_0 - \partial_j \pi^j\right) - \left(\partial_i A^i + B_0\right)\partial^j \left(\pi_j + \tfrac{1}{2}\varepsilon_{0jk} B^k\right)\Big]. \end{aligned} \tag{40}$$

Based on the equivalence between the first-class system and the original second-class theory, we replace the Hamiltonian path integral of the extended massive MCS model with that of the first-class system. The Hamiltonian path integral for the first-class system constructed in the above reads as

$$\begin{aligned} Z = \int \mathcal{D}&\left(A_\mu, B_\mu, p^\mu, \pi^\mu, \lambda^{(1)}, \lambda^{(2)}\right)\mu\left([A_\mu],[B_\mu]\right) \\ &\times \delta\left(\pi_i + \tfrac{1}{2}\varepsilon_{0ij}B^j - \tfrac{1}{2}\varepsilon_{0ij}\partial^j A^0\right)\det{}^{1/2}\left(\varepsilon_{0ij}\delta(x-y)\right) \\ &\times \exp\left\{ \mathrm{i}\int d^3x \left[(\partial_0 A_0)p^0 + (\partial_0 A_i)p^i + (\partial_0 B_0)\pi^0 + (\partial_0 B_i)\pi^i \right.\right. \\ &\left.\left. - \mathcal{H}_{GU} - \lambda^{(1)}\left(\partial_i p^i - A_0 + \partial_k \partial^k \pi_0\right) - \lambda^{(2)}\left(-\pi_0 + \tfrac{1}{2}\varepsilon_{0ij}\partial^i A^j\right)\right]\right\}, \end{aligned} \tag{41}$$

After some field redefinitions and performing some partial integrations over the auxiliary fields we find out that the argument of the exponential from the Hamiltonian path integral of the first-class system takes the form [18]

$$\begin{aligned} S_{GU} = \int d^3x \Big[&-\tfrac{1}{4}\partial_{[\mu}\bar{A}_{\nu]}\partial^{[\mu}\bar{A}^{\nu]} + \tfrac{1}{2}\varepsilon_{\mu\nu\rho}\left(\partial_\lambda \partial^\lambda \bar{A}^\mu\right)\partial^\nu \bar{A}^\rho \\ &- \tfrac{1}{2}\left(\partial_\mu \varphi - \bar{A}_\mu\right)\left(\partial^\mu \varphi - \bar{A}^\mu\right)\Big]. \end{aligned} \tag{42}$$

and describes a Stückelberg-like coupling between the scalar field φ and the 1-form $\bar{A}_\mu$.

4. CONCLUSIONS

Using gauge-unfixing method starting from massive MCS model and a higher order derivative extension of it, we constructed an equivalent first-class theory. After integrating out the auxiliary fields and performing some fields redefinitions we discover the manifestly Lorentz covariant path integrals corresponding to the Lagrangian formulation of the first-class system, which reduce to the Lagrangian path integral for Stückelberg coupling between a scalar field and a 1-form or for an appropriate extension of the phase space to the Lagrangian path integral for Chern–Simons coupling between two 1-forms.

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