ON THE STOCHASTIC ANISOTROPIC SHEARED MAGNETIC FIELD LINES DIFFUSION

MARIAN NEGREA[1], VIRGIL-NICOLAE CANCEA[2]

[1]Association Euratom-MEdC, Romania
University of Craiova, Department of Physics, 13 A.I. Cuza Street, 200585 Craiova, Dolj, Romania
E-mail: mnegrea@yahoo.com
[2]University of Craiova, Department of Physics, 13 A.I. Cuza Street, 200585 Craiova, Dolj, Romania
E-mail: cancea.virgil@ucv.ro

Received September 20, 2015

In the present paper the decorrelation trajectory method has been used in order to calculate some averaged quantities of interest for stochastic anisotropic magnetic field lines in a sheared slab magnetic configuration for several values of the magnetic Kubo number, of the shear parameter and of the stochastic anisotropy parameter. The study has shown that a rich variety of transport behaviors can be found by varying the previously parameters, mainly by varying the anisotropy parameter in the plane perpendicular to the average magnetic field.

Key words: plasma, anisotropy, turbulence, magnetic shear.

PACS: 52.35.Ra, 52.25.Fi, 05.40.-a, 02.50.-r.

1. INTRODUCTION

A main issue for fusion consists in the study of turbulence phenomena in a plasma at high temperature. In this framework, the magnetic turbulence appears as a plausible candidate for determining the anomalous transport properties of a hot magnetized plasma. The magnetic fluctuations measured by the dimensionless magnetic Kubo number K_m [1]-[7], even when small, can destroy the nested magnetic configurations in a toroidal confinement geometry, such as in a tokamak and any particle makes radial displacements, thus enhancing the radial transport. The study of transport in such stochastic sheared magnetic fields is then of great importance for the nuclear fusion.

In the present paper we restrict ourselves to the study of the properties of the magnetic lines of a sheared stochastic magnetic field alone. This kind of study was started in [3]. In order to do that, we considered the simplified model of the slab approximation including the shear. The presence of the shear influences the length of a magnetic line between two fixed z -values (z is a coordinate in the direction of the reference magnetic field) [8]. It has be shown that the magnetic shear parameter K_s [1],[3] and the magnetic Kubo number K_m have an important influence on the diffusion of the magnetic field lines. Also, many previously studies were devoted to the shearless isotropic case only *e.g.* [1]. Another series of papers considered the anisotropic shearless magnetic field lines diffusion assuming astrophysical and space plasma parameters with various level of magnetic field fluctuations (see, *e.g.* [10]-[14]). The main contribution of our work consists in the analysis of the influence of the shear and of the stochastic anisotropy in both weak and strong turbulence regimes on the behavior of the stochastic magnetic fields. Until now the influence of the shear on the diffusion coefficients was analyzed only in the quasilinear limit *i.e.* the small Kubo number regime. It was shown that the transport regime of plasma particles can depend on parameters like the turbulence level. In [15] a detailed direct numerical simulation was done concerning the stochastic magnetic fields. Another direct numerical approaches, related to diffusion problems, have been reported in [16]-[29], some ones included also some fractional diffusion development.

In the latter it was shown that for low fluctuation level the magnetic field transport is non-Gaussian while for a high fluctuation level the Gaussian diffusion regime is attained. Using the DCT method [30] we extended the analysis to a relatively high Kubo number regime $K_m > 1$. Because of the existence of the three parameters, namely the magnetic Kubo number K_m, the shear parameter K_s and the anisotropy parameter Λ, a richer class of behaviors of the averaged quantities is observed. The competition between these parameters plays an important role and seems to be decisive in the determination of the trapping effects.

The paper is organized as follows. The Langevin equations for the sheared stochastic magnetic field are established in Sec. II. In Sec. III, in the framework of DCT method, the analytical expressions of the averaged quantities of interest are obtained. The conclusions are summarized in Sec. IV.

2. THE MAGNETIC FIELD MODEL

In the present paper we consider a magnetic field of the following form [3]:

$$\mathbf{B}(\mathbf{X},\, Z) = B_0\left\{\mathbf{e}_z + \beta b_x(\mathbf{X},\, Z)\,\mathbf{e}_x + \left[\beta b_y(\mathbf{X}, Z) + L_s^{-1}X\right]\mathbf{e}_y\right\} \tag{1}$$

where $\mathbf{X} = (X,\, Y)$, β is the dimensionless parameter measuring the amplitude of the magnetic field fluctuations relative to the main constant magnetic field B_0. The shear of the magnetic field is equivalent with the existence of a gradient ∇B where $B \approx B_0\left[1 + \frac{1}{2}\left(XL_s^{-1}\right)^2\right]$ if the contributions of the magnetic field fluctuations are neglected from the magnetic field intensity B. The gradient is then $\nabla B = \frac{\partial B}{\partial X}\mathbf{e}_x = B_0\frac{1}{2}XL_s^{-2}\mathbf{e}_x$.The corresponding dimensional equations for the magnetic field lines are:

$$\frac{dX}{\beta B_0 b_x} = \frac{dY}{\beta B_0 b_y + B_0 L_s^{-1}X} = \frac{dZ}{B_0} \tag{2}$$

These equations can be rewritten by considering the Z variable as a mere parameter which plays here the role of "time":

$$\frac{dX}{dZ} = \beta b_x(\mathbf{X};\, Z)$$

$$\frac{dY}{dZ} = \beta b_y(\mathbf{X};\, Z) + L_s^{-1} X \tag{3}$$

The coefficient L_s^{-1} is the first order shear length and is defined as follow:

$$L_s^{-1} = \frac{s_0}{q_0 R_0} \tag{4}$$

where R_0 is the major radius of a tokamak, r_0 is the radius corresponding to a rational surface, *i.e.* such that $q(r_0) \equiv q_0 = m_0/n_0$, with $m_0,\ n_0$ given integers and $q(r)$ is the safety factor profile, and $s_0 \equiv s(r_0) = \left(\frac{d\ln q(r)}{d\ln r}\right)_{r=r_0}$ is the local shear parameter. The linear term depending on X in the right hand side of Eq.(1) is the linear shear term, and is the so-called "sheared slab in first order approximation" configuration which represents a local approximation valid only for $|X| \ll L_s$ and which mimics the field around a rational surface $X = 0$ existing in toroidal systems. An extension of this approximation contains also the quadratic term depending on X^2 which becomes important for the reversed shear in the case when $s(r_0) = 0$, *i.e.* when $L_s \to \infty$; however, the latter situation will not be the object of the present paper. The magnetic field fluctuations are described by the dimensionless functions $b_i(\mathbf{X},\, Z)$, $i = (x,\, y)$, taken to be Gaussian random anisotropic processes. The physical system is considered to be characterized by two different correlation lengths $\lambda_x \neq \lambda_y$ in a plane perpendicular to the main magnetic field B_0; the third characteristic length is $\lambda_z \neq \lambda_x \neq \lambda_y$ [3].

In [3] we have introduced the dimensionless coordinates $\mathbf{x} = (x,\, y),\ z$ which are related to the dimensional ones $\mathbf{X},\ Z$ by the relations:

$$x = \frac{X}{\lambda_x},\ y = \frac{Y}{\lambda_y},\ z = \frac{Z}{\lambda_z} \tag{5}$$

The magnetic field given in Eq. (1) satisfies the zero-divergence constraint $\nabla \cdot \mathbf{B} = 0$ if, *e.g.* we consider that the fluctuating magnetic field derives from a vector potential which has only the z - component:

$$\mathbf{A}(\mathbf{X},\, Z) = B_0 \lambda_x \beta\, \psi(\mathbf{x},\, z)\, \mathbf{e}_z \tag{6}$$

Eqs. (3) is then be rewritten in the following dimensionless form using Eqs.(5, 6):

$$\frac{dx(z)}{dz} = K_m b_x[\mathbf{x}(z),\, z] = \Lambda K_m \left.\frac{\partial \psi(\mathbf{x}(z),\, z)}{\partial y}\right|_{\mathbf{x}=\mathbf{x}(z)} \equiv w_x[\mathbf{x}(z),\, z]$$

$$\frac{dy(z)}{dz} = K_m \Lambda b_y[\mathbf{x}(z),\ z] + \Lambda K_s x(z) =$$
$$= -\Lambda K_m \left.\frac{\partial \psi(\mathbf{x}(z),\ z)}{\partial x}\right|_{\mathbf{x}=\mathbf{x}(z)} + \Lambda K_s x(z) \equiv$$
$$\equiv w_y[\mathbf{x}(z),\ z] + \Lambda K_s x(z) \tag{7}$$

where the following dimensionless parameters are defined, same like in [3]:

$$\text{the magnetic Kubo number } K_m = \beta \frac{\lambda_z}{\lambda_x} \tag{8}$$

$$\text{the shear parameter} \qquad K_s = \frac{\lambda_z}{L_s} \tag{9}$$

$$\text{the anisotropy parameter} \qquad \Lambda = \frac{\lambda_x}{\lambda_y} \tag{10}$$

The orders of magnitude for the correlations lengths are considered as: $\lambda_x \sim \lambda_y \sim 10^{-2}\ m$ and $\lambda_z \sim 100\ m$. Then the order of magnitude for the magnetic Kubo number is: $K_m = \beta \frac{\lambda_z}{\lambda_x} \simeq 10^{-1} \div 1$ but values of order 10^{-2} or 10 for the magnetic Kubo number are acceptable if there exist a slow variation of the quantities β, λ_x, λ_z. The level of the stochasticity of the system is contained in the definition of the magnetic Kubo number even if we have considered a fixed correlation length in the Oz direction, $\lambda_z \sim 100\ m$.

The fluctuating potential $\psi(\mathbf{x}; z)$ is assumed to be a Gaussian process with zero average. The second order moment of $\psi(\mathbf{x};\ z)$, *i.e.*, its Eulerian autocorrelation function $M(\mathbf{x},\ z)$ is assumed to have the following factorized form:

$$M(\mathbf{x},\ z) = \langle \psi(\mathbf{0},\ 0) \psi(\mathbf{x},\ z) \rangle = M_1(\mathbf{x}) M_2(z) \tag{11}$$

where:

$$M_1(\mathbf{x}) = \exp(-\frac{\mathbf{x}^2}{2}), \quad M_2(z) = \exp(-\frac{z^2}{2}) \tag{12}$$

All the necessary quantities to apply the DCT method where deduced in [1]-[7]. The system given in (7) is used in order to find specific averaged quantities for the magnetic field. These depend on three dimensionless parameters: the magnetic Kubo number K_m, the shear parameter K_s and the anisotropy parameter Λ.

3. THE AVERAGED QUANTITIES

The main idea of the DCT method concerns in the study of the Langevin system (7) in the whole space of the realizations that is subdivided into *subensembles* S, characterized by given values of the potential and of the fluctuating field at the starting point of the trajectories (see below Eq.(14)). The definition of the DCT approximation method consists practically in the following two statements:

(a) In each subensemble is defined a deterministic trajectory $\mathbf{x}^S(z)$ by the following criterion: the *Eulerian average* of the potential ψ^S in the subensemble S, calculated along this deterministic trajectory equals the *Lagrangian* average of the same potential in the subensemble S:

$$\psi^S[\mathbf{x}^S(z);\, z] = \langle \psi[\mathbf{x}(z);\, z] \rangle^S . \tag{13}$$

This deterministic trajectory is called decorrelation trajectory.

(b) The average Lagrangian velocity in the subensemble S is approximated with the average Eulerian velocity calculated along the deterministic trajectory.

We first define a set of subensembles S of the realizations of the stochastic sheared magnetic field that are defined by given values of the potential ψ and magnetic field fluctuation $\mathbf{b}$ in the point $\mathbf{x} = 0$ at the "moment" $z = 0$:

$$\psi(\mathbf{0};\, 0) = \psi^0,\;\; b_i(\mathbf{0};\, 0) = b_i^0,\, i = (x,\, y) \tag{14}$$

The correlation of the Lagrangian fluctuating fields defined in Eq.(3) can be represented as a sum over the subensembles S of the correlations calculated in each subensemble:

$$L_{ij}(z) = \int d\psi^0 d\mathbf{b}^0 P(\mathbf{b}^0,\, \psi^0) \left\langle w_i(\mathbf{0};\, 0) w_j \left[\mathbf{x}(z);\, z\right]\right\rangle^S \tag{15}$$

where

$$P(\mathbf{b}^0,\, \psi^0) = P(b_x^0)P(b_y^0)P(\psi^0) = (2\pi)^{-3/2}\,\Lambda^{-1} \exp\left[-\frac{\left(\psi^0\right)^2 + \left(\Lambda^{-1} b_x^0\right)^2 + \left(b_y^0\right)^2}{2}\right] \tag{16}$$

is the probability density of $\mathbf{b}$, ψ having the values $\mathbf{b}^0$, ψ^0 at $\mathbf{x} = 0$, and at the "moment" $z = 0$.

The equations for the decorrelation trajectory (DCT) are:

$$\frac{dx^S(z)}{dz} = K_m b_x^S \left[\mathbf{x}^S(z);\, z\right]$$

$$\frac{dy^S(z)}{dz} = K_m \Lambda b_y^S \left[\mathbf{x}^S(z);\, z\right] + \Lambda K_s x^S(z) \tag{17}$$

where $\mathbf{x}^S(0) = 0$. We have made the following approximation: *we have considered that the contribution of the subensemble-averaged shear term $\langle K_s x\,(z)\rangle^S$ in each subensemble S, are equal to its value along the deterministic trajectory, $K_s x^S(z)$.* The Lagrangian average of $x\,(z)$ in the subensemble S are then approximated by x^S.

We have used in our analysis a computer code based on the Runge-Kutta-Fehlberg $45(RKF45)$ [31], using the recipe depicted in [32]. A specified trajectory depends on the parameters that define the subensemble S: ψ^0, b_x^0 and b_y^0 and also on the magnetic Kubo number K_m, the shear parameter K_s and the anisotropy parameter Λ.

For the whole set of realizations, an average quantity A_{av} is obtained by summing up the contributions of each subensemble:

$$A_{av}(z) = \int_{-\infty}^{\infty} d\psi^0 \int_{-\infty}^{\infty} db_y^0 \int_{-\infty}^{\infty} db_x^0 P(\mathbf{b}^0,\ \psi^0) A^S\left(\mathbf{x}^S;\ z\right) \tag{18}$$

With this formula we can calculate the averages for any quantity of interest, such as: the fluctuating magnetic field components $(b_i)_{av}(z)$ $(i = x, y)$, the solutions of the equations of motion $y_{av}(z)$ and $x_{av}(z)$, the global velocities etc. In Figure 1 we represented the averaged solutions, averaged magnetic field fluctuations and the corresponding ratios x_{av}/y_{av} and $(b_x)_{av}/(b_y)_{av}$for the following values for the parameters: $\Lambda = 0.2$, $K_s = 2$ and three values for the magnetic Kubo number: $K_m = 0.5$ (blue solid line), 1 (red dotted line) 2 (black dashed line). We observe a negative minimum present in the ratio $(b_x)_{av}/(b_y)_{av}$ at $z = 3$ for $K_m = 2$. For $z \succeq 3$ practically the averaged magnetic field components become zero and the solutions become constant; they are different from zero in the poloidal direction and also in the radial one except for the relatively small magnetic turbulence given by $K_m = 0.5$ (blue solid line). In Figure 2we represented the same curves as in Figure 1 except for $\Lambda = 5$ see *e.g.* [3]. We can observe a very rapid oscillation for the ratio $(b_x)_{av}/(b_y)_{av}$ for $K_m = 2$ in the interval $z \in [0,\ 0.3]$. In Figure 3 we represented the same averaged quantities as in Figures (1) and (2) for $\Lambda = 1$ (*i.e.* for the isotropic case), all other parameters and colours being the same. In the isotropic case ($\Lambda = 1$) and a relatively weak magnetic turbulence ($K_m = 0.5$), from Figure 3 it is observed that $(b_x)_{av}(z)$ is very small when compared to $(b_y)_{av}(z)$.

We can observe the existence of a positive maximum for the ratio $(b_x)_{av}/(b_y)_{av}$ for $K_m = 2$ in the interval $z \in [0.5,\ 4]$ for $z = 1$. In Figure (4) we have represented the subensemble averaged coordinates $y_{av}(z)$ and $x_{av}(z)$, the trajectory $y_{av}(z) = f(x_{av}(z))$, the averaged flux $\psi_{av}(z)$ and the averaged components of the magnetic field. The parameters are: $\Lambda = 0.4$, $K_s = 2$ and three values for the magnetic Kubo number: $K_m = 0.5$ (blue solid line), 1 (red dotted line) 2 (black dashed line).

Different behaviors of $\psi_{av}(z)$ appear when the shear parameter is non-zero: for any value of Λ and K_m, $\psi_{av}(z)$ increases up to a maximum value and then decreases to zero; the maximum value depends on the parameters and can have the value: $[\psi_{av}(z)]^{\max} = 0.07$ if $\Lambda = 0.4,\ K_m = 2,\ K_s = 2$. The stochastic anisotropy do not plays any role in this behavior and only the value of the maximum increases if the anisotropy parameter increases. We can conclude that the trapping is influenced

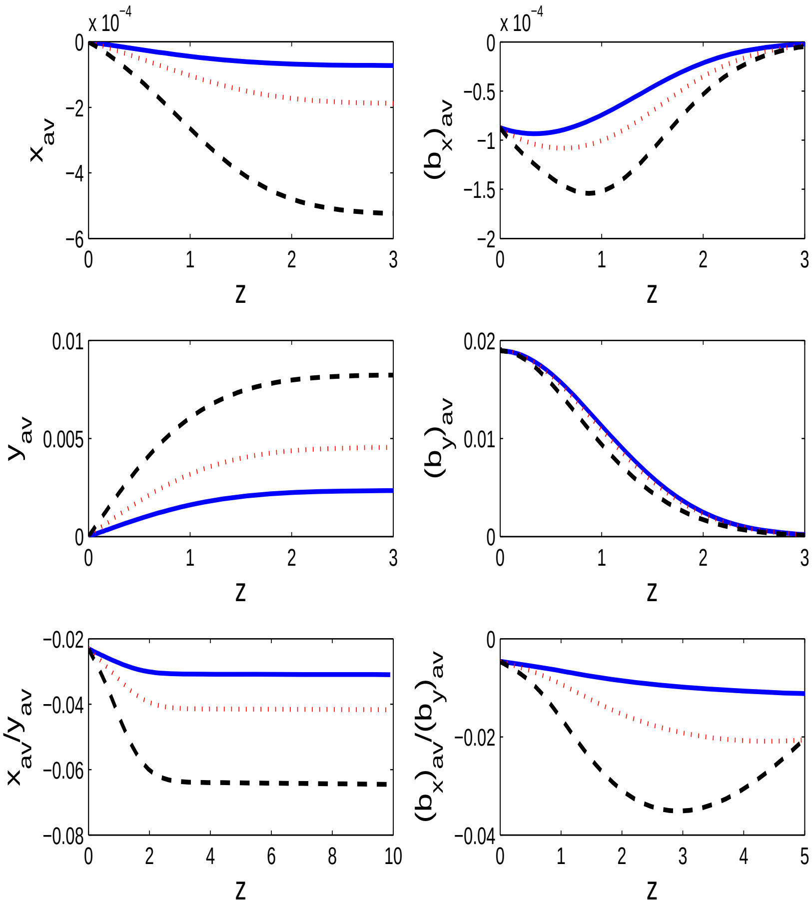

Fig. 1 – Averaged solutions, averaged magnetic field fluctuations and the corresponding ratios x_{av}/y_{av} and $(b_x)_{av}/(b_y)_{av}$for the following values for the parameters: $\Lambda = 0.2$, $K_s = 2$ and three values for the magnetic Kubo number: $K_m = 0.5$ (blue solid line), 1 (red dotted line) 2 (black dashed line).

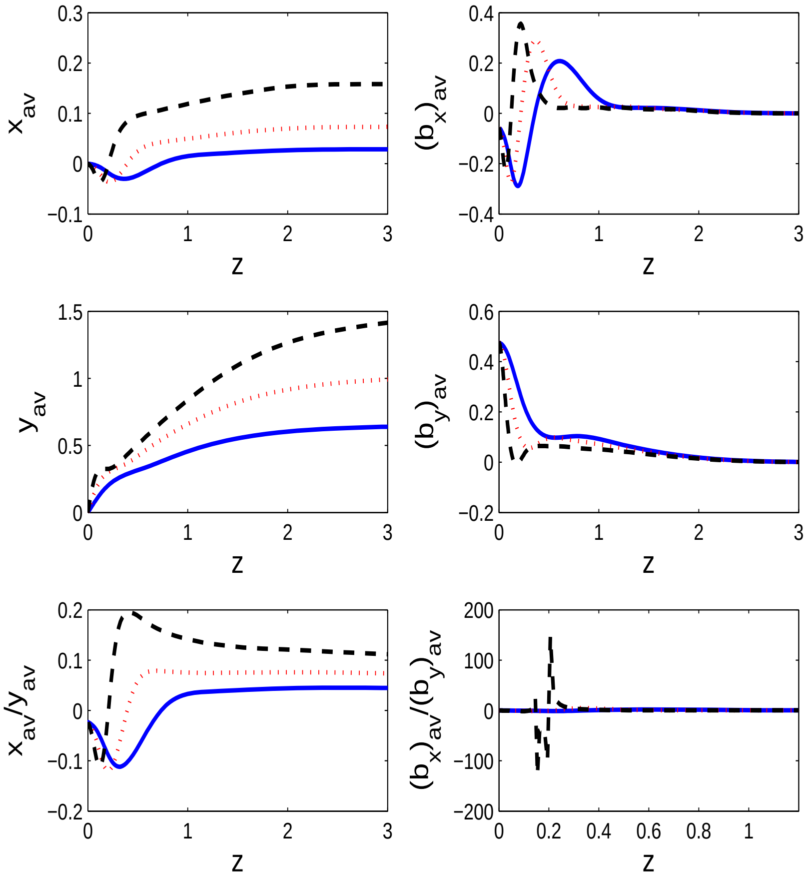

Fig. 2 – Same as in (1) except for $\Lambda = 5$.

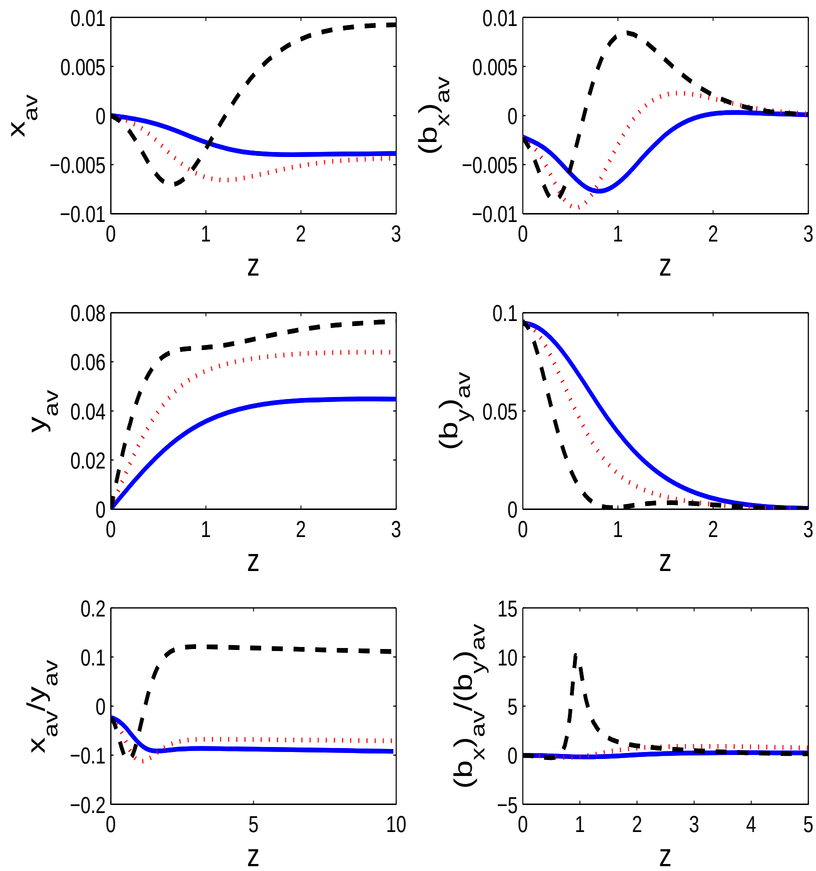

Fig. 3 – Same as in Figures 1 and 2 except for $\Lambda = 1$ (the isotropic case)

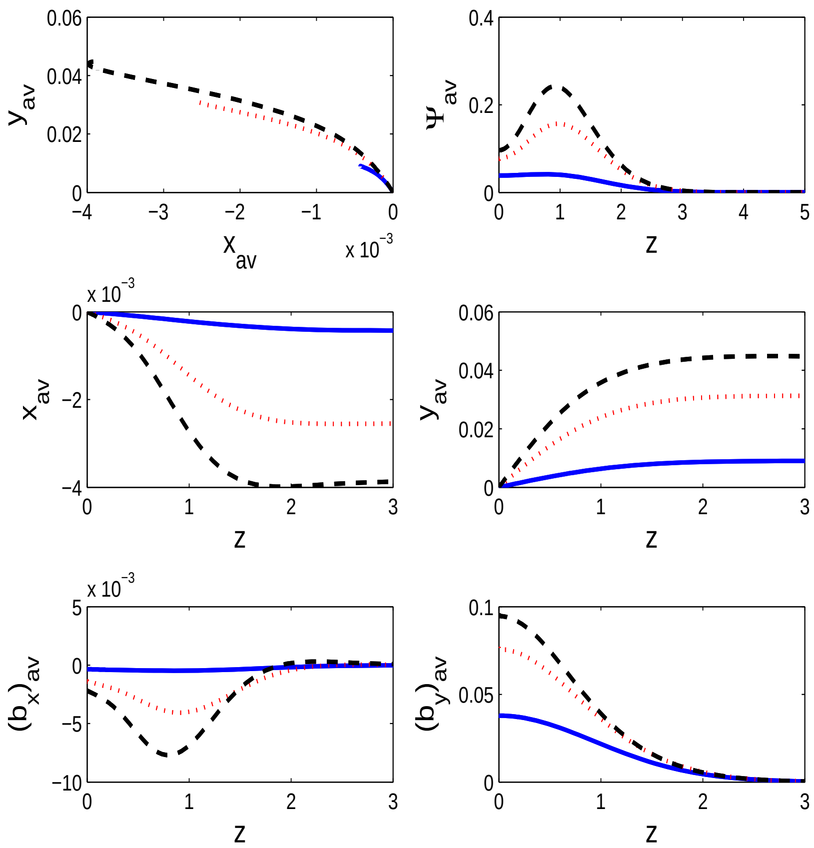

Fig. 4 – The representation of the global DCT averaged quantities for three values of the anisotropy: blue solid $\Lambda = 0.4$; red dotted $\Lambda = 0.8$; black dashed $\Lambda = 1$. The other parameters are $K_s = 2$ and $K_m = 0.5$.

by the stochastic anisotropy too.

The poloidal component $(b_y)_{av}(z)$ seems to have an exponential decay to zero taking only positive values. It is also clear from this figure that the shear parameter K_s does not influence the magnitude of $(b_x)_{av}(z)$ but changes its behavior: a small "oscillation" appears.

From this analysis we can conclude that for a quasi-limited region in the toroidal direction there exists a transport of the geometric points in the y direction and that means that a quasi-limited region exists where the positive ions, which in general follow the magnetic field lines, can be expelled from the plasma mainly in the poloidal direction.

4. CONCLUSIONS

We have studied in our paper the averaged quantities of interest for the transport of stochastic anisotropic sheared magnetic field lines in a tokamak device. In the latter the level of magnetic field fluctuations is of order $\beta \simeq 10^{-5} \div 10^{-4}$; this means that we have considered that are present not too strong magnetic fluctuations.

We have applied the decorrelation trajectory method, which was specifically designed in order to take the trapping effect into account [30]. It was applied in previous works to various plasma turbulence situations [1]-[7], or in particular to the diffusion of guiding centers in presence of a fluctuating electrostatic potential and a constant magnetic field [30] for the study of magnetic field line transport in anisotropic turbulence. The stochastic anisotropy of turbulence is introduced by taking different correlation lengths $\lambda_x \neq \lambda_y$ in a plane perpendicular to the main magnetic field B_0 and the third characteristic length is $\lambda_z \neq \lambda_x \neq \lambda_y$. The study has shown that a rich variety of transport behaviors can be found by varying the anisotropy parameter Λ in the plane perpendicular to the average magnetic field.

In most previous works the problem of the diffusion of magnetic lines is treated by starting either from Langevin equations (as in the present paper) or from a hybrid kinetic (or stochastic Liouville) equation and applying a strong approximation. The quantities analyzed are useful for the calculation of the diffusion coefficients that respect the quasilinear approximation, thus yielding the scaling $D_{xx}^{as}(K_m) \approx K_m^2$ and the Bohm scaling $D_{xx}^{as}(K_m) \approx K_m$ for large K_m. The latter scaling is known, however, to be incorrect. Indeed, the Corrsin approximation ignores the trapping effect which necessarily exists in a strongly turbulent plasma [33]-[34]. The calculation must be improved in order to take into account the trapping effect.

The following observations are important for the start of the treatment of the problem:

a) *the existence of the magnetic shear*

b) *the particular choice of the autocorrelation of the stochastic potential which* *is Gaussian* [of Ornstein-Uhlenbeck type] *in our case in comparison with the Lorentzian one chosen in* [30].

It was confirmed also (see *e.g.* [4]), that the DCT results are in agreement with those obtained by direct numerical simulations assuming an isotropic spectrum of electrostatic drift type turbulence that is Gaussian for small wavevectors and power-law k^{-3} for large wavevectors.

Acknowledgements. For MN, this work has been carried out within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training programme 2014-2018 under grant agreement No 633053. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

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