

SUPERSYMMETRIC COMPACTIFICATIONS OF M-THEORY WITH M2 BRANE POTENTIALS

ANDREI MICU

Department of Theoretical Physics, IFIN-HH,
Reactorului 30, 077125, Magurele/Ifov, Romania
E-mail: amicu@theory.nipne.ro

Received June 23, 2015

In this note we propose new flux configurations for M-theory compactifications which preserve supersymmetry and nevertheless give rise to potentials for space-filling M2 branes.

1. INTRODUCTION

Ever since their discovery, dualities represented a crucial ingredient for understanding string theory. Dualities are also a great tool for new discoveries in string theory. From this point of view, M-theory is in a privileged position. On one hand it makes an easy connection to string theory via the relation to type IIA strings and on the other hand it is one of the best solutions for studying the more complex F-theory.

F-theory has recently become an interesting topic as it has the potential to provide a unified picture which collects the low energy necessary ingredients (particle spectrum, gauge group, etc.) and also a consistent string background which also stabilizes moduli. On the other hand, F-theory lacks an independent self-consistent description and can only be studied through dualities. One of the cleanest description is via the duality with M-theory which states that F-theory compactified on a manifold X is dual to M-theory compactified on $X \times S^1$. From this point of view it is therefore very important to understand compactifications of M-theory to 3 dimensions, as these can be related to 4-dimensional compactifications of F-theory.

In this note we shall concentrate on a very specific aspect of M-theory compactifications which nevertheless opens the way to more general compactifications of M-theory. In string compactifications one often considers D-brane probes and studies the conditions under which such D-branes preserve any supersymmetry of the underlying compactification background. There are two types of conditions which have to be satisfied in order for these D-branes to be supersymmetric. One condition comes from the internal cycle on which such branes are wrapped which has to be volume minimizing in that particular class of cycles. The other condition is related to the spinors which exist on the internal manifold and the conditions which such spinors satisfy. This second condition is usually automatically satisfied in Calabi–Yau or related compactifications (such as manifolds with $SU(n)$ structure), while the first one

can easily be evaded by looking at space-time-filling D-branes which are points on the internal manifold. In this last class we can find only few examples like D2 branes in type IIA compactifications to 3d, D3 branes in type IIB compactifications to 4d or M2 branes in M-theory compactifications to 3d. Since this class of D-branes probes the entire internal manifold, such solutions can give us valuable information about the manifold on which we compactify. This information is even more valuable when we deal with compactifications on non-geometric backgrounds, like manifolds with $SU(3) \times SU(3)$ structure.

Let us analyze in more detail the D3 brane case. In flux compactifications of type IIB string theory to 4 dimensions on manifolds with $SU(3)$ structure, space-time-filling D3 branes are automatically supersymmetric and preserve half of the supersymmetries of the supergravity background. In compactifications on manifolds with $SU(3) \times SU(3)$ structure this is no longer the case [1]. It was shown that in such backgrounds D-branes are no longer BPS objects and they feel a non-trivial potential. The minima of these potentials are precisely at the places where the two spinors defining the $SU(3) \times SU(3)$ structure are parallel, ie points of $SU(3)$ structure. A similar situation should be encountered in M-theory compactifications to 3 dimensions and space-time-filling M2 branes. However, for M-theory, the most general supersymmetric compactifications to 3d which are known are compactifications with certain fluxes on manifolds with $SU(4)$ structure [2]. It is quite straightforward to check that space-time-filling M2 branes in such cases preserve all supersymmetries and therefore there is no potential these branes feel. The generalization we propose in this note aims to find new backgrounds which feature such M2-brane potentials.

We shall first study from scratch N=2 supersymmetric M-theory compactifications to 3 dimensions. It turns out that supersymmetry in M-theory compactifications is related to the existence of Majorana spinors on the internal manifold. Most of the studies so far imposed for simplicity that the internal spinors are Majorana-Weyl. On the contrary, we shall give up this additional constraint and only ask that Majorana spinors are non-vanishing, while the Majorana-Weyl components can vanish at certain points. In this way however, one can no longer talk about the global reduction of the structure group as this is intimately related to the existence of non-vanishing Majorana-Weyl spinors. We shall see that when placed in such backgrounds, M2 branes precisely match the type IIB description and in general will not preserve supersymmetry, while the points where these branes are supersymmetric are the $SU(4)$ -structure points. Finally we shall study under what conditions fully consistent solutions can be found.

2. M-THEORY COMPACTIFICATIONS TO 3 DIMENSIONS WITH N=2 SUPERSYMMETRY

In supersymmetric backgrounds, supersymmetry variations should identically vanish. For the M-theory case, the supersymmetry variation of the gravitino Ψ_M reads

$$\delta\Psi_M = \nabla_M \epsilon - \frac{1}{288} (\Gamma_M^{NPQR} - 8\delta_M^N \Gamma^{PQR}) G_{NPQR} \epsilon, \quad (1)$$

where G is a 4-form which denotes the field strength of the 3-form potential C which appears in the 11d supergravity spectrum. In the above, ϵ is the supersymmetry parameter which is given by a Majorana spinor in 11 dimensions. For compactifications to 3 dimensions we choose the total space to be a warped product space of the form

$$M_{11} = M_{1,2} \times_w K_8. \quad (2)$$

The supersymmetry parameter ϵ decomposes accordingly

$$\epsilon = \xi \otimes \eta, \quad (3)$$

where ξ and η are 3-dimensional and 8-dimensional spinors respectively. In 3 dimensions (Minkowsky signature) supersymmetries are counted by Majorana spinors. In order for the total spinor ϵ to be Majorana, the spinor η above has to be a Majorana spinor on the compact 8-dimensional space. This means that the number of supersymmetries which can be constructed in 3 dimensions is given by the number of non-vanishing Majorana spinors one can define on the internal manifold K_8 . The subtlety of 8d compactifications is that in 8 Euclidean dimensions one can define Majorana-Weyl spinors, but the spinors η need not be Majorana-Weyl. Further imposing a chirality constraint leads to the results obtained in [2]. On the other hand, requiring that the internal manifold admits only Majorana (rather than Majorana-Weyl) non-vanishing spinors leads to a generalization of the results above [3]. Each 8-dimensional Majorana spinor η decomposes into two Majorana-Weyl components $\eta = \eta_+ + \eta_-$, but these components are allowed to vanish at different points and if this happens, η_+ or η_- , which are also Majorana spinors, will not give by themselves independent supersymmetries in 3 dimensions but only η which is non-vanishing.

The existence of non-vanishing Majorana-Weyl spinors is associated to the reduction of the structure group of the frame bundle. For example, given one M-W spinor the structure group is reduced to $Spin(7)$ while for two spinors of the same chirality the structure group becomes $SU(4)$ and for two spinors of different chiralities one encounters a G_2 structure group. It is then clear that the existence of non-vanishing Majorana spinors does not lead in general to a global reduction of the structure group. For one Majorana spinor, at arbitrary points both M-W components are non-vanishing and the structure group looks like G_2 , while at points where one M-W component vanishes the structure group is only $Spin(7)$.

In the following we shall concentrate on the existence of two Majorana spinors in 8 dimensions η_1 and η_2 . In general one expects a $SU(3)$ structure group (when all the Majorana-Weyl components are non-vanishing) and $SU(4)$ or G_2 structure groups depending on the chiralities of the non-vanishing components.

2.1. M2 BRANES IN BACKGROUNDS WITH VARYING STRUCTURE GROUP

Recall that the original motivation for studying more general backgrounds in M-theory compactifications was finding configurations where space-time-filling M2 branes are not automatically supersymmetric. Here we shall briefly check that the backgrounds we propose are indeed suitable.

In general there is no reason that the supersymmetries preserved by a brane are the same as the supersymmetries preserved by the supergravity background. Indeed, a supergravity background which preserves supersymmetry is necessary a solution of the Killing spinor equation (1), while for the brane supersymmetry the brane chirality operator is the one which decides which are the preserved supersymmetries. For the case of M-theory compactifications to 3 dimensions, the brane chirality operator for space-time-filling branes is just the chirality operator on the internal manifold $\tilde{\gamma} = \gamma_9$ and the supersymmetries preserved by the brane are the ones which satisfy

$$\Gamma_{M_2}\eta \equiv \tilde{\gamma}\eta = \eta . \quad (4)$$

Therefore, supersymmetric brane configurations are the ones for which the internal spinor is chiral. It is then clear that if we start from the beginning with non-vanishing Majorana-Weyl spinors on the internal background, space-filling M2-branes are automatically preserving the same supersymmetries. On the other hand if we just start with Majorana spinors, the above equation is non-trivial and tells us that space-time-filling M2 branes are going to be supersymmetric only at those points where the Majorana spinor becomes chiral. For the case we study in this note, where we are interested in the existence of two Majorana spinors $\eta_{1,2}$ at either $SU(4)$ points – where they preserve all supersymmetries – or at the G_2 points where they preserve half of the supersymmetries.

2.2. 8D MANIFOLDS WITH VARYING STRUCTURE GROUP

Having established that the manifolds which are suitable for generating a potential for space-time-filling M2-branes are the ones which allow for non-vanishing Majorana spinors, in the following we shall try to give a characterization of such manifolds.

We shall adopt a 9-dimensional formalism where the chirality matrix in 8d, ie $\tilde{\gamma}$ is treated as an independent matrix γ_9 . One can then define the following spinor-bilinears

$$\begin{aligned}
(V_1)_m &= \eta_1^T \gamma_m \eta_1, & (V_2)_m &= \eta_2^T \gamma_m \eta_2, & (V_3)_m &= \eta_1^T \gamma_m \eta_2 = \eta_2^T \gamma_m \eta_1, \\
K_{mn} &= \eta_1^T \gamma_{mn} \eta_2 = -\eta_2^T \gamma_{mn} \eta_1, & \Psi_{mnp} &= \eta_1^T \gamma_{mnp} \eta_2 = -\eta_2^T \gamma_{mnp} \eta_1, \\
(\Phi_1)_{mnpq} &= \eta_1^T \gamma_{mnpq} \eta_1, & (\Phi_2)_{mnpq} &= \eta_2^T \gamma_{mnpq} \eta_2, & (\Phi_3)_{mnpq} &= \eta_1^T \gamma_{mnpq} \eta_2. \quad (5)
\end{aligned}$$

These forms are not independent, but satisfy certain relations (Fierz identities) which can be derived from the completeness relations for the gamma matrices. The vectors $V_{1,2,3}$ hold the key information about the number and chirality of the non-vanishing M-W components of the spinors. In general all the vectors are independent and non-vanishing and this is the signal for the $SU(3)$ case. There exist the possibility that $V_1 = V_2$ and $V_3 = 0$ which corresponds to the $SU(4)$ case and also $V_1 = -V_2$ and $V_3 \neq 0$ which corresponds to the G_2 case. These possibilities can be conveniently described by the parameter α defined as the scalar product of the vectors V_1 and V_2

$$\alpha = (V_1)_m (V_2)^m. \quad (6)$$

Since the vectors V_1 and V_2 have unit norm, the absolute value of the parameter α is less than 1. At arbitrary points on the manifold we expect that α has some value in the interval $(-1 \dots 1)$ and only at specific points reaches the values $\alpha = \pm 1$. At $\alpha = +1$, the vectors V_1 and V_2 are parallel and we are at a $SU(4)$ point, while at $\alpha = -1$ the two vectors are anti-parallel and we are at a G_2 point.

In order to have a proper understanding of the manifolds we deal with it is important to find a parametrization of the forms defined above from spinor bilinears such that the Fierz relations become more intuitive. For this purpose we define

$$V_{\pm} = V_1 \pm V_2, \quad (7)$$

which are orthogonal to one another and also orthogonal to V_3 . Then we decompose the spinor bilinears in parts along and orthogonal to the vectors and impose the Fierz identities. We find

$$K = J + \frac{1}{1-\alpha} V_- \wedge V_3, \quad (8)$$

$$\Psi = \varphi + \frac{1}{1+\alpha} J \wedge V_+ + \frac{1}{2(1-\alpha)} V_+ \wedge V_- \wedge V_3, \quad (9)$$

$$\Phi_+ = -\frac{2}{1+\alpha} J \wedge J - \frac{2}{1+\alpha} \rho \wedge V_+ - \frac{2}{1-\alpha} J \wedge V_- \wedge V_3, \quad (10)$$

$$\Phi_- = \frac{4}{1-\alpha} \varphi \wedge V_3 + \frac{2}{1-\alpha} \rho \wedge V_- + \frac{2}{1+\alpha} J \wedge V_+ \wedge V_3, \quad (11)$$

$$\Phi_3 = -\frac{1}{1-\alpha} \varphi \wedge V_- + \frac{2}{1-\alpha} \rho \wedge V_3 - \frac{1}{2(1+\alpha)} J \wedge V_+ \wedge V_-. \quad (12)$$

In the above, J can be understood as an almost complex structure ($J \cdot J \sim -\mathbf{1}$) while ϕ and $\rho \equiv J \cdot \phi$ are real and imaginary parts of a $(3, 0)$ form with respect to J . These forms live in a space which is orthogonal to the vectors V_{\pm} and V_3 and define a α -dependent $SU(3)$ structure.

So far we only discussed general aspects of manifolds which admit a pair of non-vanishing Majorana spinors. We now have to see what conditions supersymmetry imposes on these manifolds. Inserting the decomposition (3) and the corresponding splitting of gamma matrices into the susy variation (1) we find the following equations for the external and internal gravitino respectively [4]

$$Q\eta = 0, \quad D_{\alpha}\eta = \nabla_{\alpha}\eta + A_{\alpha}\eta = 0, \quad (13)$$

where we defined

$$A_{\alpha} = \lambda\gamma_{\alpha}\gamma_9 + \frac{1}{24}F_{\alpha\beta\gamma\delta}\gamma^{\beta\gamma\delta} + \frac{1}{4}\tilde{f}_{\beta}\gamma_{\alpha}^{\beta}\gamma_9, \quad (14)$$

$$Q = -\lambda\gamma_9 + \frac{1}{2}\partial_{\alpha}\Delta\gamma^{\alpha} - \frac{1}{288}F_{\alpha\beta\gamma\delta}\gamma^{\alpha\beta\gamma\delta} - \frac{1}{6}\tilde{f}_{\alpha}\gamma^{\alpha}\gamma_9. \quad (15)$$

In the above, the quantities F and f are forms on the internal manifold and denote the decomposition of the 11-dimensional flux G as

$$G = F + f \wedge Vol_3. \quad (16)$$

These equations have to be satisfied for a pair of 8-dimensional Majorana spinors $\eta_{1,2}$. Furthermore, in order to obtain more intuitive information about the manifold we have to project the spinor equations on a basis for 8-dimensional spinors and use the definitions of the spinor bilinears. Finally we can introduce the parametrization (8) in order to find what relations should be satisfied by the $SU(3)$ structure forms J , φ and ρ . Doing this by brute force and allowing all fluxes to be non-vanishing will lead to very complex equations which will be difficult to deal with. We shall rather choose another strategy. Recall that the novelty of these manifolds is that they allow different structure groups at different points. For this to be the case it is necessary that the parameter α introduced before has a non-trivial variation. Since the norm of the vectors V_{\pm} and V_3 as well for the $SU(3)$ structure forms J and φ depend on α , it is quite straightforward to identify the conditions under which α has a non-vanishing derivative. We find

$$\nabla_m \alpha = -\frac{1}{3}\rho_{pqr}F_m{}^{pqr}. \quad (17)$$

Therefore the fluxes which are responsible for the variation of α are fluxes which lift to $(3, 1)$ or $(4, 0)$ and which were forbidden in the case of static $SU(4)$ structures considered in [2]. In the following we shall concentrate on finding solutions which posses such a behavior.

3. SOLUTIONS WITH VARYING STRUCTURE GROUP

3.1. BASIC FLUX

In this section we shall study the influence of the fluxes which are responsible for the variation of the parameter α on the compactification geometry. We therefore make the following ansatz for the flux F

$$F = h \wedge \rho + g \wedge \varphi, \quad (18)$$

where h and g are one-forms on the internal manifold. This expression should be introduced in the reduced supersymmetry equations (13). Note that the Q equations will be simply algebraic constraints on the flux components, while the second equation will give information about the derivatives of various forms defined on the manifold. After tedious, but completely straightforward calculations one finds that the algebraic relations force the external 3-dimensional space to be Minkowsky and fix most of the flux components in terms of just few unknowns. In particular all the flux components orthogonal to the vectors vanish while the others are given by*

$$d\Delta \cdot V_- = \frac{1-\alpha}{3} h_-, \quad d\Delta \cdot V_3 = \frac{1-\alpha}{3} h_3, \quad g_3 = -\frac{1}{2} h_-, \quad g_- = 2h_3, \quad (19)$$

where the subscript 3 or $-$ denote the projection on the corresponding vectors, while all the other components vanish. Further, the derivative relations yield

$$d\alpha = -(1+\alpha)h_-V_- - 4(1+\alpha)h_3V_3, \quad (20)$$

$$dV_+ = \frac{1}{2}h_-V_+ \wedge V_- + 2h_3V_+ \wedge V_3, \quad dV_- = 2h_3V_- \wedge V_3, \quad dV_3 = -\frac{1}{2}h_-V_- \wedge V_3, \\ dJ = -h_-J \wedge V_- - 4h_3J \wedge V_3, \quad (21)$$

$$d\varphi = \frac{2}{1-\alpha}\rho \wedge (h_3V_- - h_-V_3) + \frac{1-5\alpha}{1-\alpha}\varphi \wedge \left(\frac{1}{4}h_-V_- + h_3V_3\right). \quad (22)$$

It is worth mentioning that the variations take place only along the vectors V_3 and V_- while when restricting them to the 6d subspace orthogonal to the vectors, dJ and $d\varphi$ vanish. This implies that the geometry in this case is a Calabi–Yau manifold fibred over a 2d space spanned by the vectors V_- and V_3 .

So far we solved the supersymmetry equations which dictate the form of the fluxes and the geometry of the internal manifold as well as the warp factor. We shall also check what further conditions are imposed by the equations of motion. It turns out however that the we have too few unknown functions at our disposal and one can not choose these functions to solve simultaneously the Bianchi identities and equations of motion.

*Here we adopted a larger definition for the word *flux*, which also includes the warp factor Δ and its derivatives

A similar result can be obtained for a slightly more general case when one allows an exterior non-vanishing flux f . The only notable difference is that the supersymmetry equations require now that the 6d subspace orthogonal to the vectors is now a manifold with $SU(3)$ structure where the intrinsic torsion classes are given in terms of the restriction of f orthogonal to the vectors. Again, one finds one equation too much when trying to impose the Bianchi identities and equations of motion.

3.2. INTEGRABILITY CONDITIONS

In this last part we want to study the reason why we fell short of finding proper solutions with just the special fluxes which induce variations of α . It is well known that in general solving the supersymmetry equations along with few other constraints (like Bianchi identities) assures that all other equations of motion (like Einstein equations) are also satisfied. From this light it may seem strange that once we were able to solve the Bianchi identities we were still not able to find a solution to the equations of motion. However, a precise statement regarding the equations of motion can only be made on a case by case basis. Such relations between supersymmetry and equations of motion comes from the integrability conditions of the supersymmetry equations.

Imposing that the supersymmetry variations (1) vanish gives a definition for the covariant derivative of the spinor ϵ . One condition for this to be a consistent definition is that

$$[\nabla_M, \nabla_N]\epsilon = \frac{1}{4}R_{MNPQ}\Gamma^{PQ}\epsilon \quad (23)$$

By some tedious gamma-matrix manipulations one can put this into the form [5]

$$\begin{aligned} A_{MN}\Gamma^N\epsilon - \frac{1}{36}B_{PQR}(\Gamma_M{}^{PQR} - 6\delta_M^P\Gamma^{QR})\epsilon \\ - \frac{1}{6!}C_{PQRST}(\Gamma_M{}^{PQRST} - 10\delta_M^P\Gamma^{QRST})\epsilon = 0, \end{aligned} \quad (24)$$

where A_{MN} is a symmetric tensor whose components are the 11-dimensional Einstein equations, B_{MNP} are the components of a 3-form which is given by the equation of motion for the flux G while C_{MNPQR} is nothing but the components of dG . We see that in this integrability condition all relevant bosonic equations of motion are present. Moreover, a consistent background where all the equations of motion are satisfied this condition is automatically satisfied. The purpose now would be to reverse this reasoning and ask what are the minimal conditions which have to be satisfied by some supersymmetric background such that it is also a solution for the equations of motion.

The relation above is completely general and we should further particularize it for the case of compactifications to 3 dimensions with N=2 supersymmetry. It is easy to see that in this case, the tensors above are non-vanishing only for certain

combinations of indices. We shall further make the simplification that the Bianchi identities are satisfied which means that the tensor C vanishes. For the tensors A and B we find that they can have either only 3d or 8d indices ie $A_{\mu\nu}$ and $B_{\mu\nu\rho}$ or A_{mn} and B_{mnp} . Imposing that the 3-dimensional space is maximally symmetric we infer that $A_{\mu\nu}$ is proportional to the 3d metric, while $B_{\mu\nu\rho}$ should be proportional to the 3d ϵ -tensor

$$A_{\mu\nu} = ag_{\mu\nu} , \quad (25)$$

$$B_{\mu\nu\rho} = b\epsilon_{\mu\nu\rho} . \quad (26)$$

We should emphasize here that the tensors A and B have a well defined form and our purpose here is not to determine this form, but rather analyze under what conditions these tensors vanish so that the equations of motion are satisfied. With this parametrization and further making the (3,8) split in the gamma-matrices and spinors, the integrability condition (24) can be written as

$$a\gamma_9\eta - \frac{1}{36}e^{-\Delta}B_{mnp}\gamma_9{}^{mnp}\eta + \frac{1}{3}e^{-\Delta}b\eta = 0, \quad (27)$$

$$A_{mn}\gamma^n\eta + \frac{1}{6}e^{-\Delta}b\gamma_9m\eta - \frac{1}{36}e^{-\Delta}B_{npq}\gamma_m{}^{npq}\eta + \frac{1}{6}e^{-\Delta}B_{mnp}\gamma^{np}\eta = 0. \quad (28)$$

These conditions can be analyzed in the same way as we did for the supersymmetry equations by projecting them on a complete set of spinors. We shall not present all equations here, but only briefly explain the results.

From the first equation one easily obtains

$$a = -\frac{(\theta \cdot V_+)}{3}e^{-\Delta}b, \quad (29)$$

which tells us that the equations given by $a = 0$ and $b = 0$ are equivalent. From the second equation, given the fact that A_{mn} is a symmetric tensor, we find

$$A_{mn} = \frac{1}{12}e^{-\Delta}(\theta \cdot V_+)bg_{mn}, \quad (30)$$

which means that once we set $b = 0$ we also have that $A_{mn} = 0$ and together with $a = 0$ implies that all the Einstein equations are satisfied. A similar analysis for the components B_{mnp} is more complicated. It can be shown however that if we set $b = 0$ and together with it also $a = 0$ and $A_{mn} = 0$, the remaining equations also require that $B_{mnp} = 0$. For this one needs to implement the $SU(3)$ parametrization and expand also the tensor B in pieces along and orthogonal to the vectors and use the $SU(3)$ relations. The conclusion of this analysis can be stated as follows:

For M-theory compactifications to 3 dimensions with $N = 2$ supersymmetry, in the presence of a flux G which satisfies the Bianchi identity $dG = 0$, all the bosonic

field equations are fulfilled once a scalar condition is satisfied. This condition can be either the external Einstein equation ($a = 0$) or the external part of the flux equation of motion ($b = 0$).

This is in good agreement with the results obtained previously where the solution we obtained by imposing the Bianchi identity was not compatible with this additional constraint. However, it looks quite reasonable that some mild, well-chosen generalization of the fluxes discussed may feature enough freedom in order to be able to also solve this additional constraint and therefore find a fully consistent solution with varying α .

REFERENCES

1. L. Martucci, JHEP **0606**, 033 (2006).
2. K. Becker, M. Becker, Nucl. Phys. B **477**, 155 (1996).
3. D. Tsimpis, JHEP **0604**, 027 (2006).
4. D. Martelli, J. Sparks, Phys. Rev. D **68**, 085014 (2003).
5. J. P. Gauntlett, S. Pakis, JHEP **0304**, 039 (2003).
6. C. Condeescu, A. Micu, E. Palti, JHEP **1404**, 026 (2014).