FOLIATED BACKGROUNDS FOR M-THEORY COMPACTIFICATIONS (II)

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We summarize the foliation approach to $\mathcal{N}=1$ compactifications of eleven-dimensional supergravity on eight-manifolds M (without boundary) down to AdS_3 spaces for the case when the internal part ξ of the supersymmetry generator is chiral on some proper subset $\mathcal{W}$ of M. In this case, a topological no-go theorem implies that the complement $M \setminus \mathcal{W}$ must be a *dense* open subset, while M admits a *singular* foliation $\bar{\mathcal{F}}$ (in the sense of Haefliger) which is defined by a closed one-form $\boldsymbol{\omega}$ and is endowed with a longitudinal G_2 structure. The geometry of this foliation is determined by the supersymmetry conditions. We also describe the topology of $\bar{\mathcal{F}}$ in the case when $\boldsymbol{\omega}$ is a Morse form.

Key words: Flux compactifications, supersymmetry, foliations.

1. INTRODUCTION

We describe the extension of the results of [1] (which were summarized in [2]) to the general case when the internal part ξ of the supersymmetry generator is allowed to become chiral on some locus $\mathcal{W} \subset M$. Assuming that $\mathcal{W} \neq M$, *i.e.* that ξ is not everywhere chiral, we showed in [3] that, at the classical level, the Einstein equations imply that the chiral locus $\mathcal{W}$ must be a set with empty interior, which therefore is negligible with respect to the Lebesgue measure of the internal space. As a consequence, the behavior of geometric data along this locus can be obtained from the non-chiral locus $\mathcal{U} \stackrel{\text{def.}}{=} M \setminus \mathcal{W}$ through a limiting process. The geometric information along the non-chiral locus is encoded [1] by a regular foliation $\mathcal{F}$ which carries a longitudinal G_2 structure and whose geometry is determined by the supersymmetry conditions in terms of the supergravity four-form field strength. When $\emptyset \neq \mathcal{W} \subsetneq M$, $\mathcal{F}$ extends to a singular foliation $\bar{\mathcal{F}}$ of the whole manifold M by adding leaves which are allowed to have singularities at points belonging to the chiral locus $\mathcal{W}$. This singular foliation "integrates" a cosmooth* singular distribution $\mathcal{D}$

*Note that $\mathcal{D}$ is *not* a singular distribution in the sense of Stefan-Sussmann (it is cosmooth rather than smooth).

(a.k.a. generalized sub-bundle of TM), defined as the kernel distribution of a closed one-form $\boldsymbol{\omega}$ which belongs to a cohomology class $\mathfrak{f} \in H^1(M,\mathbb{R})$ determined by the supergravity four-form field strength. The vanishing locus $\mathrm{Sing}(\boldsymbol{\omega})$ of $\boldsymbol{\omega}$ coincides with the chiral locus $\mathcal{W}$. In the most general case, $\bar{\mathcal{F}}$ can be viewed as a Haefliger structure [4] on M. The singular foliation $\bar{\mathcal{F}}$ carries a longitudinal G_2 structure, which is allowed to degenerate at the singular points of singular leaves. On the non-chiral locus $\mathcal{U}$, the problem can be studied using the approach of [1] or the approach advocated in [5], which makes use of two $\mathrm{Spin}(7)_\pm$ structures. The results of [1] agree with those of [5] along this locus, as shown in [3] by direct computation, upon using a certain "refined parameterization" of the flux components.

The topology of singular foliations defined by a closed one-form can be extremely complicated in general, but it is better understood when $\boldsymbol{\omega}$ is a Morse one-form. In the Morse case, the singular foliation $\bar{\mathcal{F}}$ can be described using the *foliation graph* [6] associated to a certain decomposition of M, which provides a combinatorial way to encode some important aspects of the foliation's topology — up to neglecting the information contained in the so-called *minimal components* of the decomposition, components which are expected to possess an as yet unexplored non-commutative geometric description.

We work in the same compactification set-up as in [5, 7], with the same notations and conventions as in [1, 3]. In such warped flux compactification, the supersymmetry conditions are equivalent with certain algebraic and differential constraints on some differential forms constructed from ξ.

2. GLOBALLY VALID PARAMETERIZATION OF A MAJORANA SPINOR ON M

Fixing a Majorana spinor $\xi \in \Gamma(M,S)$ which is everywhere of norm one, we consider the inhomogeneus differential form (see [3]):

$$\check{E}_{\xi,\xi} \stackrel{\text{def.}}{=} \check{E} = \frac{1}{16}\sum_{k=0}^{8} \check{\boldsymbol{E}}^{(k)}$$

where we use the following notations for the non-vanishing rank components :

$$\check{\boldsymbol{E}}^{(0)} = ||\xi||^2 = 1 \ , \ \check{\boldsymbol{E}}^{(1)} \stackrel{\text{def.}}{=} V \ , \ \check{\boldsymbol{E}}^{(4)} \stackrel{\text{def.}}{=} Y \ , \ \check{\boldsymbol{E}}^{(5)} \stackrel{\text{def.}}{=} Z \ , \ \check{\boldsymbol{E}}^{(8)} \stackrel{\text{def.}}{=} b\nu \ ,$$

with b a smooth real valued function defined on M

The Fierz identities can be fully described [8] by the conditions:

$$\check{E}^2 = \check{E} \ , \ \mathcal{S}(\check{E}) = 1 \ , \ \tau(\check{E}) = \check{E} \tag{1}$$

where $\mathcal{S}$ denotes the trace defined on the Kähler-Atiyah algebra and τ is the modified reversion, and can be shown to be equivalent with a set of relations which hold globally on M and which differ from those obtained in [1] by including the limit when

$b=\pm 1$, equivalently when $V=0$. In particular, the geometric data along $\mathcal{W}$ can be recovered from $\mathcal{U}$ through a limiting process which is described in [3]. We will not give any details of the calculations here (see [1, 3]), but will focus on describing the geometric meaning of the results.

The chirality decomposition of M. Let $S^\pm \subset S$ be the positive and negative chirality subbundles of S, which give the orthogonal decomposition $S=S^+\oplus S^-$. Decomposing a normalized spinor as $\xi=\xi^+ + \xi^-$ with $\xi^\pm \in \Gamma(M,S^\pm)$, we have:

$$||\xi||^2 = ||\xi^+||^2 + ||\xi^-||^2 = 1 \ , \ \ b = ||\xi^+||^2 - ||\xi^-||^2 \ \ ,$$

which give:

$$||\xi^\pm||^2 = \frac{1}{2}(1\pm b) \ . \tag{2}$$

Notice that b equals ± 1 at a point $p\in M$ iff $\xi_p \in S_p^\pm$. Since $||V||^2 = 1-b^2$ (implied by (1)), the one-form V vanishes at p iff ξ_p is chiral *i.e.* iff $\xi_p \in S_p^+ \cup S_p^-$. Consider the *non-chiral locus* (an open subset of M):

$$\mathcal{U} \stackrel{\text{def.}}{=} \{p\in M|\xi_p^+ \neq 0 \text{ and } \ \xi_p^- \neq 0\} = \{p\in M|V_p\neq 0\} = \{p \in M||b(p)|<1\} \ \ ,$$

and its closed complement, the *chiral locus*:

$$\mathcal{W} \stackrel{\text{def.}}{=} M\setminus \mathcal{U} = \{p\in M|\xi_p^+ = 0 \text{ or } \xi_p^- = 0\} = \{p\in M|V_p = 0\} = \{p\in M||b(p)|=1\} \ .$$

The chiral locus $\mathcal{W}$ decomposes further as a disjoint union of two closed subsets, the *positive and negative chirality loci* $\mathcal{W}=\mathcal{W}^+\sqcup\mathcal{W}^-$, where:

$$\mathcal{W}^\pm \stackrel{\text{def.}}{=} \{p\in M|\xi_p \in S_p^\pm\} = \{p\in M|b(p)=\pm 1\} = \{p\in M|\xi_p^\mp = 0\} \ \ .$$

Since ξ does not vanish on M, we have:

$$\mathcal{U}^\pm \stackrel{\text{def.}}{=} \mathcal{U}\cup\mathcal{W}^\pm = \{p\in M|\xi_p^\pm \neq 0\} \ \ .$$

3. A TOPOLOGICAL NO-GO THEOREM

We remind the reader the warped compactification ansatz for the field strength $\mathbf{G}$ of the supergravity 3-form field:

$$\mathbf{G} = \nu_3\wedge \mathbf{f} + \mathbf{F} \ \ , \quad \text{with } \ \mathbf{F} \stackrel{\text{def.}}{=} e^{3\Delta}F \ \ , \ \ \mathbf{f} \stackrel{\text{def.}}{=} e^{3\Delta} f \tag{3}$$

where ν_3 is the volume form of the AdS_3 space, $f\in\Omega^1(M)$, $F\in\Omega^4(M)$, while Δ is the warp factor. Another quantity that appears in the relations is κ, a positive real parameter describing the AdS_3 space (κ becomes zero in the Minkowski limit).

Theorem [3]. Assume that the supersymmetry conditions, the Bianchi identity and equations of motion for G as well as the Einstein equations are satisfied. Then there exist only the following possibilities:

1. The set $\mathcal{W}^+$ coincides with M and hence $\mathcal{W}^-$ and $\mathcal{U}$ are empty. In this case, ξ is a chiral spinor of positive chirality which is covariantly constant on M and we have $\kappa = f = F = 0$ while Δ is constant on M.
2. The set $\mathcal{W}^-$ coincides with M and hence $\mathcal{W}^+$ and $\mathcal{U}$ are empty. In this case, ξ is a chiral spinor of negative chirality which is covariantly constant on M and we have $\kappa = f = F = 0$ while Δ is constant on M.
3. The set $\mathcal{U}$ coincides with M and hence $\mathcal{W}^+$ and $\mathcal{W}^-$ are empty.
4. At least one of the sets $\mathcal{W}^+$ and $\mathcal{W}^-$ is non-empty but both of these sets have empty interior. In this case, $\mathcal{U}$ is dense in M and the union $\mathcal{W} = \mathcal{W}^+ \cup \mathcal{W}^-$ coincides with the topological frontier $\mathrm{Fr}(\mathcal{U}) = \mathrm{fr}(\mathcal{U}) = \bar{\mathcal{U}} \setminus \mathcal{U}$ of $\mathcal{U}$.

The proof relies on the analysis of the supersymmetry conditions (see [3]). Cases 1 and 2 correspond to the classical limit (the limit when the quantum correction required by M5-brane anomaly cancellation is negligible) of the well-known compactifications of [9]. Case 3 was studied in [1] (having been pioneered in [7] – where, however, a complete solution was not given). In this paper we concentrate on Case 4 (which was first considered in [5], though from a different perspective). Hence, from now on we assume:

$$M = \bar{\mathcal{U}} = \mathcal{U} \sqcup \mathcal{W} \ , \ \mathcal{W} = \mathrm{Fr}\mathcal{U} \ .$$

The foliation approach to this case (which we describe below) gives a handle on both local and *global* aspects of the geometry of M and shows that, due to global aspects, the relation between such compactifications and 7-dimensional compactifications of M-theory is much more subtle than one may imagine at first sight.

4. THE SINGULAR FOLIATION $\bar{\mathcal{F}}$

The one-form V determines a generalized distribution $\mathcal{D}$ (generalized subbundle of TM) which is defined through:

$$\mathcal{D}_p \stackrel{\text{def.}}{=} \ker V_p \ , \ \forall p \in M \ . \tag{4}$$

This singular distribution is *cosmooth* (rather than smooth) in the sense of [10]. The set of regular points of $\mathcal{D}$ equals the non-chiral locus $\mathcal{U}$ and we have:

$$\mathrm{rk}\mathcal{D}_p = 7 \ \text{ when } \ p \in \mathcal{U} \ ,$$
$$\mathrm{rk}\mathcal{D}_p = 8 \ \text{ when } \ p \in \mathcal{W} \ .$$

In particular, the restriction $\mathcal{D}|_{\mathcal{U}}$ is a regular Frobenius distribution. As in [1], we endow $\mathcal{D}|_{\mathcal{U}}$ with the orientation induced by that of M. One can show that the one-form:

$$\boldsymbol{\omega} \stackrel{\text{def.}}{=} 4\kappa e^{3\Delta} V \tag{5}$$

satisfies the following relations, which hold globally on M as a consequence of the supersymmetry conditions:

$$\mathrm{d}\boldsymbol{\omega} = 0 \ , \ \boldsymbol{\omega} = \mathbf{f} - \mathrm{d}\mathbf{b} \ , \ \text{where } \mathbf{b} \stackrel{\text{def.}}{=} e^{3\Delta} b \ . \tag{6}$$

As a result of the first equation, the generalized distribution $\mathcal{D} = \ker V = \ker \boldsymbol{\omega}$ determines a singular foliation $\bar{\mathcal{F}}$ of M, which degenerates along the chiral locus $\mathcal{W}$; since $\mathcal{D}$ is cosmooth rather than smooth, this singular foliation can be described as a Haefliger structure [4] (see [3] for details), thus it is not a singular foliation in the sense of Stefan-Sussmann. The second equation implies that $\boldsymbol{\omega}$ belongs to the cohomology class $\mathfrak{f} \in H^1(M,\mathbb{R})$ of $\mathbf{f}$. The restriction $\mathcal{F} \stackrel{\text{def.}}{=} \bar{\mathcal{F}}|_{\mathcal{U}}$ is a regular codimension one foliation which satisfies Theorems 1, 2 and 3 of [1] (which are local in nature); those theorems give a complete characterization of the intrinsic and extrinsic geometry of $\mathcal{F}$. Using a certain "improved parameterization" of the four-form F along the non-chiral locus $\mathcal{U}$ (a parameterization which is "adapted" to the foliation $\mathcal{F}$), a lengthy computation shows that the results of [1] agree with those of [5] along this locus; we refer the reader to [3] for details.

5. DESCRIPTION OF THE SINGULAR FOLIATION IN THE MORSE CASE

Consider the case when the closed one-form $\boldsymbol{\omega} \in \Omega^1(M)$ is Morse. This case is generic in the sense that Morse one-forms constitute a dense open subset of the set of all closed one-forms belonging to the fixed cohomology class $\mathfrak{f}$ — hence a form $\boldsymbol{\omega}$ which satisfies equations (6) can be replaced by a Morse form by infinitesimally perturbing b. Singular foliations defined by Morse 1-forms were studied, for example, in [11–13]. Let $\Pi_f = \mathrm{im}(\mathrm{per}_{\mathfrak{f}}) \subset \mathbb{R}$ be the period group of the cohomology class $\mathfrak{f}$ and $\rho(\mathfrak{f}) = \mathrm{rk}\Pi_{\mathfrak{f}}$ be its irrationality rank. A leaf $\mathcal{L}$ of $\bar{\mathcal{F}}$ is called *singular* if it intersects $\mathcal{W}$ and *regular* otherwise. Notice that $\mathcal{W}$ is a finite set when $\boldsymbol{\omega}$ is Morse. The study of Morse 1-forms is a rich subject originating in Novikov theory, which is a generalization of Morse theory from functions to forms. We refer the reader to [12] for an introduction to this subject.

5.1. TYPES OF SINGULAR POINTS

Let $\mathrm{ind}_p(\boldsymbol{\omega})$ denote the Morse index of a point $p \in \mathrm{Sing}(\boldsymbol{\omega}) = \mathcal{W}$, *i.e.* the Morse index at p of a Morse function $h_p \in \mathcal{C}^\infty(U_p,\mathbb{R})$ such that $\mathrm{d}h_p$ equals $\boldsymbol{\omega}|_{U_p}$, where U_p

is a sufficiently small vicinity of p. The Morse index does not depend on the choice of U_p and h_p. Let:

$$\mathrm{Sing}_k(\boldsymbol{\omega}) \stackrel{\text{def.}}{=} \{p \in \mathcal{W} | \mathrm{ind}_p(\boldsymbol{\omega}) = k\} \ , \ k = 1, \ldots, d$$
$$\Sigma_k(\boldsymbol{\omega}) \stackrel{\text{def.}}{=} \{p \in \mathcal{W} | \mathrm{ind}_p(\boldsymbol{\omega}) = k \text{ or } \mathrm{ind}_p(\boldsymbol{\omega}) = d-k\} \ , \ k = 1, \ldots, \left[\frac{d}{2}\right] \ .$$

Thus $\Sigma_k(\boldsymbol{\omega}) = \mathrm{Sing}_k(\boldsymbol{\omega}) \cup \mathrm{Sing}_{n-k}(\boldsymbol{\omega})$ for $k < \frac{d}{2}$ and $\Sigma_{d_0}(\boldsymbol{\omega}) = \mathrm{Sing}_{d_0}(\boldsymbol{\omega})$ when $d = 2d_0$ is even. In a small enough vicinity of $p \in \mathrm{Sing}_k(\boldsymbol{\omega})$ (which we can assume to equal U_p by shrinking the latter if necessary), the Morse lemma applied to h_p implies that there exists a local coordinate system $(x_1, \ldots, x_d)$ such that:

$$h_p = -\sum_{j=1}^{k} x_j^2 + \sum_{j=k+1}^{d} x_j^2 \ .$$

Definition. The elements of $\Sigma_0(\boldsymbol{\omega})$ are called *centers* while all other singularities of $\boldsymbol{\omega}$ are called *saddle points*. The elements of $\Sigma_1(\boldsymbol{\omega})$ are called *strong saddle points*, while all other saddle points are called *weak*.

Remark. Other names for the various types of singular points are in use in the Mathematics literature.

5.2. BEHAVIOR OF THE SINGULAR LEAVES NEAR SINGULAR POINTS

In a small enough vicinity of $p \in \mathrm{Sing}_k(\boldsymbol{\omega})$, the singular leaf $\mathcal{L}_p$ passing through p is modeled by the locus $Q_k \subset \mathbb{R}^n$ given by the equation $h_p = 0$, where p corresponds to the origin of $\mathbb{R}^n$. One distinguishes the cases (see Tables 1 and 2):

- $k \in \{0, n\}$, *i.e.* p is a *center*. Then $\mathcal{L}_p = \{p\}$ and the nearby leaves of $\mathcal{F}_p$ are diffeomorphic to S^{n-1}.
- $2 \leq k \leq n-2$, *i.e.* p is a *weak saddle point*. Then Q_k is diffeomorphic to a cone over $S^{k-1} \times S^{n-k-1}$ and $\mathbb{R}^n \setminus Q_k$ has two connected components while $Q_k \setminus \{p\}$ is connected. Removing p does not *locally* disconnect $\mathcal{L}_p$.
- $k \in \{1, n-1\}$, *i.e.* p is a *strong saddle point*. Then Q_k is diffeomorphic to a cone over $\{-1, 1\} \times S^{n-2}$ and $\mathbb{R}^n \setminus Q_k$ has three connected components while $Q_k \setminus \{0\}$ has two components. Removing p *locally* disconnects $\mathcal{L}_p$.

5.3. COMBINATORICS OF SINGULAR LEAVES

Definition. A singular leaf of $\bar{\mathcal{F}}$ which is not a center is called a *strong singular leaf* if it contains at least one strong saddle point and a *weak singular leaf* otherwise.

Table 1

Types of singular points p. The first and third figure on the right depict the case $d = 3$ for centers and strong saddles, while the second figure attempts to depict the case $d > 3$ for a weak saddle.

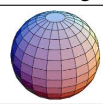
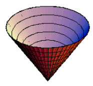
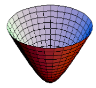
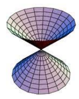
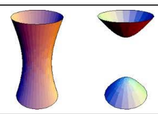
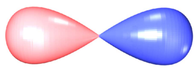
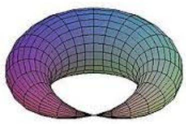
Name	Morse index	Local form of $\mathcal{L}_p$	Local form of regular leaves
Center	0 or n	$\bullet = \{p\}$	
Weak saddle	between 2 and $n-2$		
Strong saddle	1 or $n-1$		

Table 2

Types of strong saddle points.

Singularity type	Example of global shape for $\mathcal{L}_p$
Splitting	
Non-splitting	

A weak singular leaf is obtained by adjoining weak saddle points to a single special leaf of $\mathcal{F}$.

The situation is more complicated for strong singular leaves. At each $p \in \Sigma_1(\boldsymbol{\omega})$, consider the strong singular leaf $\mathcal{L}$ passing through p. The intersection of $\mathcal{L} \setminus \{p\}$ with a sufficiently small neighborhood of p is a disconnected manifold diffeomorphic to a union of two cones without apex, whose rays near p determine a connected cone $C_p \subset T_pM$ inside the tangent space to M at p (see the last row of Table 1). The set $\overset{\bullet}{C}_p \stackrel{\text{def.}}{=} C_p \setminus \{0_p\}$ (where 0_p is the zero vector of T_pM) has two connected components, thus $\pi_0(\overset{\bullet}{C}_p)$ is a two-element set. Hence the finite set:

$$\hat{\Sigma}_1(\boldsymbol{\omega}) \stackrel{\text{def.}}{=} \sqcup_{p \in \Sigma_1(M)} \pi_0(\overset{\bullet}{C}_p) \tag{7}$$

is a double cover of $\Sigma_1(\boldsymbol{\omega})$ through the projection σ that takes $\pi_0(\dot{C}_p)$ to $\{p\}$. Consider the complete unoriented graph having as vertices the elements of $\hat{\Sigma}_1(\boldsymbol{\omega})$. This graph has a dimer covering given by the collection of edges:

$$\hat{\mathcal{E}} = \{\pi_0(\overset{\bullet}{C}_p)|p \in \Sigma_1(\boldsymbol{\omega})\} \ \ , \tag{8}$$

which connect vertically the vertices lying above the same point of $\Sigma_1(\boldsymbol{\omega})$ (see Figure 1). If L is a special leaf of $\mathcal{F}$ and $p \in \Sigma_1(\boldsymbol{\omega})$ adjoins L, then the connected components of the intersection of L with a sufficiently small vicinity of p are locally approximated at p by one or two of the connected components of $\overset{\bullet}{C}_p$. The second case occurs iff p is a non-splitting strong saddle point (see Table 2). Hence L determines a subset $\hat{s}_1(L)$ of $\hat{\Sigma}_1(\boldsymbol{\omega})$ such that $\sigma(\hat{s}_1(L)) = s_1(L)$ and such that the fiber of $\hat{s}_1(L)$ above a point $p \in s_1(L)$ has one element if p is a splitting singularity and two elements if p is non-splitting.

The graph $\mathcal{E}$ has one vertex for each special leaf of $\mathcal{F}$ which adjoins some strong saddle point and an edge for each strong saddle point. Notice that this edge is a loop when the strong saddle point is a non-splitting singularity.

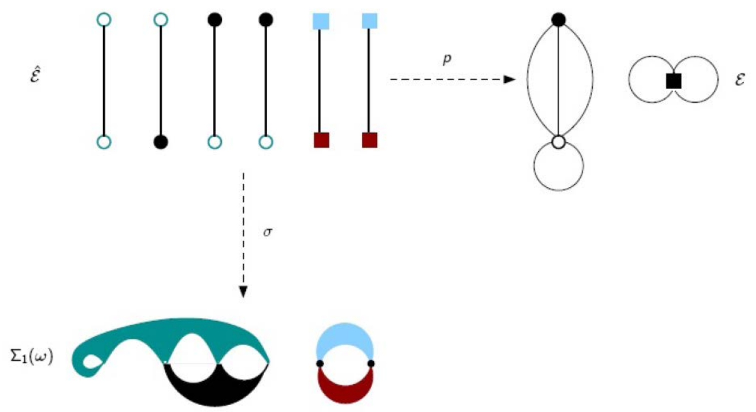


Fig. 1 – Example of the graphs $\hat{\mathcal{E}}$ and $\mathcal{E}$ for a Morse form foliation $\bar{\mathcal{F}}$ with two compact strong singular leaves. The regular foliation $\mathcal{F}$ of M^* has four special leaves, depicted using four different colors, each of which is compactifiable. At the bottom of the picture, we depict $\Sigma_1(\boldsymbol{\omega})$ as well as the schematic shape of the special leaves in the case $d = 3$. The strong singular leaves of $\bar{\mathcal{F}}$ correspond to the left and right parts of the figure at the bottom; each of them is a union of two special leaves of $\mathcal{F}$ and of singular points. Each special leaf corresponds to a vertex of $\mathcal{E}$.

In our application, the set $\mathrm{Sing}(\boldsymbol{\omega}) = \mathcal{W} = \mathcal{W}^+ \sqcup \mathcal{W}^-$ consists of positive and negative chirality points of ξ, which are the points where b attains the values $b = \pm 1$.

5.4. THE FOLIATION GRAPH

Define $C^{\max}$ to be the union of all compact leaves and $C^{\min}$ to be the union of all non-compactifiable leaves of $\mathcal{F}$ (see [3] for details). Both $C^{\max}$ and $C^{\min}$ are open subsets of M which have a common topological small frontier† F. Each of the open sets $C^{\rm max}$ and $C^{\rm min}$ has a finite number of connected components, which are called the *maximal* and *minimal* components of the set $M \setminus F = C^{\rm max} \sqcup C^{\rm min}$ and we index them as $C_j^{\rm max}$ and $C_a^{\rm min}$ such that:

$$C^{\rm max} = \sqcup_j C_j^{\rm max} \ \ , \ \ C^{\rm min} = \sqcup_a C_a^{\rm min} \ \ .$$

Let:

$$\mathbf{\Delta} \stackrel{\rm def.}{=} M \setminus C^{\rm max} = \overline{C^{\rm min}} = C^{\rm min} \sqcup F$$

be the union of all non-compact leaves and singularities. This subset has a finite number of connected components $\mathbf{\Delta}_s$.

Definition. The *foliation graph* $\Gamma_{\boldsymbol{\omega}}$ of $\boldsymbol{\omega}$ is the unoriented graph whose vertices are the connected components $\mathbf{\Delta}_s$ of $\mathbf{\Delta}$ and whose edges are the maximal components $C_j^{\rm max}$. An edge $C_j^{\rm max}$ is incident to a vertex $\mathbf{\Delta}_s$ iff a connected component of ${\rm fr} C_j^{\rm max}$ is contained in $\mathbf{\Delta}_s$; it is a loop at $\mathbf{\Delta}_s$ iff ${\rm fr} C_j^{\rm max}$ is connected and contained in $\mathbf{\Delta}_s$. A vertex $\mathbf{\Delta}_s$ of $\Gamma_{\boldsymbol{\omega}}$ is called *exceptional* (or of *type II*) if it contains at least one minimal component; otherwise, it is called *regular* (or of *type I*).

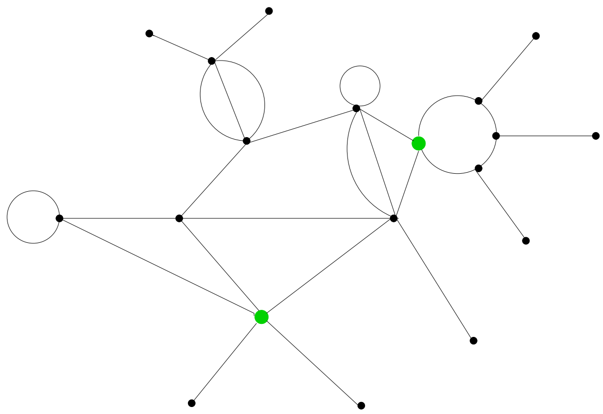


Fig. 2 – An example of foliation graph. Regular (a.k.a type I) vertices are represented by black dots, while exceptional (a.k.a. type II) vertices are represented by green blobs. All terminal vertices are regular vertices and correspond to center singularities.

It is believed [13] that the leaf space of $\bar{\mathcal{F}}$ should be described as a non-commutative space, the 'commutative part' of which is given by the foliation graph. A rigorous definition of the C^*-algebra of singular foliations in the sense of Haefliger does not appear to have been given in the Mathematics literature, so this expectation

†The *small frontier* of a set A is the set ${\rm fr}(A) \stackrel{\rm def.}{=} A \setminus {\rm Int} A$.

(a) Foliation graph when $\mathcal{W} = \emptyset$ and $\rho(\boldsymbol{\omega}) = 1$. (b) Foliation graph when $\mathcal{W} = \emptyset$ and $\rho(\boldsymbol{\omega}) > 1$.

Fig. 3 – Degenerate foliation graphs in the everywhere non-chiral case.

should be taken with a grain of salt. Much more detail about the topology of $\bar{\mathcal{F}}$ can be found in [3].

When ξ is everywhere non-chiral (Case 3 of the topological no-go theorem, *i.e.* $\mathcal{W} = \emptyset$), the foliation graph is either a circle or a single exceptional vertex (see Figure 3). It was shown in [1] that, in this case, the exceptional vertex corresponds to a non-commutative torus of dimension given by the projective irrationality rank of $\boldsymbol{\omega}$. Already in that case, one cannot think of the generic compactification of this type (which corresponds to a non-commutative leaf space) as a two-step reduction in the sense of "generalized Scherk-Schwarz compactifications with a twist" (see [14]); this is doubly true when $\mathcal{W}$ is a proper subset of M.

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