NONLINEAR CONTROL OF CHAOTIC CIRCUITS

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The paper investigates a specific type of nonlinear dynamical systems represented by electronic circuits with nonlinear elements, known as Chua circuits. Despite they are described by simple equations, with only one nonlinear function, they have a complex behavior, very rich in dynamical states and with interesting transitions from chaos to regular dynamics. The interest will be given to the problem of controlling the chaotic behavior using a technique specific for Hamiltonian systems. The main results which will be reported will concern the possibility of attaching a Lagrangian function and of transforming the Chua system into a variational one. The optimization of the dynamics will be achieved through a quadratic control term.

Key words: chaos, Chua circuits, Jacobi last multiplier technique, variational system.

1. INTRODUCTION

The control theory [1], [2] is a quite recent but very developed branch of nonlinear sciences, dealing with the study of input-feedback processes which can influence the behavior of the dynamical systems. It becomes extremely important in many fields from sciences and engineering, with a lot of specific procedures proposed along the time [3]. The control of nonlinear dynamical systems involves important concepts as stability, optimality or uncertainty. A special connection of control and optimization has been established with the concept of chaos, a huge number of papers being published in the last years on the control and synchronization of the chaotic systems. Starting from a system with non-regular dynamics, we can "optimize" its evolution, or "control" the original system. From historical perspective, the first article on chaos control was published in 1989 by Hubler [4]. He proved that it is not possible to control chaos with negative feedback technique, but with "weak chaos". In 1990 Pecora and Carroll proposed the idea of chaos synchronization [5]. In the last years, many techniques for chaos control and synchronization have been developed: *adaptive control, intelligent control based on genetic algorithms, time delay feedback approach, stochastic control*, etc. [6].

The chaos control appears as an important topic in all the branches dealing with evolutionary processes: chemistry, biology, medicine, genetics, economics, sociology, etc. [7]. From Physics perspective, optimization and control deal with systems from mechanics, optics, plasma physics, but with electric and electronic circuits, too. This last perspective represents the main approach which will be used in this paper. The paper is organized as follows: the next section will point out the simplest differential systems known as chaos generators. It will be a monographic approach with basic mathematical results on the main chaotic systems expressed through first order differential equations. The third section of the paper will deal with control techniques, presenting a procedure proposed for Hamiltonian systems that, in the fourth section, will be extended towards more pragmatic non-variational systems appearing in electronics. A Chua circuit with nonlinear terms expressed through a non yet investigated nonlinearity will be considered and numerical investigations will clearly show the effects of quadratic control terms in passing from chaos to regular dynamics. Some concluding remarks will end the paper.

2. CHAOS AND ITS CONTROL. EXAMPLES

2.1. WHAT CONTROL AND OPTIMIZATION MEAN?

The scientific concept of chaos becomes from day to day one of the most used and studied concept, with a lot of important applications. It appears to be not a paradigmatic phenomenon generated by the nonlinear interactions inside evolutionary processes, but a short-term predictable, controllable and useful thing. The control theory is a very developed branch of nonlinear sciences, extremely important in many fields and with a lot of specific procedures proposed during the time. It supposes that, starting from a system with non-regular dynamics, we can “optimize” its evolution, or to “control” the original system. In the case of chaos appearing into nonlinear dynamical systems, the control means that the chaotic dynamics is weakened or eliminated by appropriate controls. The modern engineering applications put quite recently the problem not of diminishing the chaos, but creating or amplifying it. This desire leads to the development of a new technique known as *chaos anti-control*, that is chaos is “created, maintained, or enhanced when it is healthy and useful” [8]. We gave before a classification of control techniques seen from the perspective of the control theory. Let us mention that from the perspective of the mathematical technique which is used, both the control and the anti-control of the chaos generated by nonlinear systems can be accomplished via some methods such as parameter perturbation or bifurcation monitoring. We shall present in the next section the most simple and well known nonlinear models, represented as systems of first order differential equations, whose chaotic behavior are straightly related with their bifurcation properties.

2.2. FIRST ORDER DIFFERENTIAL SYSTEMS AS CHAOS GENERATORS

From the perspective of nonlinear dynamics, the study of chaotic systems imposes to find evolutionary equations which describe the chaotic behavior. Many such equations or systems of equations were proposed and the main question becomes what is the simplest mathematical form of the equations generating chaos. For many years, people believed that Lorenz system, proposed in 1963, is the only one paradigmatic example of a three-dimensional system of first order equations with simple quadratic terms which admits a chaotic attractor. The system has the general form:

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= bx - y - xz \\ \dot{z} &= -cz + xy \end{aligned} \tag{1}$$

It generates chaos when the parameters get the values $a = 10,\ b = 28,\ c = 8/3$.

Later on, in '90s, other examples of first order differential systems with chaotic attractors started to be noted: Rossler, Chen, Lu, Lu-Chen, Ruchlidge, Sprott, Genesio, Chua systems, all of them representing various versions of first order differential systems in three dimensions. Maybe the most interesting is the Chen system proposed in 1999 [9]:

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= (b-a)x + by - xz \\ \dot{z} &= -cz + xy \end{aligned} \tag{2}$$

Chen and Ueta found that this system presents an interesting chaotic attractor for $a = 35,\ b = 28,\ c = 3$. Despite it is quite similar with Lorenz, people realized that the two systems are not topologically equivalent, in the sense that there are not nonsingular coordinate transformations converting one system to the other. This fact raised the problem of putting order in all these chaos generators, in the sense of finding criteria allowing to identify for example the systems which are equivalent with Lorenz, that is belonging to the family of "proto-Lorenz systems" as they where called in [10]. A first answer to this problem was given by Vanecek and Celikovsky [11], who proposed a classification for the first order differential systems of Lorenz type. They considered a family of the general form:

$$\begin{aligned} \dot{x} &= a_1^1 x + a_2^1 y \\ \dot{y} &= a_1^2 x + a_2^2 y + b_{13}^2 xz \\ \dot{z} &= a_3^3 z + b_{12}^3 xy \end{aligned} \tag{3}$$

In a more general and compact form, the previous system can be rewritten if we adopt the notation: $x \equiv x^1$; $y \equiv x^2$; $z \equiv x^3$ and we use Einstein's summation convention:

$$\frac{dx^i}{dt} = a^i_j x^j + b^i_{jk} x^j x^k; \;\; i,j,k = 1,2,3 \tag{4}$$

The nice characterization proposed in [11] takes into account the linear part of the previous system, that is the matrix $A = (a^i_j)$. More precisely, while for the classical Lorenz system $a^1_2 a^2_1 > 0$, Chen system satisfies a dual condition $a^1_2 a^2_1 < 0$. From this perspectives, the systems (4) can be divided in Lorenz-type and in dual to the Lorenz system. The systems with $a^1_2 a^2_1 = 0$ were also studied, as a bridge between Chen and Lorenz systems [12]. So, three different types of chaotic systems can be considered from this perspective: Lorenz, Chen and Lu.

Coming back to the classical forms 1 or 2 of the three dimensional systems generation chaos, another interesting discussion is related to the parameters appearing into the system. Taking into account the values of the three parameters, a, b, c, which correspond to chaotic attractors, a unified chaotic system of Lorenz type has been proposed [13]:

$$\begin{aligned} \dot{x} &= (25a+10)(y-x) \\ \dot{y} &= (28-35a)x-(29a-1)y-xz \\ \dot{z} &= -\frac{8+a}{3}z+xy \end{aligned} \tag{5}$$

Different values of the parameter a generates the chaotic attractors for all mentioned systems: $a = 0$ corresponds to Lorenz, $a = 0.8$ to Lu system, and $a = 1$ leads to Chen attractor.

Despite one can think that the above classifications put order among the first order differential systems generating chaos, other systems with very nice chaotic behavior, not belonging to any of these classes were pointed out. Genesio system, one of such examples, has the form:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= -cx-by-az+x^2 \end{aligned} \tag{6}$$

It exhibits a chaotic attractor at the parameter values $a = 1.2$; $b = 2.92$; $c = 6$ [14]. Another important example which does not belong to Lorenz family is Rossler system. If all the previous systems depend on three parameters, a, b, c, the Rossler system contains only two. Its form is:

$$\begin{aligned} \dot{x} &= -y-z \\ \dot{y} &= x+ay \\ \dot{z} &= b+z(x-y) \end{aligned} \tag{7}$$

The new question arising from this example is related to the minimal number of parameters that should appear in a system in order to generate chaos. A nice study on this question is presented in [15].

Let us end these notes on the three dimensional differential systems which present chaotic attractors, with another example that will be hardly investigated in the forth section of the paper. It is known as Chua system and it also does not fit any of the previous classes of chaotic systems. The explicit form of Chua system is [16]:

$$\begin{aligned} \dot{x} &= \alpha(y - f(x)) \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y. \end{aligned} \tag{8}$$

The system contains two parameters only and the nonlinearity is given by a simple function $f(x)$. The chaotic attractor corresponds to the parameters: Other interesting examples of nonlinear functions were considered and a new approach will be proposed here from exactly this perspective.

The importance of Chua system relies in the fact that it was the first chaotic system put into a "material form" through an electric circuit, showing that chaos is not "a pathological phenomenon that can exist only in mathematical abstractions" [17]. It was the moment when chaos became more than interesting for electrical engineers and for other specialists working in computer simulations or in nonlinear circuit theory.

3. CHAOS CONTROL FOR HAMILTONIAN SYSTEMS

Let us consider an integrable system described by the Hamiltonian H_0 and a chaotic system described by the "perturbed" Hamiltonian of the form:

$$H' = H_0 + V_1 \tag{9}$$

The problem of controlling the chaos is the following: to find a control term V_2 such that the dynamics of the "controlled" Hamiltonian $H = H_0 + V_1 + V_2$ becomes more regular than of the perturbed system. We will follow an algorithm proposed in [18]. It allows finding the control term as a series whose items can be explicitly and easily computed by recursion. Let $\mathcal{A}$ be the algebra of the real functions defined on the phase space. For $V \in \mathcal{A}$, the time evolution following the flow of the time independent is given by the equation:

$$\frac{dV}{dt} = \{H, V\} \equiv \{H\}V \tag{10}$$

We introduced the notation:

$$\{H\}* \equiv \{H, *\} \tag{11}$$

The equation (10) is formally solved as:

$$V(t) = e^{t\{H\}}V(0) \tag{12}$$

The vector space is the set of constants of motion, that is:

$$Ker\{H\} = \{\mathbf{C} \in \mathcal{A}, \{H\}\mathbf{C} = 0\}; Ker\{H\} \subset \mathcal{A} \tag{13}$$

Three new operators are defined in connection with the operator $\{H_0\}$ attached to the initial integrable Hamiltonian H_0:

i) The pseudo-inverse operator Γ of $\{H_0\}$, such that $\{H_0\}^2\Gamma = \{H_0\}$, which is equivalent with:

$$\{H_0, \{H_0, \Gamma V\}\} = \{H_0, V\}$$

ii) The non-resonant part $\mathcal{N}$ of $\{H_0\}$ with the action on the algebra $\mathcal{A}$ of the form:

$$\mathcal{N}V = \{H_0, \mathcal{N}\mathcal{V}\} = \{\mathcal{H}_0, V\}, \forall V \in \mathcal{A}$$

iii) The resonant part $\mathcal{R}$ of H_0 such that $\mathcal{R} = 1 - \mathcal{N}$, which is equivalent with:

$$\{H, \mathcal{R}V\} = 0, \forall V \in \mathcal{A}$$

Using this operators, it has been shown that the control term $V_2(V_1)$, depending on the perturbation V_1, can be taken as [18]:

$$V_2(V_1) = \sum_{n-1}^{\infty} \frac{(-1)^n}{(n+1)!} \{\Gamma V_1\}^n (n\mathcal{R} + 1) V_1 \tag{14}$$

This control technique has been applied in [19] for variational systems as Yang-Mills mechanical model or Henon-Heiles system and clear improvements of the chaotic behavior has been achieved by choosing the control potential V_2 in a polynomial form. The technique will be extended in the next section of the paper for controlling the chaos generated into a system which is not a variational one [20].

4. GENERALIZED CHUA CHAOTIC SYSTEMS

4.1. THE SYSTEM OF EQUATIONS FOR CHUA CIRCUIT

Chua circuit is a specific type of electronic circuit which generates stochastic signals, with important applications in communication technologies, biology, neurosciences, and in other fields. It consists of two capacitors C_1 and C_2 and one inductance L, placed on three parallel branches of the circuit (Figure 1).

The capacitors are related through a resistor with the electric resistance R or, equivalently, with the conductance $G = 1/R$. The key element of the circuit is the nonlinear resistor R_X which represents in fact a special diode with a nonlinear intensity-voltage characteristic. In his initial paper [16], Leon Chua approximated

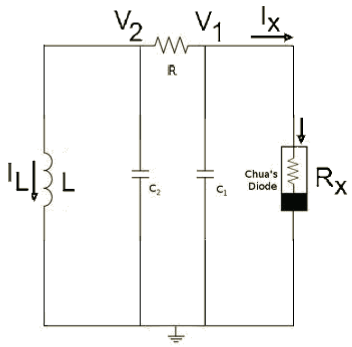

Fig. 1 – Chua circuit

this characteristic through three linear sectors, one passing through origin and having the slope A, the other two connecting with the central segment in the breaking points ± 1 and having a slope characterized by B. The concrete expression of the diode's conductivity as function of the potential V_1 has been considered as:

$$g(V_1) = AV_1 + \frac{1}{2}(A-B)[|V_1+1| - |V_1-1|] \tag{15}$$

The Kirchoff laws for this circuit leads to the equations:

$$\begin{aligned} \frac{dV_1}{dt} &= \frac{1}{RC_1}(V_2 - V_1) - \frac{1}{C_1} g(V_1) \\ \frac{dV_2}{dt} &= \frac{1}{RC_2}(V_1 - V_2) + \frac{1}{C_2} I_L \\ \frac{dI_L}{dt} &= -V_L \equiv V_2 \end{aligned} \tag{16}$$

By adopting adequate notations, the previous system can be put in the simple mathematical form (8). This system has became an important example of nonlinear system attached to an electronic circuit and many studies have been done for various choices of the nonlinear term $f(x)$. We shall propose ourselves a study of the system for a nonlinearity taken as a combination of tangent hyperbolic functions.

4.2. CHUA AS A VARIATIONAL SYSTEM

The main idea of this paper is to study the improvement of the chaotic behavior of Chua circuit using the technique for variational systems mentioned in the previous section. To do that we have to find a way to transform Chua system into a variational one, by attaching it a Lagrangian function. We shall do it by using the very old but somehow ignored *Jacobi last multilayer* technique [21]. This technique makes a direct link between solutions and Lie symmetries of differential equations and allows, it is true for even order differential equations, to attach a whole family of Lagrangians to the considered equation. Let us consider a second order differential equation defined in the $(1+1)-$dimensional space $\{q,t\}$ having the form:

$$\ddot{q} = f(t,q,\dot{q}). \tag{17}$$

If this equation can be seen as the Euler-Lagrange equation attached to a Lagrangian function $L(t,q,\dot{q})$, then there is a Jacobi multiplyer related to the Lagrangian by the relation:

$$M(t,q,\dot{q}) = \frac{\partial^2 L}{\partial \dot{q}^2} \tag{18}$$

The multiplyer satisfies the equation [22]:

$$\frac{d}{dt}(\log M) + \frac{\partial f}{\partial \dot{q}} = 0 \tag{19}$$

Conversely, if we find solutions of (19), one can attach Lagrangians to the equation (17), by a double integration of (18). They will have the quite evident form [22]:

$$L = \iint M(t,q,\dot{q})d\dot{q}d\dot{q} + L'(t,q)\dot{q} + L''(t,q)$$

We are trying now to extend the above technique to Chua system (8). As it is a first order differential system, we will look for Lagrangians linear in velocities:

$$L = F(x,y,z,t)\dot{x} + G(x,y,z,t)\dot{y} + H(x,y,z,t)\dot{z} - V(x,y,z,t) \tag{20}$$

We have to find appropriate expressions for the unknown functions respectively denoted by $F(x,y,z), G(x,y,z)$, $H(x,y,z)$ and for the potential $V(x,y,z)$. The Euler-Lagrange equations corresponding to (20) for the three independent variables x, y, and respectively z are, respectively:

$$(G_x - F_y)\dot{y} + (H_x - F_z)\dot{z} = V_x + F_t \tag{21}$$

$$(F_y - G_x)\dot{x} + (H_y - G_z)\dot{z} = V_y + G_t \tag{22}$$

$$(F_z - H_x)\dot{x} + (G_z - H_y)\dot{y} = V_z + H_t \tag{23}$$

Supplementary equations are generated as constraints, by taking into account the equality of the mixte second order derivatives of $V(x,y,z)$. We get a system of six equations which can be solved in specific cases only. We shall consider the case when:

$$F_z = -F_y = G_x = -G_z = H_y = -H_x = \mu \tag{24}$$

In these circumstances, the system (21)-(23) generates the unique equation:

$$\frac{d}{dt}\left(\ln \mu(x + f(x))\right) = 0 \tag{25}$$

It is the generalization of (19) allowing us to determine the Lagrangian attached to the 3-dimensional Chua system. A direct integration of the previous relation gives:

$$\ln\mu(x+f(x)) = a = cons\tan t$$

$$\mu = \frac{1}{x+f(x)}\exp a \tag{26}$$

A possible first integral of the motion could be taken as:

$$\mathcal{F} = \left(\frac{\alpha\beta(\alpha-1)}{\alpha+\beta}+\ln y\right)x - \left(\frac{\alpha\beta(\beta+1)}{\alpha+\beta}-\ln y\right)z - y \tag{27}$$

We note that the nonlinearity we found has the form of $\ln\, y^{xz}$. As we shall see below, a quadratic polynomial could be used as control term in this case.

4.3. CONTROL TERMS FOR CHUA SYSTEM

Let us consider now that the nonlinear function from (8) and (26) has the form:

$$f(x) = A[\tanh\alpha(x+A/B)+\tanh\beta(x-A/B)] - Bx - Cx^2 \tag{28}$$

Numerical investigations show that the system behaves as a chaotic one. The chaos grows when A is growing and it does not strongly depend on the values of B. On the contrary, the chaotic behavior is strongly influenced by the quadratic term. This becomes clear in the picture below (Figure 2), where the behavior of the system at different values of C and for similar initial conditions, $x_0 = -0.01, y_0 = 0, z_0 = 0$, is presented. We also fixed the other parameters at the values $A = 1.5, B = 1, \alpha = 1, \beta = 0.6$. One can notice that for $C = 0$ (case from top), the system is strongly chaotic, while it turns into more regular ones when stronger controls are considered: $C = 5$ (middle) and $C = 10$ (down). This last term in (28) has the role of control. Its effect is increasing with its weight.

5. CONCLUSIONS

The paper investigated the behavior of one of the most emblematic examples of chaotic electric circuit. The idea was to consider Chua circuit as an example of quasi-variational system and to apply a control technique specific for Hamiltonian systems. Two main results were reported: (i) a possible first integral which can be attached to Chua system and can be considered as a suitable Lagrangian has been deduced in the form of (27); (ii) for the case when the nonlinear Chua function is chosen as (28), the last quadratic term could be seen as a control term of the system, improving the regularity of the dynamics. The result is quite similar with those reported for

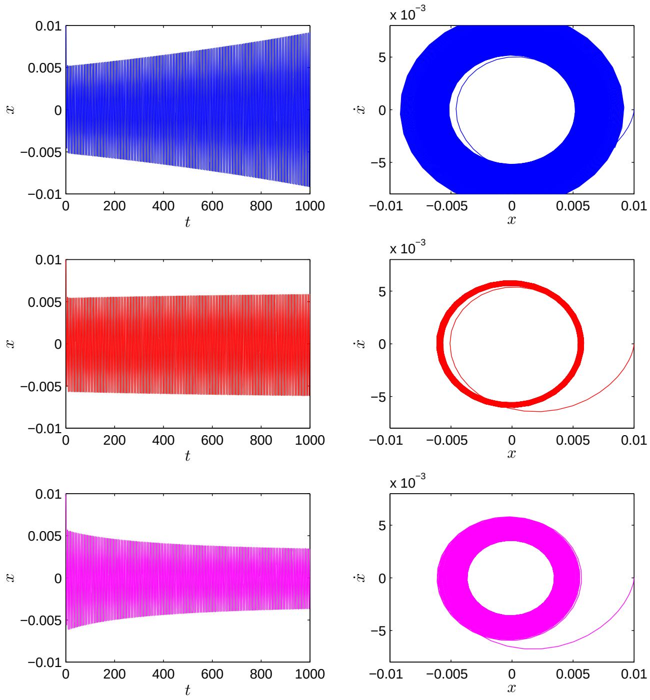

Fig. 2 – Control of chaos through a quadratic term

important example of variational systems as the mechanical Yang-Mills or Henon-Heiles models. Forthcoming studies will be devoted to the same Chua circuit but in a more compact form known as jerk system.

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