INTEGRABILITY OF A FAMILY OF 2-DIM CUBIC SYSTEMS WITH DEGENERATE INFINITY

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We study a family of cubic systems with degenerate infinity introduced in the paper [Chavarriga *et al.*, Integrability of cubic systems with degenerate infinity, Differential Equations Dynam. Systems 6 (1998), 425-438]. For systems of the family having a monodromic singularity at the origin the sets in the space of parameters corresponding to the systems with a local analytic first integral are found.

Key words: analytic differential equations, partial integrability, variety, center on a center manifold, analytic normalization.

1. INTRODUCTION

The planar analytic differential system

$$\dot{x} = -y + f(x,y), \qquad \dot{y} = x + g(x,y), \tag{1}$$

with f and g being polynomials without constant and linear terms, has either a center or a focus at the origin. In the first case all trajectories are ovals, and in the second case they spiral towards to the singular point in either positive or negative time. Thus, there arises a problem to distinguish between these two cases. This is a challenging problem called the center problem (or center–focus problem), which attracts a lot of attention starting from the pioneering works of Dulac [6] and Kapteyn [11, 12], who solved the problem for the case when f and g are quadratic polynomials. Later on, Malkin [16] presented the necessary and sufficient conditions for the origin of system (1) to be a center in the case when f and g are homogeneous polynomials of degree three. Despite of there are hundreds of papers devoted to studies of the problem it is completely solved only for very few natural families (see, *e.g.* [2, 8, 21] and references given there).

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By the Poincaré–Lyapunov theorem [15, 17, 18] (see also [21] for a detailed proof) system (1) has a center at the origin if and only if it admits a first integral of the form

$$\Phi(x,y) = x^2 + y^2 + \sum_{i+j\geq 3} \psi_{ij} x^i y^j, \tag{2}$$

where the series converge in some neighborhood of the origin in $\mathbb{R}^2$. Moreover, the existence of a center for system (1) is equivalent to the existence of any analytic first integral of system (1) (not necessary of the form (2)) defined in a neighborhood of the origin (it follows, for instance, from the results of [24, 26]).

An interesting polynomial family of the form

$$\begin{aligned} \dot{x} &= -y + p_2(x,y) + x r_2(x,y) = P(x,y), \\ \dot{y} &= x + q_2(x,y) + y r_2(x,y) = Q(x,y), \end{aligned} \tag{3}$$

where

$$\begin{aligned} p_2 &= a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ q_2 &= b_{20}x^2 + b_{11}xy + b_{02}y^2, \\ r_2 &= r_{20}x^2 + r_{11}xy + r_{02}y^2, \end{aligned}$$

has been studied in [1, 2]. We note that system (3) has its infinity filled up with singularities and it is a polynomial differential system of degree 2 in the projective space, see [3, 23, 25] for more details.

The authors of [2] have shown that system (3) has a center at the origin if and only if after some transformations it can be written in one of four forms in polar coordinates. However, from their results it is impossible (at least we do not see a way) to determine the conditions on the coefficients of polynomials p_2, q_2 r_2, which fulfillment yields the existence of a center at the origin of system (3). In this paper we will obtain such coefficient conditions. We also will show that all integrable systems in family (3) are Darboux integrable, that is, they admit either a Darboux first integral or a Darboux integrating factor, which are constructed from invariant curves of the vector field using the well known procedure (see *e.g.* [13, 14]), which we briefly recall for convenience of the reader.

We recall that a Darboux polynomial of system (1) is a polynomial $F(x,y)$ in the ring $\mathbb{C}[x,y]$ of polynomials with complex coefficients in x,y, such that

$$\frac{\partial F}{\partial x}(-y + f(x,y)) + \frac{\partial F}{\partial y}(x + g(x,y)) = KF,$$

where $K(x,y) \in \mathbb{C}[x,y]$ is a polynomial called the *cofactor* associated to the Darboux polynomial F. A simple computation shows that if there are Darboux polynomials $F_1, F_2, \ldots, F_k$ with the cofactors $K_1, K_2, \ldots, K_k$, respectively, satisfying $\sum_{i=1}^{k} \alpha_i K_i = 0$ for some $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, then $\Psi(x,y) = F_1^{\alpha_1} \cdots F_k^{\alpha_k}$ is a first integral of system (1), and that if

$$\sum_{i=1}^{k}\alpha_i K_i + f'_x + g'_y = 0, \tag{4}$$

then system (1) admits the integrating factor $\mu(x,y) = F_1^{\alpha_1}\cdots F_k^{\alpha_k}$.

2. CENTER CONDITIONS FOR SYSTEM (3)

To compute the necessary conditions for existence of a first integral of the form (2) for system (3) we look for a function $\Phi_{2m+1}(x,y) = x^2+y^2+\sum_{j+k=3}^{2m+1}\phi_{jk}x^jy^k$ such that

$$\frac{\partial\Phi_{2m+1}}{\partial x}P + \frac{\partial\Phi_{2m+1}}{\partial y}Q = g_1(a,b,r)(x^2+y^2)^2 + g_2(a,b,r)(x^2+y^2)^3 + \dots, \tag{5}$$

where (a,b,r) stands for the 9–tuple of the parameters of system (3), *i.e.*, $(a,b,r) = (a_{20},a_{11},\dots,r_{02})$. Equating the coefficients of the same monomials on both sides of (5) we compute the coefficients $\psi_{jk}(a,b,r)$ in (2) and the polynomials denoted by $g_1(a,b,r)$, $g_2(a,b,r)$,... called the *focus quantities* of system (3) (see *e.g.* [19, 21, 22] for more details about the calculation of focus quantities). For system (3) we have computed the first 8 focus quantities $g_1,\dots,g_8$. The first one is

$$g_1 = -2\,(b_{02}b_{11} - a_{02}a_{11} - a_{11}a_{20} - 2a_{02}b_{02} + 2a_{20}b_{20} + b_{11}b_{20} - 4r_{02} - 4r_{20}).$$

The size of the polynomials g_i's growths exponentially, so we do not present them here, but the interested reader can easily compute the quantities using any available computer algebra system. The system of algebraic equations

$$g_1 = \dots = g_8 = 0 \tag{6}$$

gives us the necessary conditions for integrability. To show that these conditions are also sufficient conditions for integrability we need to find the irreducible decomposition of the variety (the zero set) of the ideal

$$B = \langle g_1,\dots,g_8\rangle \tag{7}$$

and to prove that the systems corresponding to the components of the irreducible decomposition have an analytic first integral around the origin.

Since the algebraic system (6) is very complicated we are not able to find the decomposition of the corresponding variety in full generality. However, we observe that if $a_{02}+a_{20}\neq 0$, then by the linear transformation

$$x' = x + \frac{b_{02}+b_{20}}{a_{02}+a_{20}}y, \quad y' = y - \frac{b_{02}+b_{20}}{a_{02}+a_{20}}x$$

we obtain a system of the form (3) with $b_{02}+b_{20}=0$, and if $a_{02}+a_{20}=0$, then we obtain a system of such form interchanging x and y. Thus, without loss of generality we can assume that in system (3) $b_{02} = -b_{20}$. Then we can prove the following statement.

Theorem 1 *System* (3) *with* $b_{02} = -b_{20}$ *has a center at the origin if one of the following conditions holds:*

a) $a_{20} + a_{02} = r_{02} + r_{20} = 0,$

b) $a_{11} + 2b_{20} = a_{02}b_{20} - r_{02} = a_{20}b_{20} - r_{20} = r_{02}a_{20} - r_{20}a_{02} = 0,$

c) $2a_{11}a_{20} - a_{11}b_{11} - 12a_{20}b_{20} - 2b_{11}b_{20} + 16r_{20} = 3a_{11}^2 + 4a_{20}^2 - 4a_{20}b_{11} - 3b_{11}^2 - 4a_{11}b_{20} - 4b_{20}^2 + 16r_{11} = a_{02}a_{11} + a_{11}a_{20} - 2a_{02}b_{20} - 2a_{20}b_{20} + 4r_{02} + 4r_{20} = 0,$

d) $-3a_{02}^2 + a_{11}^2 - 10a_{02}a_{20} - 3a_{20}^2 + 2a_{02}b_{11} - 2a_{20}b_{11} + b_{11}^2 + 4a_{11}b_{20} + 4b_{20}^2 = 5a_{02}a_{11} + 7a_{11}a_{20} - a_{11}b_{11} + 10a_{02}b_{20} - 2a_{20}b_{20} - 2b_{11}b_{20} + 16r_{20} = 17a_{02}^2 + 30a_{02}a_{20} + 9a_{20}^2 + 2a_{02}b_{11} + 6a_{20}b_{11} - 3b_{11}^2 - 8a_{11}b_{20} - 8b_{20}^2 + 8r_{11} = a_{02}a_{11} + a_{11}a_{20} - 2a_{02}b_{20} - 2a_{20}b_{20} + 4r_{02} + 4r_{20} = 0$

Proof. We first tried to find the irreducible decomposition of the variety $V(B)$ of ideal (7) over the field $\mathbb{Q}$ of rational numbers using the routine `minAssGTZ` of the `primdec` library [5] of the computer algebra system SINGULAR [10], however we were unable to complete calculations using our computational facilities. As it was for the first time observed in [7] modular computations can help in such situations. We then performed computations in the field $\mathbb{Z}_{32003}$ using the approach suggested in [20] and obtained the conditions a)–d) given in the statement of the theorem.

We now prove that under each of conditions a)–d) the corresponding system (3) has an analytic first integral near the origin.

Case a). When the first condition of the theorem is fulfilled the system is written in the form

$$\begin{aligned} \dot{x} &= -y + a_{20}x^2 + a_{11}xy - a_{20}y^2 + x(r_{20}x^2 + r_{11}xy - r_{20}y^2), \\ \dot{y} &= \ \ x + b_{20}x^2 + b_{11}xy - b_{20}y^2 + y(r_{20}x^2 + r_{11}xy - r_{20}y^2). \end{aligned} \tag{8}$$

It is possible to compute four invariant lines and their cofactors for this system, but the expressions are huge and useless. So, we prove the existence of a Darboux first integral for system (8) without computing the integral explicitly.

First, we observe that if

$$F = 1 + h_1x + h_2y$$

is an invariant line of system (8) with the cofactor

$$K = c_1x + c_2y + c_3x^2 + c_4xy + c_5y^2, \tag{9}$$

then $c_1 = h_2,\ c_2 = -h_1,\ c_3 = a_{20}h_1 + b_{20}h_2 - h_1h_2$ and

$$c_5 = -c_3. \tag{10}$$

Under these conditions we see that

$$a_{20}h_1 + b_{20}h_2 - h_1h_2 - r_{20} = 0. \quad (11)$$

If

$$b_{20}a_{20} - r_{20} \neq 0, \quad (12)$$

then (11) yields

$$h_1 = \frac{r_{20} - b_{20}h_2}{a_{20} - h_2}.$$

Further calculations give $c_3 = -c_5 = r_{20}$ and

$$\begin{aligned} c_4 = \frac{1}{(-a_{20}+h_2)^2} \big(& a_{20}^2 b_{11} h_2 - a_{11}a_{20}b_{20}h_2 - a_{20}^2 h_2^2 - 2a_{20}b_{11}h_2^2 \\ & + a_{11}b_{20}h_2^2 + b_{20}^2 h_2^2 + 2a_{20}h_2^3 + b_{11}h_2^3 - h_2^4 \\ & + a_{11}a_{20}r_{20} - a_{11}h_2r_{20} - 2b_{20}h_2r_{20} + r_{20}^2 \big), \end{aligned} \quad (13)$$

where h_2 is a root of the equation

$$\begin{aligned} & - a_{20}^2 b_{11}h_2 + a_{11}a_{20}b_{20}h_2 + a_{20}^2 h_2^2 + 2a_{20}b_{11}h_2^2 - a_{11}b_{20}h_2^2 \\ & - b_{20}^2 h_2^2 - 2a_{20}h_2^3 - b_{11}h_2^3 + h_2^4 + a_{20}^2 r_{11} - 2a_{20}h_2r_{11} + h_2^2 r_{11} \\ & - a_{11}a_{20}r_{20} + a_{11}h_2r_{20} + 2b_{20}h_2r_{20} - r_{20}^2 = 0. \end{aligned} \quad (14)$$

Using (14) we obtain from (13) that $c_4 = r_{11}$.

Assume that the discriminant of the polynomial in (14) with respect to h_2 is different from zero (the discriminant is a polynomial of degree 12 in the parameters of system (8)). Then equation (14) has four different roots, which we denote by u_1, u_2, u_3, u_4. Further, from the equation

$$a_1K_1 + a_2K_2 + a_3K_3 + a_4K_4 = 0, \quad (15)$$

with K_i's being the cofactors associated to the invariant lines, we obtain a linear system in variables a_1, a_2, a_3, a_4 with the coefficient matrix

$$\begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ -\frac{r_{20}-b_{20}u_1}{a_{20}-u_1} & -\frac{r_{20}-b_{20}u_2}{a_{20}-u_2} & -\frac{r_{20}-b_{20}u_3}{a_{20}-u_3} & -\frac{r_{20}-b_{20}u_4}{a_{20}-u_4} \\ r_{20} & r_{20} & r_{20} & r_{20} \\ r_{11} & r_{11} & r_{11} & r_{11} \end{pmatrix}$$

Since the determinant of the matrix is equal to zero, there are real or complex numbers a_1, a_2, a_3, a_4 not all equal to zero satisfying (15). Thus, in this case the system has a real or complex Darboux integral. In the latter case both the imaginary and real parts of the complex integral are real first integrals of the systems. Note that since the linear Darboux polynomials do not vanish near the origin, the first integral is analytic at the origin.

Since the set of systems (3) which admit a first integral of the form (2) is a real algebraic variety and it is a closed set in the Zariski topology (see *e.g.* [21]), system (8) also has a first integral of the form (2) if condition (12) does not hold or the discriminant of the polynomial in (14) is equal to zero.

Case b). In this case system (3) has the form

$$\begin{aligned}\dot{x} &= -y + a_{20}x^2 - 2b_{20}xy + a_{02}y^2 + x(a_{20}b_{20}x^2 + r_{11}xy + a_{02}b_{20}y^2)\\ \dot{y} &= \ \ x + b_{20}x^2 + b_{11}xy - b_{20}y^2 + y(a_{20}b_{20}x^2 + r_{11}xy + a_{02}b_{20}y^2).\end{aligned} \tag{16}$$

System (16) has two invariant lines,

$$F_1 = 1 + b_{20}x + \frac{1}{2}\left(b_{11} - \sqrt{b_{11}^2 - 4b_{20}^2 - 4r_{11}}\right)y$$

and

$$F_2 = 1 + b_{20}x + \frac{1}{2}\left(b_{11} + \sqrt{b_{11}^2 - 4b_{20}^2 - 4r_{11}}\right)y$$

with the cofactors

$$K_1 = \frac{1}{2}\left(b_{11} - \sqrt{b_{11}^2 - 4(b_{20}^2 + r_{11})}\right)x - b_{20}y + a_{20}b_{20}x^2 + r_{11}xy + a_{02}b_{20}y^2$$

and

$$K_2 = \frac{1}{2}\left(b_{11} + \sqrt{b_{11}^2 - 4(b_{20}^2 + r_{11})}\right)x - b_{20}y + a_{20}b_{20}x^2 + r_{11}xy + a_{02}b_{20}y^2,$$

respectively, which allow us to construct the Darboux integrating factor

$$\mu = F_1^{\alpha_1}F_2^{\alpha_2},$$

where

$$\alpha_1 = \frac{2a_{20} - 2\sqrt{b_{11}^2 - 4\left(b_{20}^2 + r_{11}\right)} - b_{11}}{\sqrt{b_{11}^2 - 4\left(b_{20}^2 + r_{11}\right)}}$$

and

$$\alpha_2 = \frac{-2a_{20} - 2\sqrt{b_{11}^2 - 4\left(b_{20}^2 + r_{11}\right)} + b_{11}}{\sqrt{b_{11}^2 - 4\left(b_{20}^2 + r_{11}\right)}}.$$

Case c). In this case in system (3)

$$b_{02} = -b_{20}, \quad r_{20} = \frac{1}{16}(-2a_{11}a_{20} + a_{11}b_{11} + 12a_{20}b_{20} + 2b_{11}b_{20}),$$

$$r_{11} = \frac{1}{16}(-3a_{11}^2 - 4a_{20}^2 + 4a_{20}b_{11} + 3b_{11}^2 + 4a_{11}b_{20} + 4b_{20}^2)$$

and

$$r_{02} = \frac{1}{16}(-4a_{02}a_{11} - 2a_{11}a_{20} - a_{11}b_{11} + 8a_{02}b_{20} - 4a_{20}b_{20} - 2b_{11}b_{20}).$$

The system admits an invariant line

$$F = 1 + \frac{1}{4}(-a_{11} + 2b_{20})x + \frac{1}{4}(2a_{20} + b_{11})y$$

with the cofactor

$$K = \frac{1}{16}\left(b_{11}x(4 + a_{11}x + 2b_{20}x) - 4a_{20}^2 xy + 3b_{11}^2 xy - b_{11}(a_{11} + 2b_{20})y^2 \right.$$
$$- (a_{11} - 2b_{20})y(-4 + 3a_{11}x + 2b_{20}x + 4a_{02}y) +$$
$$\left. 2a_{20}(x(4 - a_{11}x + 6b_{20}x) + 2b_{11}xy - (a_{11} + 2b_{20})y^2)\right).$$

Using (4) we obtain that $\mu = F^{-4}$ is an integrating factor of the system.

Case d). In this case the system is written as

$$\begin{aligned} \dot{x} &= -y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + x\tilde{R}(x,y) = P(x,y), \\ \dot{y} &= \ \ x + b_{20}x^2 + b_{11}xy - b_{20}y^2 + y\tilde{R}(x,y) = Q(x,y), \end{aligned} \tag{17}$$

where

$$\tilde{R}(x,y) = \frac{1}{16}(a_{11}(-7a_{20} + b_{11}) + 2(a_{20} + b_{11})b_{20} - 5a_{02}(a_{11} + 2b_{20}))x^2 +$$
$$\frac{1}{8}\left(-17a_{02}^2 - 9a_{20}^2 - 6a_{20}b_{11} + 3b_{11}^2 - 2a_{02}(15a_{20} + b_{11}) + 8a_{11}b_{20} + 8b_{20}^2\right)xy +$$
$$\frac{1}{16}((3a_{20} - b_{11})(a_{11} + 2b_{20}) + a_{02}(a_{11} + 18b_{20}))y^2$$

and

$$a_{11} = \pm\sqrt{3a_{02}^2 + 10a_{02}a_{20} - 2a_{02}b_{11} + 3a_{20}^2 + 2a_{20}b_{11} - b_{11}^2} - 2b_{20}.$$

We consider only the case when

$$a_{11} = a = \sqrt{3a_{02}^2 + 10a_{02}a_{20} - 2a_{02}b_{11} + 3a_{20}^2 + 2a_{20}b_{11} - b_{11}^2} - 2b_{20},$$

the other case is treated similarly.

We first look for an invariant line

$$F_1 = 1 + h_2x + h_3y$$

of system (17) with the cofactor of the form (9).

Computations yield

$$F_1 = 1 + \frac{4a_{20}b_{20} - aa_{02} - aa_{20} + 4a_{02}b_{20}}{4(a_{02} + a_{20})}x + \frac{1}{4}(b_{11} - 5a_{02} - 3a_{20})y$$

and

$$K_1 = \frac{1}{16}\left(4(-5a_{02}-3a_{20}+b_{11})x - 5aa_{02}x^2 - 7aa_{20}x^2 + ab_{11}x^2 + 16a_{20}b_{20}x^2 \right.$$
$$+ 4ay + 16ab_{20}xy + aa_{02}y^2 + 3aa_{20}y^2 - ab_{11}y^2 + 16b_{20}y(-1+a_{02}y)$$
$$\left. -2\left(17a_{02}^2 + 3(3a_{20}-b_{11})(a_{20}+b_{11}) + 2a_{02}(15a_{20}+b_{11}) + 8b_{20}^2\right)xy\right).$$

We then look for an invariant curve of degree two,

$$H_2 = 1 + h_1x + h_2y + h_3x^2 + h_4xy + h_5y^2, \tag{18}$$

with the cofactor of the form (9). It turns out the system for determining coefficients $c_1, c_2, h_1, h_2, h_3, h_4, h_5$ is so complicated that we are not able to complete computations using our computational facilities. However, computing invariant curves for some fixed values of parameters of system (17) we observe that for these values the system admits an integrating factor of the form

$$\mu = H_1 H_2^{-5/2}. \tag{19}$$

Thus, we conjecture that (17) has an invariant curve of degree two and an integrating factor of the form (19) also for generic values of parameters. But then we have that the cofactor K_2 of the curve H_2 is

$$K_2 = \frac{2}{5}(K_1 + P'_x + Q'_y).$$

Using this cofactor we easily find that any system (17) admits the invariant curve of degree two,

$$H_2 = 1 + \left(-\frac{a}{2} + 2b_{20}\right)x\left(-\frac{a_{02}}{2} + \frac{a_{20}}{2} + \frac{b_{11}}{2}\right)y +$$
$$+\frac{1}{16}\left(15a_{02}^2 + 26a_{02}a_{20} + 7a_{20}^2 + 2a_{02}b_{11} + 6a_{20}b_{11} - b_{11}^2 - 8ab_{20} + 16b_{20}^2\right)x^2 +$$
$$\frac{1}{8}(-3aa_{02} - 5aa_{20} - ab_{11} - 4a_{02}b_{20} + 4a_{20}b_{20} + 4b_{11}b_{20})xy +$$
$$\frac{1}{16}\left(-11a_{02}^2 - 18a_{02}a_{20} - 3a_{20}^2 - 6a_{02}b_{11} - 2a_{20}b_{11} + b_{11}^2\right)y^2,$$

and the integrating factor (19).

The integrating factor which we constructed in this case is not defined if $a_{02} + a_{20} = 0$, and the integrating factor of Case b) is not defined if $b_{11}^2 - 4\left(b_{20}^2 + r_{11}\right) = 0$. However, similar as in Case a), since the set of integrable systems is closed in the Zariski topology, we conclude that the first integrals of the form (2) also exist when the conditions mentioned above hold. ■

To summarize, we have shown that under each of four conditions of Theorem 1 the corresponding system is locally analytically integrable. However, since the modular computations were involved we cannot claim that conditions of Theorem 1 represent the full list of conditions for local analytic integrability, since some components of the decomposition of the variety $V(B)$ can be lost under modular computation. To verify that conditions a)–d) of Theorem 1 are all necessary conditions for local integrability one can find the irreducible decomposition of $V(B)$ over the field of rational numbers (which appears unfeasible with present computational facilities) or to check whether the conditions of Theorem 1 are equivalent to conditions obtained in [2] (which also is a challenging problem).

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