LAGRANGIANS AND HAMILTONIANS RELATED TO FOLIATIONS

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Hamiltonians related to foliations, analogous to Riemannian foliations, are studied in the paper. One prove that each of the following data: a bundle–like Hamiltonian, a transverse hyperregular Hamiltonian, a hyperregular Hamiltonian foliated cocycle or a geodesic orthogonal property are equivalent to the fact that a foliation have to be a Riemannian one. Relations with the analogous Lagrangian case, considered previously by the authors, are studied.

Key words: Foliation, bundle–like Hamiltonian/Lagrangian,
transverse Hamiltonian/Lagrangian.

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1. INTRODUCTION

The existence of a transverse Riemannian metric to a foliation is not always possible, but only for a Riemannian foliation, that is a natural generalization of a Riemannian manifold. Geometric structures related to Riemannian foliations are studied by many authors; see, for example, [1, 2, 8, 18, 20] and the bibliography therein. But some other transverse structures can be considered related to a foliation.

Following E. Ghys' Appendix E of [8], Miernowski and Mozgawa raised in [6, Theorem 3.2] the question *if any Finslerian foliation* (see [3–6, 11]) *is a Riemannian foliation*, known as *Ghys' conjecture*. We gave in [12] a general solution to this problem, proving more: *a foliation that allows a positive allowed transverse Lagrangian is a Riemannian foliation*. Similar problems related to the first order case are studied in [13, 14] and in the case of higher order spaces are studied in [15–17].

A connection between (regular) foliations and Hamiltonians, in order to recover Riemannian foliations, is studied in the paper. Foliations that allow special Hamiltonians (bundle–like or transverse) and that have analogous properties to that of Riemannian foliations are considered. Explicitly, one prove that anyone of the following data is equivalent to the fact that a given foliation is a Riemannian foliation: a bundle–like Hamiltonian; a transverse hyperregular Lagrangian (Proposition 1); a hyperregular Hamiltonian foliated cocycle (Proposition 4); the transverse geodesic condition in Proposition 3.

Lagrangians, Hamiltonians and epimorphisms on vector bundles are considered and studied in the next section. One consider the general case of an epimorphism, since this case is involved in the study of the vertical bundle of a submersion and it is a natural setting to consider together bundle-like and transverse Lagrangians and Hamiltonians.

Some Lagrangian and Hamiltonian data on foliations, analogous to Riemannian data on Riemannian foliations, are discussed in the last section, where the main results are Proposition 7 and its special case Proposition 8, that give some results on positive transverse Hamiltonians, related to our previous paper [12] in Lagrangian case.

2. LAGRANGIANS, HAMILTONIANS AND EPIMORPHISMS

We use definitions and basic facts on foliations from [8], but different notations. We denote, relatively to $\mathcal{F}$, by $TM/\tau\mathcal{F} = \nu F \xrightarrow{\pi_{\nu F}} M$ the *transverse bundle* of $\mathcal{F}$ and by $\bar{\Pi} : TM \to \nu F$ the canonical vector bundle projection (called here the *transverse epimorphism*).

On the domain U of a chart in the foliated atlas, we denote by $\pi_U : U \to \bar{U}$ the canonical projection which has the fibers exactly the leaves of $\mathcal{F}_U$ and also by $\Pi_U = (\pi_U)_* : TU \to T\bar{U}$ its differential. A real function $f : M \to I\!R$ (or $f : U \to I\!R$) is *basic* if it is constant on leaves, *i.e.* $X(f) = 0$, for every $X \in \mathcal{X}(\mathcal{F})$, *i.e.* tangent to the leaves of $\mathcal{F}$. A section $\bar{s} \in \Gamma(\nu\mathcal{F})$ (called a *transverse vector field*) is *basic* provided that if it is restricted to U, then $\bar{s} = \Pi_U(s) : U \to \nu\mathcal{F}_U$, where $s \in \mathcal{X}(U)$ and $[s,X] = 0$, for every $X \in \mathcal{X}(\mathcal{F}_U)$, where $[\cdot,\cdot]$ is the Lie bracket. The section s is called a *foliated vector field*. A section $\bar{\omega} \in \Gamma(\nu^*\mathcal{F})$ (called a *transverse covector field*) is *basic* if $\bar{\omega}(\bar{s})$ is a basic function for every basic $\bar{s}$. The differential form ω induced on M is called a *foliated covector field*. Using the tensor product, one can define a *basic tensor* on $\nu\mathcal{F}$, of any order. For example, a bilinear form φ on the fibers on $\nu\mathcal{F}$ is *basic* if $\varphi(\bar{s}_1, \bar{s}_2)$ is basic for every basic local sections $\bar{s}_1$ and $\bar{s}_2$. A foliation $(M,\mathcal{F})$ lifts naturally to a foliation $(\nu F, \mathcal{F}_{\nu\mathcal{F}})$ on the normal bundle, such that the natural projection $\nu\mathcal{F} \xrightarrow{\pi_{\nu F}} M$ is a covering when it is restricted from leaves to leaves. In order to consider transverse Lagrangians $L : \nu\mathcal{F} \to I\!R$ that are continuous on $\nu\mathcal{F}$ and differentiable on $\nu\mathcal{F}_* = \nu\mathcal{F}\backslash\{\bar{0}\}$ (or $L : \nu\mathcal{F}_* \to I\!R$, where $\nu\mathcal{F}_* \subset \nu\mathcal{F}$ is a open fibered submanifold), one must restrict the foliation $(\nu\mathcal{F}, \mathcal{F}_{\nu\mathcal{F}})$ to a foliation $(\nu\mathcal{F}_*, \mathcal{F}_{\nu\mathcal{F}_*})$ on $\nu\mathcal{F}_* = \nu\mathcal{F}\backslash\{\bar{0}\}$, where $\{\bar{0}\}$ is the image of the null section (see below, in this section, a more general case of foliated vector bundles).

The *basic Hessian* of a Lagrangian $L : \nu\mathcal{F} \to I\!R$ on $\nu\mathcal{F}$ is the symmetric bilinear form on the fibers of $\pi^*_{\nu\mathcal{F}}\nu\mathcal{F}$, defined by $Hess(L)(X,Y) = X(Y(L))$, for X and Y local induced basic sections on an open $\pi^{-1}_{\nu\mathcal{F}}(U)$, then extended by linearity for $X,Y \in \Gamma(\pi^*_{\nu\mathcal{F}}\nu\mathcal{F}_U)$ and then on $\nu\mathcal{F}$. Using local coordinates, $g_{\bar{u}\bar{v}}(x^w,x^{\bar{w}},y^{\bar{w}}) = \dfrac{\partial^2 L}{\partial y^{\bar{u}}\partial y^{\bar{v}}}(x^w,x^{\bar{w}},y^{\bar{w}})$ are the *local coefficients* of the basic Hessian.

A Riemannian foliation can be given using different but equivalent definitions (see [8] or [20]). Our goal is to extend some of them to foliations that are related to some special Lagrangians.

Let us suppose that two foliations $\mathcal{F}_E$ and $\mathcal{F}_M$ are given on two manifolds E and M respectively. A *foliated map* $f : E \to M$ is a differentiable map that sends leaves to leaves.

We say that a fibered manifold (*i.e.* a surjective submersion) $\pi : \mathcal{E} \to M$ is a *foliated fibered manifold* (an *f.f.m.* for short) if π is a foliated map that is a local diffeomorphism on leaves, *i.e.* the fiber structure is transverse according to the two foliations. Using local coordinates, it reads that there are foliated atlases on M and $\mathcal{E}$ adapted to the fibered manifold structure as well, having the following properties: if $(x^u,x^{\bar{u}})$ are local foliated coordinates on M and $(x^u,x^{\bar{u}},y^\alpha)$ are the corresponding coordinates on $\mathcal{E}$, then the change rules of these coordinates are $x^{u'} = x^{u'}(x^u,x^{\bar{u}})$, $x^{\bar{u}'} = x^{\bar{u}'}(x^{\bar{u}})$, $y^{\alpha'} = y^{\alpha'}(x^{\bar{u}},y^\alpha)$. Notice that the coordinates y^α are transverse. A local section $s : U \subset M \to \mathcal{E}$ of the f.f.m. $\mathcal{E}$ that is a foliated map is called a *foliated section*. If (U,φ) is a foliated chart, $U \subset M$, then s has the local form $(x^u,x^{\bar{u}}) \to (x^u,x^{\bar{u}},s^\alpha(x^{\bar{u}}))$, *i.e.* its local functions s^α are basic.

We say that a vector bundle $\pi : E \to M$ is a *foliated vector bundle (f.v.b.)* if it is an f.f.m. (*i.e.* regarded as a fibered manifold).

Let $\pi : E \to M$ be an f.v.b. A *Lagrangian* on E is a differentiable map $L : E' \to I\!R$, where $E' \subset E$ is an open fibered submanifold. In order to keep some regularity conditions, one can ask that L extends to a continuous $L : E \to I\!R$; for example, $E' = E\backslash\{\bar{0}\}$, where $\{\bar{0}\}$ is the image of the null section, or $E' = E\backslash s(M)$ is the image of a global section $s : M \to E$. The *vertical hessian* of L is the bilinear form defined on the fibers of the vertical bundle of E by $\left(\dfrac{\partial^2 L}{\partial y^a \partial y^b}\right)$, using some local coordinates; it is easy to see that the definition does not depend on coordinates.

If $\pi : E \to M$ is an f.v.b., then any other tensor (vector) bundle $\pi^r_s : T^r_s(E) \to M$ is a foliated one, with a foliation $\mathcal{F}_{T^r_s(E)}$. In particular $\pi' = \pi^0_1 : E^* = T^0_1(E) \to M$, the dual bundle of E, is a foliated vector bundle.

A Hamiltonian on E is a differentiable map $H : F \to I\!R$, where $F \subset E^*$ is an open fibered submanifold; thus a Hamiltonian on E is a Lagrangian on E^*.

If one consider that the vector bundle is foliated, then we say that a Lagrangian $L : E' \to I\!R$ or a Hamiltonian $H : F \to I\!R$ is *transverse* if it is a basic function according to the induced foliation $\mathcal{F}_{E'}$ or $\mathcal{F}_F$ respectively.

In particular, the normal bundle $\nu\mathcal{F}$ and its dual $\nu^*\mathcal{F}$ are foliated vector bundles, with foliations $\mathcal{F}_{\nu\mathcal{F}}$ and $\mathcal{F}_{\nu^*\mathcal{F}}$ respectively. A *transverse Lagrangian* or a *transverse Hamiltonian* is considered foliated according with the canonical lifted foliations on $\nu\mathcal{F}$ or its dual $\nu^*\mathcal{F}$ respectively.

Let us consider an epimorphism of vector bundles $f : E \to E''$, *i.e.* f is a morphism of vector bundles that is a surjection on E'' and as well on fibers. Then $E' = \ker f \subset E$ is vector subbundle. Then the fibers of f are leaves of a simple foliation (*i.e.* the fibers of a surjective submersion). Consequently we use in the sequel local coordinates adapted to the epimorphism f and this simple foliation: (x^i) on M, $(x^i, y^a, y^{\bar{a}})$ on E and $(x^i, y^{\bar{a}})$ on E'', such that f has the local form $(x^i, y^a, y^{\bar{a}}) \to (x^i, y^{\bar{a}})$ and the local coordinates on E' are $(x^i, y^\alpha, 0)$. We denote by $(x^i, p_a, p_{\bar{a}})$ and $(x^i, p_{\bar{a}})$ the local coordinates on E^* and $\acute{E}''^*$ respectively; then the coordinates on $(E')^0$ are $(x^i, 0, p_{\bar{\alpha}})$. Since the local form of the co-morphism is $(x^i, p_{\bar{\alpha}}) \xrightarrow{f^*} (x^i, 0, p_{\bar{\alpha}})$, it shows the local form of the equality $(E')^0 = f^*(E''^*)$.

Let $H : E^* \to I\!R$ be a Hamiltonian on E. Then $H'' = H \circ f^* : E''^* \to I\!R$ is a Hamiltonian on E'', that we call the *projection* of H by f. If the local form of H is $(x^i, p_a, p_{\bar{a}}) \xrightarrow{H} H(x^i, p_a, p_{\bar{a}})$, then the local form of H'' is $(x^i, p_{\bar{a}}) \xrightarrow{H} H(x^i, 0, p_{\bar{a}})$. The co-Legendre map of H is given by $\mathcal{H} : E^* \to E$, $\mathcal{H}(x^i, p_a, p_{\bar{a}}) = (x^i, \dfrac{\partial H}{\partial p_a}, \dfrac{\partial H}{\partial p_{\bar{a}}})$. Consequently, the co-Legendre map of H'' is given by $\mathcal{H}'' : E''^* \to E''$, $\mathcal{H}''(x^i, p_{\bar{a}}) = (x^i, \dfrac{\partial H''}{\partial p_{\bar{a}}}(x^i, p_{\bar{a}}) = \dfrac{\partial H}{\partial p_{\bar{a}}}(x^i, 0, p_{\bar{a}}))$. It follows that $\mathcal{H}'' = f \circ \mathcal{H} \circ f^*$. We say that H is *regularly related* to the epimorphism f if H'' is hyperregular, *i.e.* $\mathcal{H}''$ is a diffeomorphism.

Let us denote by $E^*_* \subset E^*$, $E''^*_* \subset E''^*$ the open submanifold of non-null vectors. Then we can relax the conditions on H, asking that H is continuous on E^* and differentiable (smooth) on E^*_*. If it is the case, then H is *regularly related* to the epimorphism f if the projection H'' of H is hyperregular, *i.e.* $\mathcal{H}'' : E^*_* \to \mathcal{H}''(E^*_*)$ is a diffeomorphism.

3. HAMILTONIANS AND FOLIATIONS

We say that a Hamiltonian $H : T^*M \to I\!R$ is

– *almost bundle–like* for the foliation $(M, \mathcal{F})$ if it is regularly related to the canonical projection $\Pi : TM \to \nu\mathcal{F}$ and its vertical Hessian is positively defined;

– *bundle–like* for the foliation $(M, \mathcal{F})$ if it is almost bundle-like and its projected Hamiltonian is basic according to the foliation $\mathcal{F}_{\nu^*\mathcal{F}}$.

A *transverse positive Hamiltonian* to a foliation $(M, \mathcal{F})$ is a Hamiltonian $H'' : \nu^*\mathcal{F} \to I\!R$ that is positively defined, hyperregular and a basic function for the lifted normal foliation $(\nu^*\mathcal{F}_*, \mathcal{F}_{\nu^*\mathcal{F}_*})$.

In fact, we have the following result.

Proposition 1 *If a Hamiltonian $H : T^*M \to I\!R$ is regularly related to the canonical projection $\Pi : TM \to \nu\mathcal{F}$ and its projected Hamiltonian H'' is positively defined, then there is an almost bundle-like Hamiltonian $H' : T^*M \to I\!R$ such that both H and H' have the same projection by f. If H'' is a transverse (positive) Hamiltonian, then H' is a bundle-like Hamiltonian.*

Proof. Consider a Riemannian metric g on the fibers of T^*M. According to the canonical projection $I^* : T^*M \to T^*M$, there is an orthogonal decomposition

$$T^*M = \ker I^* \oplus (\ker I^*)^{\perp}. \tag{1}$$

Let us denote by Π_1 and Π_2 the corresponding orthogonal projectors. Using local coordinates $(x^u, x^{\bar{u}}, p_u, p_{\bar{u}})$ on T^*M, we denote by $(g^{uv}, g^{u\bar{v}} = g^{\bar{v}u}, g^{\bar{u}\bar{v}})$ the components of g in the point of M that has as coordinates $(x^w, x^{\bar{w}})$. Then the projection I^* has the local form $(p_u, p_{\bar{u}}) \xrightarrow{I^*} p_u$ and the decomposition (1) has the local form

$$p_u dx^u + p_{\bar{u}} dx^{\bar{u}} = (p_{\bar{u}} + N^u_{\bar{u}} p_u)\, dx^{\bar{u}} + p_u \left(dx^u - N^u_{\bar{u}} dx^{\bar{u}}\right),$$

where $N^u_{\bar{u}} = g^{u\bar{v}} g_{\bar{u}\bar{v}}$ and $(g_{\bar{u}\bar{v}}) = (g^{\bar{u}\bar{v}})^{-1}$. It is easy to see that $\ker I^* = \Pi^*(\nu^*\mathcal{F})$. We define

$$H'\left(x^u, x^{\bar{u}}, p_u, p_{\bar{u}}\right) = H\left(x^u, x^{\bar{u}}, 0, p_{\bar{u}} + N^u_{\bar{u}} p_u\right) + \frac{1}{2}\tilde{g}^{uv} p_u p_v, \tag{2}$$

where $\tilde{g}^{uv} = g\,(dx^u - N^u_{\bar{u}} dx^{\bar{u}}, dx^v - N^v_{\bar{v}} dx^{\bar{v}}) = g^{uv} - N^u_{\bar{u}} g^{\bar{u}v} - N^v_{\bar{v}} g^{\bar{v}u} - N^u_{\bar{u}} N^v_{\bar{v}} g^{\bar{u}\bar{v}}$.

It is easy to see that both H and H' have the same projection by f, $H''(x^u, x^{\bar{u}}, p_{\bar{u}}) = H\,(x^u, x^{\bar{u}}, 0, p_{\bar{u}})$; according to the hypotheses, H'' is hyperregular and positively defined. If we denote by $h^{\bar{u}\bar{v}} = \frac{\partial^2 H}{\partial p_{\bar{u}} \partial p_{\bar{v}}}\,(x^u, x^{\bar{u}}, 0, p_{\bar{u}} + N^u_{\bar{u}} p_u)$, then the vertical Hessian of H' is

$$\begin{pmatrix} h^{\bar{u}_1\bar{v}_1} N^u_{\bar{u}_1} N^v_{\bar{v}_1} + \tilde{g}^{uv} & h^{\bar{v}\bar{v}_1} N^v_{\bar{v}_1} \\ N^u_{\bar{u}_1} h^{\bar{u}\bar{u}_1} & h^{\bar{u}\bar{v}} \end{pmatrix}$$

and it is easy to prove that it is positively defined. Indeed, let us denote
$E = \left(h^{\bar{u}_1\bar{v}_1} N^u_{\bar{u}_1} N^v_{\bar{v}_1} + \tilde{g}^{uv}\right) P_u P_v + N^u_{\bar{u}_1} h^{\bar{u}\bar{u}_1} P_u P_{\bar{u}} + h^{\bar{v}\bar{v}_1} N^v_{\bar{v}_1} P_v P_{\bar{v}} + h^{\bar{u}\bar{v}} P_{\bar{u}} P_{\bar{v}} =$
$h^{\bar{u}\bar{v}} (P_{\bar{u}} + N^u_{\bar{u}} P_u)(P_{\bar{v}} + N^v_{\bar{v}} P_v) + \tilde{g}^{uv} P_u P_v$.

Then it is easy to see that $E \geq 0$ and if $E = 0$, then $P_u = P_{\bar{u}} = 0$. Thus H' is almost bundle-like.

According to their definitions, if H'' is basic according to the foliation $\mathcal{F}_{\nu\mathcal{F}}$, then H' is bundle-like. □

Using the proof of Proposition 1, one can easily prove the following result.

Proposition 2 *A foliation allows a positive transverse Hamiltonian H'' iff it allows a positive bundle–like Hamiltonian H.*

If $H : T^*M \to I\!R$ fulfils the hypothesis of Proposition 1, then $H' : T^*M \to I\!R$, given by the conclusion of this Proposition, is called a Hamiltonian in a *standard form*.

It is easy to see that $\ker I^* = \Pi^*(\nu^*\mathcal{F}) = (\tau\mathcal{F})^\circ$ is the polar of $\tau\mathcal{F}$.

Since H'' is supposed to be hyperregular, then one can consider its corresponding dual Lagrangian $L'' : \nu\mathcal{F} \to I\!R$. Then the equation

$$\frac{\partial H}{\partial p_{\bar{u}}}\left(x^u, x^{\bar{u}}, 0, p_{\bar{u}}\right) = y^{\bar{u}}$$

has the unique solution $p_{\bar{u}} = L_{\bar{u}}(x^u, x^{\bar{u}}, y^{\bar{u}})$. One have $L''(x^u, x^{\bar{u}}, y^{\bar{u}}) = L_{\bar{u}} y^{\bar{u}} - H''(x^u, x^{\bar{u}}, L_{\bar{u}})$.

Let us find now the dual Lagrangian of H'. By a straightforward calculus, one obtain

$$\begin{aligned} L'\left(x^u, x^{\bar{u}}, y^u, y^{\bar{u}}\right) =& y^u p_u + y^{\bar{u}} p_{\bar{u}} - H'\left(x^u, x^{\bar{u}}, p_u, p_{\bar{u}}\right) = \\ & L''\left(x^u, x^{\bar{u}}, y^{\bar{u}}\right) + \frac{1}{2} g_{uv}\left(y^u - N^u_{\bar{u}} y^{\bar{u}}\right)\left(y^v - N^v_{\bar{v}} y^{\bar{v}}\right). \end{aligned} \tag{3}$$

According to [12], L' is a bundle-like Lagrangian for the foliation $\mathcal{F}$.

Since

$$TM = T\mathcal{F} \oplus (T\mathcal{F})^\perp,$$

it follows that

$$y^u \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}} = \left(y^u - N^u_{\bar{u}} y^{\bar{u}}\right)\frac{\partial}{\partial x^u} + y^{\bar{u}}\left(\frac{\partial}{\partial x^{\bar{u}}} + N^u_{\bar{u}} \frac{\partial}{\partial x^u}\right),$$

thus the formula (3) has a geometrical meaning (*i.e.* it has the same form in every system of coordinates).

The following result gives a test to check if an almost bundle-like Hamiltonian is a bundle-like Hamiltonian one.

Proposition 3 *Let $H : T^*M \to I\!R$ be an almost bundle-like Hamiltonian in standard form. Then H is bundle–like iff for any local foliated covector field ω on a $U \subset M$ that is also a local section of the polar of $\tau\mathcal{F}$ on U, then $U \ni x \to H(x, \omega_x) \in I\!R$ is a basic function on U.*

Proof. Using local coordinates, the condition given in hypothesis reads that the projected Hamiltonian H'' is basic for $\mathcal{F}_{\nu\mathcal{F}}$, thus the conclusion follows. □

If $p : E \to M$ is an f.v.b., then we consider (if anything else is assumed) $E' = E_*$; in the sequel, a Lagrangian on E is a differentiable map $L : E_* \to I\!R$. We say that a bilinear form h on the fibers of E is *foliated* if for every foliated (local) sections s_1, $s_2 : U \to E$ that are h-orthogonal to the fibers of p, then $h_{|U}(s_1, s_2)$ is a basic function (on U).

If $p : E \to M$ is an f.f.m. (in particular, a f.v.b. or an open fibered submanifold of a f.v.b.), then its vertical bundle $p_v : VE \to E$ is an f.v.b. We say that the Lagrangian $L : E \to I\!R$ is *foliated* if its vertical hessian is foliated. Using local foliated coordinates $(x^u, x^{\bar{u}})$ on M and $(x^u, x^{\bar{u}}, y^a)$ on E, it reads that the local components

$\dfrac{\partial^2 L}{\partial y^a \partial y^b}$ of the vertical hessian depend only on $(x^{\bar{u}}, y^a)$.

Let $\pi : E \to M$ and $\pi' : E' \to M'$ be vector bundles, $f_0 : M \to M'$ be a differentiable map and $f : E \to M'$ be an f_0-morphism; we denote these data as $(f_0, f) : (E, \pi, M) \to (E', \pi', M')$, or as $(E, \pi, M) \stackrel{(f_0,f)}{\to} (E', \pi', M')$. One can consider also co-morphisms $[f_0, f] : (E, \pi, M) \to (E', \pi', M')$. We recall that in a co-morphism $f_x : E'_{f_0(x)} \to E_x$ is linear, $(\forall) x \in M$.

If $(f_0, f) : (E, \pi, M) \to (E', \pi', M')$ is a morphism and f is an isomorphism on fibers, then one can consider a co-morphism $[f_0, f^{-1}] : (E, \pi, M) \to (E', \pi', M')$ of vector bundles.

If $[f_0, f] : (E, \pi, M) \to (E', \pi', M')$ is a co-morphism and f is an isomorphism on fibers, then one can consider a morphism $(f_0, f^{-1}) : (E, \pi, M) \to (E', \pi', M')$ of vector bundles. Notice that if $L' : E' \to I\!R$ is a Lagrangian, then $L = L' \circ f^{-1} : E \to I\!R$ is a Lagrangian on E.

A dual construction can be considered as follows. If $(f_0, f) : (E, \pi, M) \to (E', \pi', M')$ is a morphism and f is an isomorphism on fibers, then one can consider the dual co-morphism $[f_0, f^*] : (E^*, \pi, M) \to (E'^*, \pi', M')$ that is also an isomorphism on fibers. Notice that if $H' : E'^* \to I\!R$ is a Hamiltonian, then $H = H' \circ f^{*-1} : E^* \to I\!R$ is a Hamiltonian on E.

These applies in particular to the case when $\varphi : M \to M'$ is a diffeomorphism and $(\varphi, \varphi_*) : (TM, \pi, M) \to (TM', \pi', M')$ is the differential map and $[\varphi, \varphi^*] : (T^*M, \pi_0, M) \to (T^*M', \pi'_0, M')$ is the co-differential map. Then $(\varphi, \varphi^{*-1}) : (T^*M, \pi_0, M) \to (T^*M', \pi'_0, M')$ is a morphism of vector bundles.

If $H : T^*M \to I\!R$ and $H' : T^*M' \to I\!R$ are two Hamiltonians, then a *co-isometry* of H and H' is a diffeomorphism $\varphi : M \to M'$ such that $H = H' \circ \varphi^{*-1}$.

A foliation $(M, \mathcal{F})$ can be defined equivalently by a foliated cocycle (for more details, see any of [2, 8, 20]). For short, this shows that the local transverse models $\bar{U}$ considered in the local definition of a foliation (see the very beginning of Section 2), can be considered as open subsets of the same differentiable manifold S (called a *transverse model*). Notice that, only for a simple foliation $\mathcal{F}$ there is a global submersion $M \to S$ that have as fibers the leaves of $\mathcal{F}$.

Given a transverse model S, a *foliated cocycle* on M is a set $(U_i, \pi_i, \gamma_{ij})_{i,j \in I}$ such that $\{U_i\}_{i \in I}$ is an open cover of M, $\{\pi_i : U_i \to S\}_{i \in I}$ are submersions such that the following conditions hold:

– the leaves of $\mathcal{F}_{|U_i}$ and π_i are the same;

– for every $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$, then γ_{ij} is a local diffeomorphism of open subsets of S and for each $x \in U_i \cap U_j$, $f_i(x) = (\gamma_{ij} \circ f_j)(x)$.

A compatibility condition between global structures on M and S can occur, as follows.

Consider that (M,g) and $(S,\bar{g})$ are Riemannian manifolds. Then a foliated cocycle is *Riemannian* if γ_{ij} are local Riemannian isometries (see [2, Cap. III, Sect. 1.4], or [18, Cap. IV, Sect. 4]). This can be considered as an equivalent definition of a Riemannian foliation. We extend below this construction in the Hamiltonian case.

Analogously to the Finslerian case (as in [5] or [6]), or to the Lagrangian case (as in [12]), a foliated cocycle is *Hamiltonian* if there is a Hamiltonian $\bar{H}: T^*S \to I\!R$ on the transverse model such that for every $i,j \in I$ with $U_i \cap U_j \neq \emptyset$, then γ_{ij} is a local Hamiltonian co-isometry of $\bar{H}$.

Proposition 4 *A foliation $(M,\mathcal{F})$ allows a Hamiltonian foliated cocycle iff it allows a transverse Hamiltonian $H'': \nu^*\mathcal{F} \to I\!R$.*

Proof. If $(U_i,\pi_i,\gamma_{ij})_{i,j\in I}$ is a Hamiltonian foliated cocycle, then one defines a transverse Hamiltonian $H'': \nu^*\mathcal{F} \to I\!R$ as follows. One defines the Hamiltonian $H''_i: \nu^*\mathcal{F}_{U_i} \to I\!R$ as induced by the Hamiltonian $\bar{H}$ on S, using $f_i: U_i \to S$ and the isomorphism on fibers $F_i: \nu\mathcal{F}_{U_i} \to TS$, induced by differentials. One prove by a straightforward verification that the definition of H''_i does not depend on i, giving a global a transverse Hamiltonian H''. One can prove easily the converse fact, that given H'' one can recover $\bar{H}$, using a foliated cocycle for $\mathcal{F}$. $\Box$

The solutions of a Hamilton equations on a symplectic manifold M are given by the integral curves of a Hamiltonian vector field. In particular, T^*M is a symplectic manifold. We say that a fibered submanifold $E' \subset T^*M$ is called *sub-symplectic* according to a Hamiltonian H, if an integral curve is a solution of the Hamilton equations and it is tangent in a point to E', then it is contained entirely in E'.

Proposition 5 *The polar $(T\mathcal{F})^\circ \subset T^*M$ is sub-symplectic according to an almost bundle-like Hamiltonian H' in the standard form; the Hamiltonian H' is bundle-like iff the restriction of its Hamiltonian vector field $X_{H'}$ to $(T\mathcal{F})^\circ$ is foliated according to the induced foliation on $(T\mathcal{F})^\circ$.*

Proof. The proof is a consequence of the local form (2) of H'. $\Box$

If $L: E \to M$ is a Lagrangian and $\alpha,\beta \in \mathcal{F}(M)$, $0<\alpha<\beta$, then we say that L is (α,β)*–positively defined* if the Hessian of L is positively defined on the non-void set $E_{\alpha,\beta} = \{e_x \in E_x : L(\bar{0}_x) = 0 < \alpha(x) < L(e_x) < \beta(x)\}$.

Let $\pi: E \to M$ be a foliated vector bundle. We say that a transverse Lagrangian $L: E \to I\!R$ is *positively allowed* if the transverse Lagrangian has the property that are two smooth functions $\alpha,\beta: M \to (0,\infty)$, $\alpha<\beta$, basic for the foliation $\mathcal{F}$ on M, such that L is (α,β)–positively defined.

If a transverse Lagrangian L has the properties:

1) L is positive definite (*i.e.* its basic Hessian is positively defined) and $L(x,y) \geq 0 = L(x,0)$, $(\forall)x \in M$ and $y \in E_x$;

2) the transverse Lagrangian has the property that there is a smooth function $\varphi: M \to (0,\infty)$, basic for the foliation $\mathcal{F}$ on M, such that for every $x \in M$ there is $y \in E_x$

with $L(x,y)=\varphi(x)$, then taking $\alpha=\frac{1}{2}\varphi$ and $\beta=\varphi$, it follows that L is positively allowed. This is the definition of a positively allowed Lagrangian in [12]–[17].

An other particular case is that of a transverse positively definite Finslerian $F:E\to I\!R$. As pointed above, it is always positively allowed, where $0<\alpha<\beta$ can be any positive constant functions on M. The interesting (non-trivial) case occurs when F is continuous on E and it is differentiable on E_*.

We state now a result proved in [15, Proposition 2.2], that we use in the following.

Let us consider a foliation $\mathcal{F}$ on M and two arbitrary foliated vector bundles $E_1 \stackrel{\pi_1}{\to} M$ and $E_2 \stackrel{\pi_2}{\to} M$. We can consider the induced vector bundle $\pi_0=\pi_2^*\pi_i$: $\pi_1^*E_1\to E_2$; this is a foliated vector bundle, according to the canonical foliation $\mathcal{F}_{E_2}$ on E_2.

Proposition 6 *Let us suppose that the vector bundle E_2 allows a positively allowed transverse Lagrangian L and the foliated vector bundle $\pi_0:\pi_2^*E_1\to E_{2*}$ allows a transverse Riemannian metric b on fibers. Then the foliated bundle E_1 allows a transverse Riemannian metric.*

The idea is averaging the Riemannian metric b on fibers of E_2, using a measure that in the Finsler case is the Bussemann-Hausdorff measure (see also [19, Section 5.1]).

If $\pi:E\to M$ is an f.v.b., then the kernel of the canonical projection $\nu\mathcal{F}_E\to\nu\mathcal{F}$ is a vector subbundle $V\nu\mathcal{F}_E\subset\nu\mathcal{F}_E$, called the *vertical normal bundle* of E. It is a foliated vector bundle, isomorphic with the induced vector bundle π^*E.

In the case when $E_1=E_2$, we obtain as a particular case of Proposition 6, the necessary condition of the equivalence stated by the following result.

Proposition 7 *A foliated vector bundle $\pi:E\to M$ has a positively allowed transverse Lagrangian L iff it allows a transverse Riemannian metric.*

A simple case of Proposition 7 is the following result, using instead E the foliated vector bundles $\nu\mathcal{F}$ and $\nu^*\mathcal{F}$.

Proposition 8 *For a foliation $\mathcal{F}$ on M, the following conditions are equivalent:*

1. *The foliation $\mathcal{F}$ is Riemannian.*
2. *The foliation $\mathcal{F}$ has a positively allowed transverse Lagrangian L on $\nu\mathcal{F}$.*
3. *The foliation $\mathcal{F}$ has a positively allowed transverse Hamiltonian H on $\nu^*\mathcal{F}$.*

Notice that the equivalence between 2. and 3. follows *via* their equivalences to 1., since in the general case it can not be deduced from the duality between L and H. By Proposition 8, it follows that assuming a positively allowed transverse Hamiltonian exists for a given foliation, then each condition in Propositions 2, 3 and 4 is equivalent to the fact that the foliation is Riemannian.

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