

MKDV EQUATIONS RELATED TO THE $D_4^{(2)}$ ALGEBRA

V. S. GERDJKOV¹, D. M. MLADENOV², A. A. STEFANOV^{2,3,4}, S. K. VARBEV^{2,5}

¹Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences,
72 Tzarigradsko chaussee, Sofia 1784, Bulgaria

²Theoretical Physics Department, Faculty of Physics, Sofia University "St. Kliment Ohridski",
5 James Bourchier Blvd, 1164 Sofia, Bulgaria

³Institute of Mathematics and Informatics, Bulgarian Academy of Sciences,
Acad. Georgi Bonchev Str., Block 8, 1113 Sofia, Bulgaria

⁴Faculty of Mathematics, Sofia University "St. Kliment Ohridski",
5 James Bourchier Blvd, 1164 Sofia, Bulgaria

⁵Institute of Solid State Physics, Bulgarian Academy of Sciences,
72 Tzarigradsko chaussee, Sofia 1784, Bulgaria

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We present a one-parameter families of mKdV-type equations related to $\mathfrak{g} \simeq D_4^{(1)}$ and $\mathfrak{g} \simeq D_4^{(2)}$. They are a set of partial differential equations, integrable *via* the inverse scattering method. They admit a Hamiltonian formulation and the corresponding Hamiltonians are also given. The Riemann-Hilbert problems for the two Lax operators are formulated on a set of $2h$ rays l_ν . We show that to each ray l_ν one can relate a subalgebra of \mathfrak{g} which is direct sum of $sl(2)$ subalgebras.

Key words: Soliton equations, Integrable models, Kac-Moody algebras.

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1. INTRODUCTION

The discovery of the inverse scattering method by Gardner, Green, Kruskal and Miura [1] and the subsequent development by Lax [2] led to the birth of modern soliton science. What followed was an explosion of interest in soliton equations and non-linear systems in general (a good review on the history of solitons can be found in [3, 4]), and the development of the techniques needed to solve them. It is worth mentioning some of the milestones in this development. One such example is the paper by Ablowitz, Kaup, Newell and Segur [5], where the integrability of the Zaharov-Shabat system is proved. Another important aspect, namely that there is a connection between soliton equations and Kac-Moody algebras was discovered by Drinfeld and Sokolov [6, 7]. Quite independently the notion of the reduction group was introduced by Mikhailov [8] and used to construct new integrable systems, starting with the family of 2-dimensional Toda field theories. These two papers provide a natural ground for analyzing the connections between the Lax operators, the Kac-Moody algebras and the properties of the related nonlinear evolution equations (NLEE).

This paper is a part of a larger series of articles, where the connection between soliton equations and the Kac-Moody algebras based on the classical Lie algebra D_4 is examined. Here we present only what is commonly called "modified Korteweg - de Vries" (mKdV) equations related to the affine Kac-Moody algebras $D_4^{(1)}$ and $D_4^{(2)}$. They are a set of non-linear partial differential equations, integrable *via* the inverse scattering method. They possess soliton solutions, which can be constructed, for example by using the dressing method (for details on the dressing method, see [8, 9]).

The equations can be derived by a number of different techniques, for example by using Drinfeld-Sokolov reduction [6] or with the help of recursion operators [10–12]. They can also be computed by directly solving the corresponding recursion relations.

The paper is organized as follows. In Section 2 we give some basic preliminaries, needed for the derivation of the equations and formulate the results from our previous paper [16] concerning the mKdV equations related to the $D_4^{(1)}$ algebra. In Section 3 we derive the equations related to $D_4^{(2)}$ and in Section 4 we briefly address the spectral properties for both Lax operators. Section 4 contains some concluding remarks.

2. PRELIMINARIES

We assume that the reader is familiar with the basic facts about finite and infinite dimensional Lie algebras. An introductory treatment can be found in [6, 13]. We will only highlight the information needed for the present paper.

Let \mathfrak{g} be a finite-dimensional simple Lie algebra. By ad_X we denote the linear operator defined by

$$\text{ad}_X(Y) = [X, Y], \quad X, Y \in \mathfrak{g}. \quad (1)$$

This operator has a kernel and can only be inverted on its image. We denote that inverse by ad_X^{-1} . If X is diagonalizable then ad_X^{-1} can be expressed as a polynomial of ad_X . Let $\langle \cdot, \cdot \rangle$ be the Killing-Cartan form on \mathfrak{g} which is defined by

$$\langle X, Y \rangle = \text{tr}(\text{ad}_X \text{ad}_Y). \quad (2)$$

Let φ be an automorphism of \mathfrak{g} of finite order. Every such φ introduces a grading in \mathfrak{g} by

$$\mathfrak{g} = \bigoplus_{k=0}^{s-1} \mathfrak{g}^{(k)}, \quad (3)$$

such that

$$\varphi(X) = \omega^k X, \quad \omega = \exp\left(\frac{2\pi i}{s}\right), \quad \forall X \in \mathfrak{g}^{(k)}, \quad (4)$$

Table 1

A realization of the Coxeter automorphisms and Coxeter numbers for $D_4^{(1)}$, $D_4^{(2)}$ and $D_4^{(3)}$. Here S_{α_i} denotes reflection with respect to the simple root α_i , R is the second order outer automorphism that exchanges α_3 and α_4 , and T is the third order outer automorphism (a triality transformation) that sends $\alpha_1 \rightarrow \alpha_3 \rightarrow \alpha_4$.

Algebra	Coxeter automorphism	Coxeter number	Exponents
$D_4^{(1)}$	$C_1 = S_{\alpha_2} S_{\alpha_1} S_{\alpha_3} S_{\alpha_4}$	6	1, 3, 5, 3
$D_4^{(2)}$	$C_2 = S_{\alpha_2} S_{\alpha_1} S_{\alpha_3} R$	8	1, 3, 5, 7
$D_4^{(3)}$	$C_3 = S_{\alpha_2} S_{\alpha_1} T$	12	1, 5, 7, 11

where s is the order of φ . Obviously the grading condition holds

$$[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}] \subset \mathfrak{g}^{(k+l)}, \quad (5)$$

where $k+l$ is taken modulo s . As is well known, Kac-Moody algebras are graded algebras (for a definitions and details see [6, 14, 15]). We will use the definition of a Kac-Moody algebra given in [6]. In our case the gradings are done using the Coxeter automorphisms given in Table 1.

A basis compatible with the grading is given by averaging the standard Cartan-Weyl basis of D_4 (for more details see [16]) over the action of the corresponding Coxeter automorphism

$$\begin{aligned} \mathcal{E}_i^{(k)} &= \sum_{s=0}^5 \omega_1^{-sk} C_1^s(E_{\alpha_i}), & \mathcal{H}_j^{(k)} &= \sum_{s=0}^5 \omega_1^{-sk} C_1^s(H_j), & \text{for } D_4^{(1)}, \\ \tilde{\mathcal{E}}_i^{(k)} &= \sum_{s=0}^7 \omega_2^{-sk} C_2^s(E_{\alpha_i}), & \mathcal{H}_j^{(k)} &= \sum_{s=0}^7 \omega_2^{-sk} C_2^s(H_j), & \text{for } D_4^{(2)}, \end{aligned} \quad (6)$$

where $\omega_1 = \exp(2\pi i/6)$, $\omega_2 = \exp(2\pi i/8)$. H_i are the basis elements of the Cartan subalgebra of D_4 and E_{α_i} is the Weyl generator corresponding to the simple root α_i . Note that $\mathcal{H}_j^{(k)}$ is non-vanishing only if k is an exponent. This means that the dimension of each subspace $\mathfrak{g}^{(k)}$ is equal to $r + m_s$ where m_s is the multiplicity of the exponent s and r is the rank of the corresponding algebra.

Remark 1 *It is important to note, that basis in (6) is correct provided we have chosen the Coxeter automorphism in its dihedral form. For the $D_4^{(1)}$ it is*

$$C = w_1 w_2, \quad w_1 = S_{\alpha_2}, \quad w_2 = S_{\alpha_1} S_{\alpha_3} S_{\alpha_4}. \quad (7)$$

where $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$ and $\alpha_4 = e_3 + e_4$ are the simple roots $D_4^{(1)}$. With this choice one can check that through each simple root passes just one orbit of C [15].

The explicit form of (6) for $D_4^{(1)}$ and $D_4^{(2)}$ can be found in [16] and [17] correspondingly.

2.1. LAX PAIR AND RECURSION RELATIONS

We will consider a Lax pair of the following form:

$$\begin{aligned} L &= i\partial_x + Q(x, t) - \lambda J, \\ M &= i\partial_t + V^{(0)}(x, t) + \lambda V^{(1)}(x, t) + \lambda^2 V^{(2)}(x, t) - \lambda^3 K, \end{aligned} \quad (8)$$

i.e. the potentials of L and M are elements of some affine Kac-Moody algebra (in our case $D_4^{(1)}$ and $D_4^{(2)}$). The Lax equation

$$[L, M] = 0 \quad (9)$$

must hold identically with respect to λ . If the inverse scattering problem for L is uniquely solvable, then we have an effective tool not only to solve the relevant NLEE, but also a strong indication of its complete integrability. For more details in this respect in the special cases of \mathbb{Z}_h -reductions see [10–12, 18].

In fact equation (9) implies the following set of recursion relations

$$\begin{aligned} \lambda^4 : & \quad [J, K] = 0, \\ \lambda^3 : & \quad [J, V^{(2)}] + [Q, K] = 0, \\ \lambda^s : & \quad i\frac{\partial V^{(s)}}{\partial x} + [Q, V^{(s)}] - [J, V^{(s-1)}] = 0, \\ \lambda^0 : & \quad -i\frac{\partial Q}{\partial t} + i\frac{\partial V^{(0)}}{\partial x} + [Q(x, t), V^{(0)}] = 0. \end{aligned} \quad (10)$$

where $s = 1, 2$. They can be solved with the use of recursion operators (see [10, 11]), which can be used to construct the whole hierarchy of soliton equations. For the cases of $D_4^{(1)}$ and $D_4^{(2)}$ those hierarchies can be found in [16, 17]. Here we will only present the first members - a set of mKdV equations.

The resulting equations can be presented in Hamiltonian form as follows:

$$\partial_t q_i = \{q_i, H\} \quad (11)$$

with a Poisson bracket given by

$$\{F, G\} = \int_{\mathbb{R}^2} \omega_{ij}(x, y) \frac{\delta F}{\delta q_i} \frac{\delta G}{\delta q_j} dx dy, \quad (12)$$

where we sum over repeating indexes. The Poisson structure tensor is

$$\omega_{ij}(x, y) = \frac{1}{2} \delta_{ij} (\partial_x \delta(x - y) - \partial_y \delta(x - y)). \quad (13)$$

In this case (11) reduces to

$$\partial_t q_i = \partial_x \frac{\delta H}{\delta q_i}. \quad (14)$$

The Hamiltonian densities can be found at the end of the corresponding sections.

2.2. MKDV EQUATIONS RELATED TO $D_4^{(1)}$

The equations related to $D_4^{(1)}$ can be found in [19], but for the sake of completeness, besides the equations here we will also give the elements of the potential of L and M . We will use the Lax pair given by (8). The potential of L is parametrized by

$$Q = \begin{pmatrix} 0 & q_1 & q_1 & -q_4 & -q_3 & q_2 & -q_2 & 0 \\ -q_1 & 0 & q_2 & -q_3 & -q_4 & q_1 & 0 & -q_2 \\ -q_1 & -q_2 & 0 & q_3 & q_4 & 0 & q_1 & -q_2 \\ q_4 & q_3 & -q_3 & 0 & 0 & q_4 & q_4 & -q_3 \\ q_3 & q_4 & -q_4 & 0 & 0 & q_3 & q_3 & -q_4 \\ -q_2 & -q_1 & 0 & -q_4 & -q_3 & 0 & q_2 & -q_1 \\ q_2 & 0 & -q_1 & -q_4 & -q_3 & -q_2 & 0 & q_1 \\ 0 & q_2 & q_2 & q_3 & q_4 & q_1 & -q_1 & 0 \end{pmatrix}, \quad (15)$$

and $J = \text{diag}(1, \omega_1, \omega_1^5, 0, 0, -\omega_1^5, -\omega_1, -1)$ with $\omega_1 = \exp(2\pi i/6)$.

As for M , the elements $V^{(k)}$ are given as a linear combination of the basis elements in $\mathfrak{g}^{(k)}$ with some coefficients $v_i^{(k)}$ and $K = \text{diag}(a, -a, -a, b, -b, a, a, -a)$. Solving (10) gives for $V^{(2)}$ we obtain:

$$v_1^{(2)} = 2\omega a q_1, \quad v_2^{(2)} = 0, \quad v_3^{(2)} = -\omega(a+b)q_3, \quad v_4^{(2)} = -\omega(a-b)q_4. \quad (16)$$

Next for $V^{(1)}$ we get

$$\begin{aligned} v_1^{(1)} &= \frac{2a}{2-\sqrt{2}} \left(2(1-\omega^3)\partial_x q_1 + \frac{\sqrt{2}}{4} \left(\omega(\sqrt{2}-3) - \omega^2(\sqrt{2}+3) \right) q_2 q_3 \right. \\ &\quad \left. - \frac{\sqrt{2}}{4} (1-\omega^3) q_1 q_2 \right), \\ v_2^{(1)} &= \frac{a}{2} \left(-4\sqrt{2}\partial_x q_2 + (q_3 - q_1) \left((1+\sqrt{2})q_1 + (1-\sqrt{2})q_3 \right) \right), \\ v_3^{(1)} &= i \frac{2a}{2+\sqrt{2}} \left(2(1+\omega^3)\partial_x q_3 - \frac{\sqrt{2}}{4} \left(\omega(\sqrt{2}+3) + \omega^2(\sqrt{2}-3) \right) q_1 q_2 \right. \\ &\quad \left. + \frac{\sqrt{2}}{4} (1+\omega^3) q_2 q_3 \right), \\ v_4^{(1)} &= \frac{a}{4} \left((1+\sqrt{2})q_1^2 + q_2^2 + (1-\sqrt{2})q_3^2 \right). \end{aligned} \quad (17)$$

Note that $v_5^{(1)}$ can formally be interpreted as "particle density" and

$$D = \int_{-\infty}^{\infty} \left(a q_1^2 - \frac{1}{2}(a+b)q_3^2 - \frac{1}{2}(a-b)q_4^2 \right) dx \quad (18)$$

is an integral of motion. Finally the solution for $V^{(0)}$ is

$$\begin{aligned} v_1^{(0)} &= 2a(\partial_x^2 q_1 - \sqrt{3}q_1 \partial_x q_2) - \sqrt{3}((3a+b)q_4 \partial_x q_3 + (3a-b)q_3 \partial_x q_4) \\ &\quad - 3q_1(2a q_2^2 - (a-b)q_3^2 - (a+b)q_4^2), \\ v_2^{(0)} &= \sqrt{3}a \partial_x q_1^2 - \frac{\sqrt{3}}{2}(a+b) \partial_x q_3^2 - \frac{\sqrt{3}}{2}(a-b) \partial_x q_4^2 \\ &\quad - 3q_2(2a q_1^2 - (a+b)q_3^2 - (a-b)q_4^2), \\ v_3^{(0)} &= -(a+b)(\partial_x^2 q_3 - \sqrt{3}q_3 \partial_x q_2) + \sqrt{3}((3a+b)q_4 \partial_x q_1 + 2bq_1 \partial_x q_4) \\ &\quad - 3q_3(2a q_4^2 - (a-b)q_1^2 - (a+b)q_2^2), \\ v_4^{(0)} &= -(a-b)(\partial_x^2 q_4 - \sqrt{3}q_4 \partial_x q_2) + \sqrt{3}((3a-b)q_3 \partial_x q_1 - 2bq_1 \partial_x q_3) \\ &\quad - 3q_4(2a q_3^2 - (a-b)q_2^2 - (a+b)q_1^2). \end{aligned} \quad (19)$$

And finally, the equations are

$$\begin{aligned} \partial_t q_1 &= \partial_x \left(2a(\partial_x^2 q_1 - \sqrt{3}q_1 \partial_x q_2) - \sqrt{3}((3a+b)q_4 \partial_x q_3 + (3a-b)q_3 \partial_x q_4) \right. \\ &\quad \left. - 3q_1(2a q_2^2 - (a-b)q_3^2 - (a+b)q_4^2) \right), \\ \partial_t q_2 &= \partial_x \left(\sqrt{3}a \partial_x q_1^2 - \frac{\sqrt{3}}{2}(a+b) \partial_x q_3^2 - \frac{\sqrt{3}}{2}(a-b) \partial_x q_4^2 \right. \\ &\quad \left. - 3q_2(2a q_1^2 - (a+b)q_3^2 - (a-b)q_4^2) \right), \\ \partial_t q_3 &= \partial_x \left(-(a+b)(\partial_x^2 q_3 - \sqrt{3}q_3 \partial_x q_2) + \sqrt{3}((3a+b)q_4 \partial_x q_1 + 2bq_1 \partial_x q_4) \right. \\ &\quad \left. - 3q_3(2a q_4^2 - (a-b)q_1^2 - (a+b)q_2^2) \right), \\ \partial_t q_4 &= \partial_x \left(-(a-b)(\partial_x^2 q_4 - \sqrt{3}q_4 \partial_x q_2) + \sqrt{3}((3a-b)q_3 \partial_x q_1 - 2bq_1 \partial_x q_3) \right. \\ &\quad \left. - 3q_4(2a q_3^2 - (a-b)q_2^2 - (a+b)q_1^2) \right). \end{aligned}$$

They admit a Hamiltonian whose density is linear combination $\mathcal{H} = a\mathcal{H}_1 + b\mathcal{H}_2$ with

arbitrary constants a and b of the following densities:

$$\begin{aligned} \mathcal{H}_1 = & q_1 \partial_x^2 q_1 - \frac{1}{2} (q_3 \partial_x^2 q_3 + q_4 \partial_x^2 q_4) + \frac{\sqrt{3}}{3} (2(q_1 q_2 + 3q_3 q_4) \partial_x q_1) \\ & + \frac{\sqrt{3}}{3} ((q_3^2 + q_4^2 - 2q_1^2) \partial_x q_2 + (q_2 q_3 - 3q_1 q_4) \partial_x q_3 + (q_2 q_4 - 3q_1 q_3) \partial_x q_4) \\ & - \frac{3}{2} (q_2^2 (2q_1^1 - q_3^2 - q_4^2) + q_3^2 q_4^2 - q_1^2 q_4^2 - q_1^2 q_3^2), \end{aligned} \quad (20)$$

$$\begin{aligned} \mathcal{H}_2 = & \frac{1}{2} (-q_3 \partial_x^2 q_3 + q_4 \partial_x^2 q_4) + \frac{3}{2} ((q_1^2 - q_2^2)(q_3^2 - q_4^2)) \\ & - \frac{\sqrt{3}}{3} ((3q_1 q_4 - q_2 q_3) \partial_x q_3 + (q_4^2 - q_3^2) \partial_x q_2 - (3q_1 q_3 - q_2 q_4) \partial_x q_4). \end{aligned} \quad (21)$$

One can get rid of one of the constants (say, a by replacing $t \rightarrow \frac{t}{a}$). However the effective Hamiltonian density will depend on the ratio b/a . Thus we have a one parameter family of MKdV-type Hamiltonians. This is deeply related to the fact that 3 is a double-valued exponent of $D_4^{(1)}$.

3. MKDV EQUATIONS RELATED TO $D_4^{(2)}$

Again, the Lax pair is given by (8), with the potential

$$Q = \begin{pmatrix} 0 & q_1 & -q_1 & -q_2 & -q_2 & q_3 & q_3 & 0 \\ -q_1 & 0 & q_2 & q_1 & q_3 & -q_2 & 0 & q_3 \\ q_1 & -q_2 & 0 & q_3 & q_1 & 0 & -q_2 & -q_3 \\ q_2 & -q_1 & -q_3 & 0 & 0 & q_1 & -q_3 & -q_2 \\ q_2 & -q_3 & -q_1 & 0 & 0 & q_3 & -q_1 & -q_2 \\ -q_3 & q_2 & 0 & -q_1 & -q_3 & 0 & q_2 & q_1 \\ -q_3 & 0 & q_2 & q_3 & q_1 & -q_2 & 0 & q_1 \\ 0 & -q_3 & q_3 & q_2 & q_2 & -q_1 & -q_1 & 0 \end{pmatrix}, \quad (22)$$

and $J = \text{diag}(1, -\omega^3, \omega, -i, i, -\omega, -\omega^3, -1)$.

The elements $V^{(k)}$ are given as a linear combination of the basis elements in $\mathfrak{g}^{(k)}$ with some coefficients $v_i^{(k)}$, and $K = aJ^3$. Solving (10) gives us:

For $V^{(2)}$ we have

$$v_1^{(2)} = \frac{1-i}{2-\sqrt{2}} a q_1, \quad v_2^{(2)} = a q_2, \quad v_1^{(3)} = \frac{i-1}{2+\sqrt{2}} a q_3. \quad (23)$$

For $V^{(1)}$ we get

$$\begin{aligned}
v_1^{(1)} &= \frac{2a}{2-\sqrt{2}} \left(2(1-\omega^3)\partial_x q_1 + \frac{\sqrt{2}}{4} \left(\omega(\sqrt{2}-3) - \omega^2(\sqrt{2}+3) \right) q_2 q_3 \right. \\
&\quad \left. - \frac{\sqrt{2}}{4} (1-\omega^3) q_1 q_2 \right), \\
v_2^{(1)} &= \frac{a}{2} \left(-4\sqrt{2}\partial_x q_2 + (q_3 - q_1) \left((1+\sqrt{2})q_1 + (1-\sqrt{2})q_3 \right) \right), \\
v_3^{(1)} &= i \frac{2a}{2+\sqrt{2}} \left(2(1+\omega^3)\partial_x q_3 - \frac{\sqrt{2}}{4} \left(\omega(\sqrt{2}+3) + \omega^2(\sqrt{2}-3) \right) q_1 q_2 \right. \\
&\quad \left. + \frac{\sqrt{2}}{4} (1+\omega^3) q_2 q_3 \right), \\
v_4^{(1)} &= \frac{a}{4} \left((1+\sqrt{2})q_1^2 + q_2^2 + (1-\sqrt{2})q_3^2 \right).
\end{aligned} \tag{24}$$

Solving for $V^{(0)}$ we obtain

$$\begin{aligned}
v_1^{(0)} &= \frac{2a}{2-\sqrt{2}} \left(8(1+\sqrt{2})\partial_x^2 q_1 + 3(-q_1 + (1-\sqrt{2})q_3)\partial_x q_2 \right. \\
&\quad \left. + 6(1-\sqrt{2})q_2\partial_x q_3 \right) + \frac{a}{4} (q_3^3 - 6q_1 q_3^2 - 6q_2^2 q_3 + 3q_1^2 q_3 - 3q_1 q_2^2), \\
v_2^{(0)} &= a \left(8\partial_x^2 q_2 + 3\sqrt{2} \left(((1+\sqrt{2})q_1 + q_3)\partial_x q_1 \right. \right. \\
&\quad \left. \left. - (q_1 + (1-\sqrt{2})q_3)\partial_x q_3 \right) \right) - \frac{a}{4} (12q_1 q_2 q_3 + q_2^3 + q_2 q_3^2 + q_1^2 q_2), \\
v_3^{(0)} &= \frac{2a}{2+\sqrt{2}} \left(8(1-\sqrt{2})\partial_x^2 q_3 + 3(1+\sqrt{2})q_1 - q_3 \right) \partial_x q_2 \\
&\quad + 6(1+\sqrt{2})q_2\partial_x q_1 + \frac{a}{4} (q_1^3 - 6q_1^2 q_3 - 6q_1 q_2^2 + 3q_1 q_3^2 - 3q_2^2 q_3).
\end{aligned} \tag{25}$$

And finally, the λ -independent terms in the Lax representation provide us with

the equations

$$\begin{aligned}
\partial_t q_1 &= \frac{2a}{2-\sqrt{2}} \partial_x \left(8(1+\sqrt{2}) \partial_x^2 q_1 + 3(-q_1 + (1-\sqrt{2})q_3) \partial_x q_2 \right. \\
&\quad \left. + 6(1-\sqrt{2})q_2 \partial_x q_3 \right) + \frac{a}{4} \partial_x (q_3^3 - 6q_1 q_3^2 - 6q_2^2 q_3 + 3q_1^2 q_3 - 3q_1 q_2^2), \\
\partial_t q_2 &= a \partial_x \left(8 \partial_x^2 q_2 + 3\sqrt{2} \left(((1+\sqrt{2})q_1 + q_3) \partial_x q_1 \right. \right. \\
&\quad \left. \left. - (q_1 + (1-\sqrt{2})q_3) \partial_x q_3 \right) \right) - \frac{a}{4} \partial_x (12q_1 q_2 q_3 + q_2^3 + q_2 q_3^2 + q_1^2 q_2), \\
\partial_t q_3 &= \frac{2a}{2+\sqrt{2}} \partial_x \left(8(1-\sqrt{2}) \partial_x^2 q_3 + 3((1+\sqrt{2})q_1 - q_3) \partial_x q_2 \right. \\
&\quad \left. + 6(1+\sqrt{2})q_2 \partial_x q_1 \right) + \frac{a}{4} \partial_x (q_1^3 - 6q_1^2 q_3 - 6q_1 q_2^2 + 3q_1 q_3^2 - 3q_2^2 q_3).
\end{aligned} \tag{26}$$

The above equations admit a hamiltonian with the following density

$$\begin{aligned}
\mathcal{H} &= \frac{1}{16} a \left(\left(\sqrt{2} + \frac{4}{3} \right) \left(2q_1 \partial_x^2 q_1 - (\partial_x q_1)^2 \right) + \frac{1}{3} \left(2q_2 \partial_x^2 q_2 - (\partial_x q_2)^2 \right) \right. \\
&\quad \left. - \left(\sqrt{2} - \frac{4}{3} \right) \left(2q_3 \partial_x^2 q_3 - (\partial_x q_3)^2 \right) + (2+\sqrt{2}) (q_2 \partial_x q_1^2 - 2q_1^2 \partial_x q_2) \right. \\
&\quad \left. + (2-\sqrt{2}) (q_2 \partial_x q_3^2 - 2q_3^2 \partial_x q_2) + 6\sqrt{2} q_2 (q_3 \partial_x q_1 - q_1 \partial_x q_3) \right. \\
&\quad \left. + (4q_1^3 q_3 - 6q_1^2 q_2^2 - 12q_1^2 q_3^2 - 24q_1 q_3 q_2^2 - 6q_2^2 q_3^2 + 4q_1 q_3^3 - q_2^4) \right).
\end{aligned} \tag{27}$$

4. ON THE SPECTRAL PROPERTIES OF THE LAX OPERATORS

Here we just briefly formulate the spectral properties of the Lax operators for the class of smooth potentials $Q(x)$ vanishing fast enough for $x \rightarrow \pm\infty$. Detailed derivations, concerning the construction of their fundamental analytic solutions, scattering matrix, etc. will be published elsewhere.

It is well known that solving the direct and the inverse scattering problems for the Lax operator L is based on the construction of the fundamental analytic solution $\chi_\nu(x, t, \lambda)$ which can be viewed as a solution to a Riemann-Hilbert problem (RHP). The spectral properties of the generic $n \times n$ Zakharov-Shabat system with complex-valued J have been analyzed in [18, 20, 21]. Skipping the details here we formulate the main results for constructing the FAS for the Lax operators considered above. Doing this properly requires special care treating the reduction conditions.

First of all we have to establish the regions of analyticity of $\chi_\nu(x, t, \lambda)$ and of $\xi_\nu(x, t, \lambda) = \chi_\nu(x, t, \lambda) e^{iJ\lambda x}$. These regions are in fact sectors of the complex λ -plane. The rays l_ν which separates the two sectors $\Omega_{\nu+1}$ and Ω_ν are defined by the

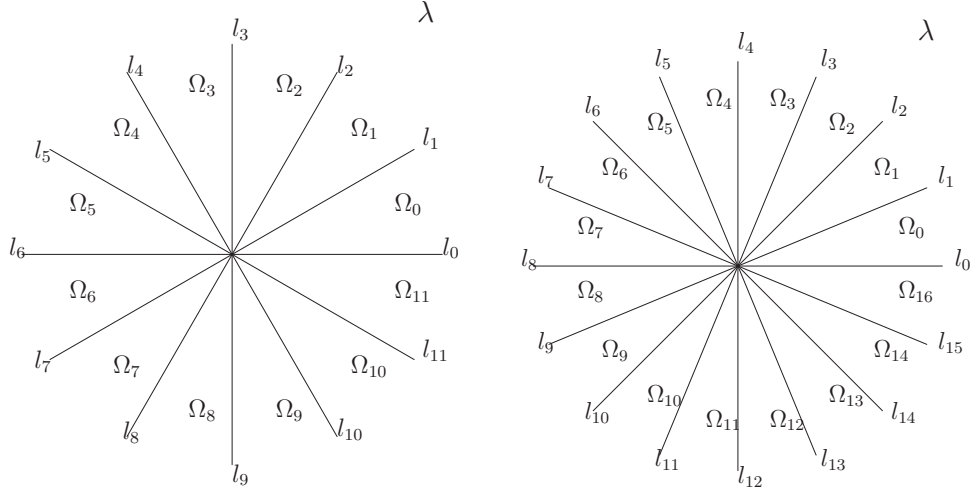


Fig. 1 – The contours $\cup_{\nu=0}^{h-1} l_\nu$ of the RHP and the analyticity sectors Ω_ν of the FAS for the Lax operators: the case of $D_4^{(1)}$ - left panel; the case of $D_4^{(2)}$ - right panel;

condition:

$$\text{Im } \lambda \alpha(J) = 0, \quad (28)$$

where α is a root of the algebra \mathfrak{g} and $J \in \mathfrak{g}^{(1)} \cap \mathfrak{h}$. Note that J carries with itself the reduction condition; indeed, $C(J) = \omega J$, where $\omega^h = 1$. Putting in eq. (28) $\lambda = |\lambda|e^{i\varphi}$ we easily obtain for each α a set of simple linear equations involving φ and the arguments of the eigenvalues of J . So for each of the Lax operators we obtain that the relevant RHP can be formulated on a set of $2h$ rays l_ν closing angles π/h , see Figure 1.

The solutions of this simple equation allows one to find that to each line $l_\nu \cup l_{\nu+6}$ we can relate a subset of the set of positive roots that are mutually orthogonal. The result is given in Table 2.

Skipping the details, we briefly reformulate the general results obtained in [10, 11] for this particular Lax operators.

A) First, in each of the sectors one can construct FAS $\chi_\nu(x, t, \lambda)$, $\lambda \in \Omega_\nu$ and calculate its limits for $x \rightarrow \pm\infty$ along the lines l_ν ; more specifically

$$\begin{aligned} \lim_{x \rightarrow -\infty} e^{i\lambda Jx} \xi_\nu(x, t, \lambda) e^{-i\lambda Jx} &= S_\nu^+(t, \lambda), \\ \lim_{x \rightarrow \infty} e^{i\lambda Jx} \xi_\nu(x, t, \lambda) e^{-i\lambda Jx} &= T_\nu^-(t, \lambda) D_\nu^+(\lambda), \end{aligned} \quad \forall \lambda \in l_\nu e^{+i0}, \quad (29)$$

Table 2

Subsets of positive roots related to the lines $l_\nu \cup l_{\nu+6}$ for $D_4^{(1)}$.

$l_0 \cup l_6$ $e_2 - e_3, e_1 \pm e_4$	$l_1 \cup l_7$ $e_1 - e_3$	$l_2 \cup l_8$ $e_1 - e_2, e_3 \pm e_4$	$l_3 \cup l_9$ $e_2 + e_3$	$l_4 \cup l_{10}$ $e_1 + e_3, e_2 \pm e_4$	$l_5 \cup l_{11}$ $e_1 + e_2$
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Table 3

Subsets of positive roots related to the lines $l_\nu \cup l_{\nu+8}$ for $D_4^{(2)}$.

$l_0 \cup l_8$ $e_2 + e_3$	$l_1 \cup l_9$ $e_1 + e_2, e_3 + e_4$	$l_2 \cup l_{10}$ $e_1 + e_4$	$l_3 \cup l_{11}$ $e_2 + e_4, e_1 - e_3$
$l_4 \cup l_{12}$ $e_2 - e_3$	$l_5 \cup l_{13}$ $e_1 - e_2, e_3 - e_4$	$l_6 \cup l_{14}$ $e_1 - e_4$	$l_7 \cup l_{15}$ $e_2 - e_4, e_1 + e_3$

and

$$\begin{aligned} \lim_{x \rightarrow -\infty} e^{i\lambda Jx} \xi_{\nu-1}(x, t, \lambda) e^{-i\lambda Jx} &= S_\nu^-(t, \lambda), \\ \lim_{x \rightarrow -\infty} e^{i\lambda Jx} \hat{\xi}_{\nu-1}(x, t, \lambda) e^{-i\lambda Jx} &= T_\nu^+(t, \lambda) D_\nu^-(\lambda), \end{aligned} \quad \forall \lambda \in l_\nu e^{-i0}, \quad (30)$$

where S_ν^\pm , T_ν^\pm and D_ν^\pm are elements of the subgroup \mathcal{G}_ν whose Lie algebra \mathfrak{g}_ν has as positive roots the subset of roots related to l_ν , see Table 2.

As a minimal set of scattering data we can use the limits of the FAS along both sides of the ray l_0 and along $l_{11}e^{i0}$ and l_1e^{-i0} . The rest of the scattering data can be recovered from the minimal set by acting with the Coxeter automorphism.

An important fact following from these consideration is formulated as

Lemma 1 *Each of the subalgebras \mathfrak{g}_ν is a direct sum of $sl(2)$ subalgebras.*

Proof 1 *We will consider the subalgebra \mathfrak{g}_0 related to l_0 . According to table 2 \mathfrak{g}_0 has as positive roots $e_2 - e_3$, $e_1 + e_4$ and $e_1 - e_4$. Each of these roots generates an $sl(2)$ subalgebra of $D_4^{(1)}$. It is easy to check that these roots are orthogonal to each other, which means that their sums are not roots [15]. Therefore \mathfrak{g}_0 is a direct sum of three $sl(2)$ subalgebras. All other cases are considered and proved analogously.*

B) The RHP satisfied by FAS is formulated as:

$$\begin{aligned} \xi_\nu(x, t, \lambda) &= \xi_{\nu-1}(x, t, \lambda) G_\nu(x, t, \lambda), \quad \lambda \in l_\nu \\ G_\nu(x, t, \lambda) &= e^{-i\lambda Jx} \hat{S}_\nu^-(t, \lambda) S_\nu^+(t, \lambda) e^{i\lambda Jx}, \quad \nu = 0, \dots, 2h-1. \end{aligned} \quad (31)$$

Similarly, we can consider the spectral properties of the Lax operator related to $D_4^{(2)}$. Since the element J is different (see (22)), then the solutions of the equation (28) is also different. Now the spectrum of L consists of 8 straight lines closing angles $\pi/8$, see the right panel of Figure 1.

The roots related to each line are given in Table 3.

5. CONCLUSIONS

We have presented the mKdV-type equations related to $D_4^{(1)}$ and $D_4^{(2)}$. As was shown, the equations related to $D_4^{(1)}$ are effectively a one-parameter family of equations, which is a consequence of the fact that 3 is a double-valued exponent of $D_4^{(1)}$. However $D_4^{(2)}$ has only simply-valued exponents, so the relevant $mKdV$ system does not contain arbitrary parameters.

The direct and inverse spectral problems for both L can be reformulated in terms of a RHP problems. We briefly reviewed their properties and define minimal sets of scattering data.

The RHP are formulated on a set of $2h$ rays l_ν . We show that to each ray l_ν one can relate a subalgebra of \mathfrak{g} which is direct sum of $\mathfrak{sl}(2)$ subalgebras.

The connection between integrable equations and infinite dimensional Lie algebras is deep and intriguing. The present paper is a small step towards the completion of the study of this connection. A more challenging task would be the study of the properties of the solutions of the equations, starting with the soliton solutions and their interactions. This next step will be done in the near future.

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A. APPENDIX

The simple Lie algebra $D_4 \equiv \mathfrak{so}(8)$ is usually represented by a 8×8 antisymmetric matrices. In this realization the Cartan subalgebra is not diagonal, so we will use a realization for which every $X \in D_4$ satisfies

$$SX + (SX)^T = 0, \quad (32)$$

where the matrix S is given by

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (33)$$

This way the Cartan subalgebra is given by diagonal matrices.

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