STATISTICAL APPROACH OF MODULATION INSTABILITY IN THE CLASS OF NLS EQUATIONS

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In this review the modulation instability of NLS type equations is investigated using the statistical approach developed by Alber (1978). Two representative equations are studied, namely the well known cubic NLS eq. and a derivative NLS one. Improving the existing results in the literature (for dNLS eq.) only the Gaussian approximation is used in the decoupling of higher order two point correlation functions that appear in writing the kinetic equation satisfied by the two-point correlation function $\rho(1,2) =< \Psi(x_1)\Psi^*(x_2) >$. Explicit results are obtained for a Lorentzian initial distribution function, the instability being restricted to the long wave lengths region of the perturbations and the long range correlations in the initial state. This seams to be an universal behavior in this class of equations.

Key words: Modulation instability, mKdV equation, statistical approach.

1. INTRODUCTION

The Modulation Instability (MI), also known as Benjamin-Feir-Leighthill instability, is probably the most known instability in the nature. As mentioned recently by Zakharov and Ostrovsky (2009) "*there are between one and two million entries on "Modulation Instability", and even more on "Self modulation" in, e.g. Yahoo.*" Then why a new paper on this subject? We shall mention two reasons for this. Firstly, the statistical approach used represents only a very small fraction from the million paper existing. A second reason will be seen a little later.

The old history of the subject begins in the middle of the sixties of the last century, in hydrodynamics/nonlinear optics in western countries and Soviet Union. The beginnings is beautifully reviewed in the mentioned paper of Zakharov and Ostrovski [1]. Let us remember from this huge list few references, namely [2–4] for MI in thermodynamics and [5–8] for MI in electrodynamics/nonlinear dispersive systems. For more details we refer to [1] and to few monographs where applications in various domains of physics can be found [9–12].

Two distinct representatives of NLS equations will be discussed, namely the well known cubic NLS equation

$$i\frac{\partial\Psi}{\partial t} + \alpha\frac{\partial^2\Psi}{\partial x^2} + \beta|\Psi|^2\Psi = 0 \tag{1}$$

and a derivative NLS equation (dNLS)

$$i\frac{\partial\Psi}{\partial t}+\alpha\frac{\partial^2\Psi}{\partial x^2}+i\beta|\Psi|^2\frac{\partial\Psi}{\partial x}=0 \tag{2}$$

Both of them describe completely integrable systems. The first described the propagation of quasi-monochromatic wave in a weak nonlinear medium, being a well known representative of a completely integrable system [13–15]. The complete integrability of the second was proved in [16, 17] (for α and β of opposite signs).

The MI can be discussed in two different and complementary ways. The first and the most common one, is a deterministic approach (DAMI). In its simplest form one considers a Stokes solution (a plane wave with constant amplitude, but with an amplitude dependent frequency) of the nonlinear equation, $\Psi_S = a\exp[i(kx-\omega t)]$, with $\omega=\alpha k^2-\beta|a|^2$ for NLS eq. (1), and $\omega=\alpha k^2+\beta|a|^2k$ for dNLS eq. (2). This Stokes solution is unstable at small modulations of the amplitude, $\Psi = \Psi_S(1+\epsilon\phi(x,t))$, and looking for plane wave solutions $\phi = A\exp[i(Qx-\Omega t)]+B^*\exp[-i(Qx-\Omega^* t)]$ the instability is associated with the positive imaginary part of Ω. Solving the linear equation satisfied by $\phi(x,t)$ one gets

$$\Omega_i = |\alpha Q|\sqrt{2\frac{\beta}{\alpha}|a|^2-Q^2} \tag{3}$$

for the NLS case, and

$$\Omega_i = |\alpha Q|\sqrt{2\frac{\beta}{\alpha}|a|^2k-Q^2} \tag{4}$$

for dNLS one. In the NLS case the instability takes place if α and β have the same sign (focusing case of NLS eq.) and is restricted to the long wave length region $Q^2 \le 2\frac{\beta}{\alpha}|a|^2$. For dNLS eq. the instability exists if α and β have opposite signs (integrability case of dNLS eq.) and the instability region is growing with the wave vector of the carrying wave, $Q^2 \le 2|\frac{\beta}{\alpha}||a|^2k$.

Practically in the same time with the deterministic approach of MI, a complementary approach, based on the study of wave-wave energy transfer within a broad spectrum in a nearly homogeneous medium developed, especially in hydrodynamics [18–20]. But in the present paper we are interested in the statistical approach of MI (SAMI) introduced by Alber [21], who tried to make a bridge between the deterministic and random schools. One starts from the kinetic equation satisfied by the two-point correlation function $\rho(x_1,x_2) =<\Psi(x_1)\Psi^*(x_2)>$, where $<...>$ denotes an ensemble average. In a linear approximation in a (k,x) space an implicit dispersion relation is found and MI is determined for various initial distribution functions.

Developed further in [22] the method was applied to discuss different physical problems as quantum like description of coherent instability in charged particle beams [23–25], surface gravity waves [26, 27], plasma physics [28–30] and nonlinear optics [31–36].

In a series of papers we extended SAMI to discrete systems [37, 38], coupled nonlinear equations [39], derivative NLS equations [40, 41], cylindrical and spherical NLS equations [42, 43].

Now we can discuss the second reason to writing the present paper. As will be seen further in both cases, NLS and dNLS, only one approximation is done in writing the kinetic equation satisfied by the two-point correlation function $\rho(1,2)$, namely a Gaussian approximation for the decoupling of some higher order two-point correlation functions. This procedure is improving our previous results on the dNLS case [40, 41], where a supplementary limiting procedure was used. In this way the whole class of NLS equations can be treated with the same degree of approximation.

The present paper is organized as follows. In the next two sections the SAMI for NLS and dNLS equations will be presented and explicit calculations will be done for a Lorentzian initial distribution. As mentioned before the only approximation done in both cases is the Gaussian decoupling of higher order two-point correlation functions. A final section of conclusions and further possible developments is closing the paper.

2. SAMI FOR NLS EQUATION

We start by writing a kinetic equation for the two-point correlation function $\rho(1,2) = \langle \Psi(x_1)\Psi^*(x_2)\rangle$. This is done in several steps in the following ways [21]:

- write the NLS eq. for $x = x_1$ and multiply by $\Psi^*(x_2)$;
- write the complex conjugated NLS eq. for $x = x_2$ and multiply by $\Psi(x_1)$;
- add the two resulting equations and take an ensemble average;
- the last step is to decouple the higher order two-point correlation functions using a "Gaussian approximation", namely

$$\left\langle |\Psi(x_1)|^2 \Psi(x_1)\Psi^*(x_2)\right\rangle \simeq 2n(1)\rho(1,2)$$

$$\left\langle |\Psi(x_2)|^2 \Psi(x_1)\Psi^*(x_2)\right\rangle \simeq 2n(2)\rho(1,2)$$

where

$$n(x_1) = n(1) = \left\langle |\Psi(x_1)|^2 \right\rangle .$$

We get

$$i\frac{\partial \rho(1,2)}{\partial t} + \alpha\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right)\rho(1,2) + 2\beta(n(1)-n(2))\rho(1,2) = 0. \quad (5)$$

The next step is to introduce the relative coordinate $x = x_1 - x_2$ and the center of mass one, $X = \frac{1}{2}(x_1 + x_2)$ and to make a Fourier transform with respect to x.

$$\begin{aligned} F(k,X,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ikx}\rho(x,X,t)\mathrm{d}x \\ \rho(x,X,t) &= \int_{-\infty}^{+\infty} e^{-ikx}F(k,X,t)\mathrm{d}k. \end{aligned} \tag{6}$$

This procedure is known as the Wigner-Moyal transform [44] (also known as the Klimontovich statistical average method [45]). Here $F(k,X,t)$ is the Wigner function and it is easily seen to be a real function.

The kinetic equation (5) transforms into

$$\frac{\partial F(k,X,t)}{\partial t} + 2\alpha k\frac{\partial F(k,X,t)}{\partial X} + 4\beta n(X,t)\sin\left(\frac{1}{2}\overleftarrow{\frac{\partial}{\partial X}}\overrightarrow{\frac{\partial}{\partial k}}\right)F(k,X,t) = 0. \tag{7}$$

Here the $\sin(...)$ operator is defined by its Taylor expansion and the arrows are indicating the direction in which the derivatives are acting.

One assumes that in the initial state the system is homogeneous and isotropic in space and do not depend on time. As consequence the initial two-point correlation function will be a function depending only on $|x|$, namely

$$\rho_0(x_1,x_2) = \rho_0(|x|) \tag{8}$$

and its Fourier transform $f(k)$ will be an even function on k.

A linear stability analysis around this initial state will be easily done. We write

$$\begin{aligned} F(k,X,t) &= f(k) + \epsilon F_1(k,X,t), \\ n(X,t) &= n_0 + \epsilon n_1(X,t), \end{aligned} \tag{9}$$

where

$$\begin{aligned} n_0 &= \int_{-\infty}^{+\infty} f(k)\mathrm{d}k, \\ n_1(X,t) &= \int_{-\infty}^{+\infty} F_1(k,X,t)\mathrm{d}k. \end{aligned} \tag{10}$$

Then the linear equation satisfied by $F_1(k,X,t)$ is

$$\frac{\partial F_1(k,X,t)}{\partial t} + 2k\alpha\frac{\partial F_1(k,X,t)}{\partial X} + 4\beta n_1(X,t)\sin\left(\frac{1}{2}\overleftarrow{\frac{\partial}{\partial X}}\overrightarrow{\frac{\partial}{\partial k}}\right)f(k) = 0. \tag{11}$$

A plane wave solution of eq. (11) will be considered

$$\begin{aligned} F_1(k,X,t) &= g(k)e^{i(QX-\Omega t)}, \\ n_1(X,t) &= Ge^{i(QX-\Omega t)}, \\ G &= \int_{-\infty}^{+\infty} g(k)\mathrm{d}k \end{aligned} \tag{12}$$

and the instability is associated with $\mathrm{Im}\,\Omega > 0$. Straightforward calculations leads to the following implicit dispersion relation

$$\begin{aligned} 1+\frac{\beta}{\alpha Q}\int_{-\infty}^{+\infty}\frac{f(k+\frac{Q}{2})-f(k-\frac{Q}{2})}{k-\omega}\mathrm{d}k &= 0, \\ \omega &= \frac{\Omega}{2\alpha Q}. \end{aligned} \tag{13}$$

Considering (α, Q) both positive quantities, the instability requires $\mathrm{Im}\,\omega > 0$.

Different initial distributions $f(k)$ will be considered. If f is a δ-function distribution, $f(k) = n_0\delta(k)$, from (13) we get

$$\Omega_i = \alpha Q\sqrt{4\frac{\beta}{\alpha}n_0 - Q^2}$$

which is exactly the result (3) obtained in the deterministic approach of MI ($n_0 \to \frac{1}{2}|a|^2$).

More interesting is a Lorentzian initial distribution function

$$f(k) = \frac{n_0}{\pi}\frac{p}{k^2+p^2}, \tag{14}$$

corresponding to an exponential decaying ρ_0

$$\rho_0 = n_0 e^{-p|x|}.$$

The integral in (13) is easily done in the complex k-space and one obtains

$$\omega_i = \sqrt{\frac{\beta}{\alpha}n_0 - \frac{Q^2}{4}} - p. \tag{15}$$

The instability is again restricted to the long wave length region and for long correlations in the initial state (small p)

$$\frac{Q^2}{4} < \frac{\beta}{\alpha}n_0 - p^2. \tag{16}$$

The effect of statistical properties of the medium (parameter p of the initial distribution function) is to diminish the instability range corresponding to a DAMI analysis. If the correlation in the initial state is shorter than a critical value, corresponding to $p^2 = p_0^2 = \frac{\beta}{\alpha}n_0$ the instability vanishes.

3. SAMI FOR DNLS EQUATION

To obtain the kinetic equation satisfied by the two-point correlation function $\rho(1,2)$ in the case of dNLS equation (2) we follow the same steps as in the previous section. One obtains

$$i\frac{\partial\rho(1,2)}{\partial t}+\alpha\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}\right)\rho(1,2)+i\beta(n(1)\frac{\partial}{\partial x_1}+n(2)\frac{\partial}{\partial x_2})\rho(1,2)+$$

$$i\beta(m^*(1)+m(2))\rho(1,2)=0. \tag{17}$$

Here beside the quantity $n(x)=\left\langle|\Psi(x)|^2\right\rangle$ we introduced

$$m(x)=\left\langle\Psi(x)\frac{\partial\Psi^*(x)}{\partial x}\right\rangle. \tag{18}$$

It is a complex quantity appearing naturally in the decoupling of higher order two-point correlation functions, namely

$$\left\langle|\Psi(x_1)|^2\frac{\partial\Psi(x_1)}{\partial x_1}\Psi^*(x_2)\right\rangle\simeq n(1)\frac{\partial\rho(1,2)}{\partial x_1}+m^*(1)\rho(1,2),$$

$$\left\langle|\Psi(x_2)|^2\frac{\partial\Psi^*(x_2)}{\partial x_2}\Psi(x_1)\right\rangle\simeq n(2)\frac{\partial\rho(1,2)}{\partial x_2}+m(2)\rho(1,2).$$

This decoupling procedure is different from the procedure used in our previous papers [40, 41][†]. We consider it to be the most correct one, as only the Gaussian approximation was used. This new quantity is related to $n(x)$. Indeed, from the definition (18) we have

$$m(x)+m^*(x)=\frac{\partial n(x)}{\partial x}. \tag{19}$$

A second relation (giving the imaginary part of $m(x)$) is obtained considering the conservation law for dNLS eq.

$$i\frac{\partial n(x)}{\partial t}+\alpha\frac{\partial}{\partial x}\left[\left\langle\frac{\partial^2\Psi}{\partial x^2}\Psi^*\right\rangle-\left\langle\Psi\frac{\partial^2\Psi^*}{\partial x^2}\right\rangle\right]+i\frac{\beta}{2}\frac{\partial}{\partial x}\left\langle|\Psi|^4\right\rangle=0,$$

which can be written

$$\frac{1}{\alpha}\frac{\partial n(x)}{\partial t}+\frac{\partial}{\partial x}i(m(x)-m^*(x))+2\frac{\beta}{\alpha}n(x)\frac{\partial n(x)}{\partial x}=0, \tag{20}$$

(the Gaussian approximation was used to write $\left\langle|\Psi|^4\right\rangle\simeq 2n^2(x)$.)

[†]In our previous papers we considered

$$\left\langle\Psi(x_1)\frac{\partial\Psi^*(x_1)}{\partial x_1}\right\rangle=\lim_{x_2\to x_1}\frac{\partial\rho(1,2)}{\partial x_1},\quad \left\langle\Psi(x_2)\frac{\partial\Psi^*(x_2)}{\partial x_2}\right\rangle=\lim_{x_1\to x_2}\frac{\partial\rho(1,2)}{\partial x_2}$$

containing an additional limit procedure. This new decoupling procedure was first used by A.T. Grecu in his PhD Thesis (Univ. Bucharest, 2010).

The next step is to apply the Wigner-Moyal transform in the eq. (17). Expanding all the quantities $n(1)$, $n(2)$, $m(1)$, $m(2)$ in Taylor series around the central point $X = \frac{1}{2}(x_1 + x_2)$ and taking a Fourier transform, after straightforward calculations one obtains

$$\frac{\partial F(k,X,t)}{\partial t} + 2\alpha k \frac{\partial F(k,X,t)}{\partial X} + \beta n(X,t) \cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)\frac{\partial F}{\partial X}$$
$$-2\beta n(X,t)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)(kF) + \beta\frac{\partial n(X,t)}{\partial X}\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)F \tag{21}$$
$$-i\beta(m(X) - m^*(X))\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)F = 0.$$

In the linear approximation the eq. (21) becomes

$$\frac{\partial F_1(k,X,t)}{\partial t} + 2\alpha\left(k + \frac{\beta}{2\alpha}n_0\right)\frac{\partial F_1}{\partial X} - 2\beta n_1(X,t)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)h(k) +$$
$$+\beta\frac{\partial n_1}{\partial t}\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)f(k) - i\beta(m(X) - m^*(X))\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)f(k) = 0, \tag{22}$$

where $h(k) = kf(k)$.

In the same linear approximation the conservation law (20) writes

$$\frac{1}{\alpha}\frac{\partial n_1(X,t)}{\partial t} + \frac{\partial}{\partial X}i(m(X) - m^*(X))_1 + 2\frac{\beta}{\alpha}n_0\frac{\partial n_1}{\partial X} = 0. \tag{23}$$

Now using (23) the quantity $i(m(X) - m^*(X))_1$ can be eliminated from (22). We remain with the following linear equation satisfied by $F_1(k,X,t)$

$$\frac{\partial^2 F_1(k,X,t)}{\partial t \partial X} + 2\alpha\left(k + \frac{\beta}{2\alpha}n_0\right)\frac{\partial^2 F_1}{\partial X^2} - 2\beta\frac{\partial n_1(X,t)}{\partial X}\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)h(k) +$$
$$\beta\frac{\partial^2 n_1}{\partial X^2}\cos\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)f(k) + \frac{\beta}{\alpha}\left(\frac{\partial n_1}{\partial t} + 2\beta n_0\frac{\partial n_1}{\partial X}\right)\sin\left(\frac{1}{2}\frac{\overleftarrow{\partial}}{\partial X}\frac{\overrightarrow{\partial}}{\partial k}\right)f(k) = 0. \tag{24}$$

Considering a plane wave solution (12) after few transformations and an integration over k space the following implicit dispersion relation is obtained

$$1 + \frac{\beta}{2\alpha}\left[\left(\omega - \frac{\beta}{2\alpha}n_0\right)I - J + K\right] = 0, \tag{25}$$

where $\omega = \frac{\Omega}{2\alpha Q} - \frac{\beta}{2\alpha} n_0$, and the integrals I, J, K are given by

$$I = \frac{1}{Q}\int_{-\infty}^{+\infty} \frac{f(k+\frac{Q}{2}) - f(k-\frac{Q}{2})}{\omega - k} \mathrm{d}k = -\int_{-\infty}^{+\infty} \frac{f(k)}{(\omega-k)^2 - \frac{Q^2}{4}} \mathrm{d}k$$

$$J = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{f(k+\frac{Q}{2}) + f(k-\frac{Q}{2})}{\omega - k} \mathrm{d}k = \int_{-\infty}^{+\infty} \frac{(\omega-k)f(k)}{(\omega-k)^2 - \frac{Q^2}{4}} \mathrm{d}k \tag{26}$$

$$K = \frac{1}{Q}\int_{-\infty}^{+\infty} \frac{h(k+\frac{Q}{2}) - h(k-\frac{Q}{2})}{\omega - k} \mathrm{d}k = -\int_{-\infty}^{+\infty} \frac{kf(k)}{(\omega-k)^2 - \frac{Q^2}{4}} \mathrm{d}k$$

As in the previous section the instability is associated with positive imaginary part of ω (one assumes both α and Q to be positive quantities).

It is easily seen that for an initial δ-function distribution $f(k) = n_0\delta(k)$, the dispersion relation (25) has only real solutions, and consequently no instability exists.

For a Lorentzian initial distribution (14) the integrals I, J, K are easily done in k-complex plane. One gets

$$I = \frac{-n_0}{(\omega+ip)^2 - \frac{Q^2}{4}}, \quad J = \frac{(\omega+ip)n_0}{(\omega+ip)^2 - \frac{Q^2}{4}}, \quad K = \frac{ipn_0}{(\omega+ip)^2 - \frac{Q^2}{4}}. \tag{27}$$

It is convenient to introduce the new complex variable $z = \omega + ip$. Then the dispersion relation (25) becomes the following second order algebraic equation in z

$$z^2 - \frac{Q^2}{4} + \gamma(2z + \gamma - 2ip) = 0. \tag{28}$$

Here we considered $\alpha > 0$, $\beta < 0$ (integrable case of dNLS eq.) and have denoted $\gamma = |\frac{\beta}{2\alpha}|n_0$. Separating the real and the imaginary part of eq. (28) ($z = \xi + i\eta$, $\eta = \omega_i + p$) we get

$$\xi^2 - \eta^2 - \frac{Q^2}{4} + 2\gamma\xi + \gamma^2 = 0,$$
$$\xi\eta = -\gamma(\eta - p).$$

Eliminating the real part we remain with the following equation satisfied by η

$$\eta^4 + \frac{Q^2}{4}\eta^2 - \gamma^2 p^2 = 0,$$

from which the positive imaginary part of ω is given by

$$\omega_i = \frac{1}{\sqrt{2}}\sqrt{\sqrt{(\frac{Q^2}{4})^2 + 4\gamma^2 p^2} - \frac{Q^2}{4}} - p. \tag{29}$$

This is greater than zero in the long wave-length limit

$$\frac{Q^2}{4} < \gamma^2 - p^2, \tag{30}$$

and as in the NLS case the influence of the (long range) correlation parameter p is to reduce the instability domain.

4. CONCLUSIONS

Let us briefly summarize the mean results of this paper. The SAMI was discussed for two representative nonlinear Schrödinger equations, the usual cubic NLS equation (1) and a derivative NLS equation (2). In both cases only one one approximation was done, namely a Gaussian decoupling of higher order two point correlation functions. For dNLS equation this was possible supplementing the kinetic equation (17) for the two point correlation function $\rho(x_1,x_2)$ with the corresponding conservation law (20) for the dNLS equation (2). This approach can be easily generalized to the most general NLS equation

$$i\frac{\partial\Psi}{\partial t}+\alpha\frac{\partial^2\Psi}{\partial x^2}+\beta|\Psi^2|^2\Psi+i\beta_1|\Psi|^2\frac{\partial\Psi}{\partial x}+i\beta_2\frac{\partial|\Psi|^2}{\partial x}\Psi=0. \tag{31}$$

Depending on the parameters α, β, β_1, β_2 different scenarios can be drawn, but the general picture of MI restricted to the long wave length region seems to remain valid. Also the influence of the statistical properties of the medium, reflected by the initial distribution function $f(k)$, is to restrict the domain where MI exists to long range correlation region of the initial state (for small values of p if $f(k)$ is a Lorentzian distribution). The extension of similar considerations to the case when the equation (31) contains also a third order dispersion term $\alpha_1\frac{\partial^3\Psi}{\partial x^3}$ is still unsolved. The treatment given in [36] to such an equation is unsatisfactory as an incomplete decoupling procedure is used (terms similar with $m(x)$ (18) were omitted).

The next step in the development of SAMI should be to go beyond the Gaussian approximation. Such an extension is motivated by the use of NLS equation to discuss the waves in deep waters, especially to explain the generation of freak waves in oceans [26]. As was pointed by several authors (see [46] for an recent discussion) the Gaussian approximation is not sufficient and several other statistical quantities are needed to characterize the properties of the medium. The first one to be taken into account is the kurtosis, $ku(x,t)=|\Psi(x,t)|^4-2n^2(x,t)$, who is obviously zero for Gaussian process. A first attempt to go beyond Gaussian approximation was done in [47], where kinetic equations were written also for higher order (fourth order) two-point correlation functions. Treating these equations in the Gaussian approximation a kind of perturbation approach was developed. But in this treatment no higher order moments, like kurtosis, was introduced, and now our interest is to give a better approach of the problem.

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