THEORY FOR QUARTET CONDENSATION IN FERMI SYSTEMS WITH APPLICATIONS TO NUCLEAR MATTER*

P. SCHUCK[1,2,3]

[1]Institut de Physique Nucléaire, CNRS, UMR8608, Orsay, F-91406, France
[2]Université Paris-Sud, Orsay, F-91505, France
[3]Laboratoire de Physique et de Modélisation des Milieux Condensés, CNRS et, Université Joseph Fourier, 25 Av. des Martyrs, BP 166, F-38042 Grenoble Cedex 9, France

Received February 24, 2015

α clustering and α condensation in lighter nuclei is presently strongly and increasingly discussed in the literature both from the experimental side as from the theoretical one. In proto-neutron stars a macroscopic condensate of α particles may occur. A discussion of the present status of the theory for quartet condensation in general and for α particle condensation in nuclear matter in particular will be presented.

1. INTRODUCTION

Since 2001, when the idea of a possible existence of α condensate type of states in $n\alpha$ nuclei was advanced and formulated for the first time [1], many exciting new results, theoretical and experimental ones, have been produced. In this contribution, we would like to dwell on theoretical aspects of quartet condensation in general and in particular in nuclear matter.

Let us start with the reminder that nuclear clustering and in particular α clustering would not exist if we did not have in nuclear physics *four* different types of fermions (proton/neutron spin up/down), all attracting one another. We should be aware of the fact that this is a rather singular situation in fermionic many body systems. However, the possibility of future trapping of four different kinds of cold fermionic atoms may open a new field of cluster physics with similar features as in nuclei. Also the possibility of the condensation of bi-excitons in semi-conductors may be a field of interest in this context. In a mean field description of an isolated α particle (what with, *e.g.*, Skyrme forces gives reasonable results, if the c.o.m. motion is treated correctly) the four fermions can occupy the lowest 0S-level. Would there be only neutrons, only two of them can be in the 0S-level, the other two neutrons would have to be in the energetically very penalizing 0P-state. That is why α particles exist, tetra-neutrons not. The ensuing fact is that α particles are very strongly

*Paper presented at the conference "Advanced many-body and statistical methods in mesoscopic systems II", September 1-5, 2014, Brasov, Romania.

This work is part of a larger collaboration on nuclear clusters with Y. Funaki, H. Horiuchi, Z. Ren, G. Roepke, A. Tohsaki, Chang Xu, T. Yamada, and Bo Zhou.

Rom. Journ. Phys., Vol. 60, Nos. 5-6, P. 811–818, Bucharest, 2015

bound ($E/A \sim 7$ MeV), almost as strong as the most bound nucleus which is ^{56}Fe ($E/A \sim 8$ MeV). In addition the first excited state of the α particle ($\sim$ 20 MeV) is by factors higher than that of any other nucleus. The α particle can, therefore, be considered as an almost inert ideal bosonic particle. As we will see in the discussion below, in spite of its strong binding, α particle condensation can only exist in the so-called BEC (Bose-Einstein Condensation) phase which implies low density. There is no analogue to the BCS phase of pairing where the Cooper pairs can have very large extensions, strongly overlapping with one another, still being fully anti-symmetrized. This is the reason why α condensation only can be present at low densities where the α particles do not overlap strongly (this holds, if the system consists of protons and neutrons and α's. If other clusters as t, ^{3}He,d are around, the situation may change, see below).

These considerations apply for nuclear matter as well as for finite nuclei. The Hoyle state in ^{12}C which can to a good approximation be described as a product of three α particles occupying all the lowest 0S state of their bosonic mean field has a density which is by a factor 3-4 lower than the one of the ground state of ^{12}C. In the ground state there exist α-type of correlations but there is no condensation phenomenon. Here, we want to focus our considerations on infinite matter.

2. α PARTICLE CONDENSATION IN INFINITE MATTER

The in medium four-body equation can be written in the following form [2]

$$\begin{aligned}(E_{\alpha,\mathbf{K}} - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\Psi^{\alpha,\mathbf{K}}_{1234} = &(1 - f_1 - f_2)v_{121'2'}\Psi^{\alpha,\mathbf{K}}_{1'2'34} + \\ &(1 - f_1 - f_3)v_{131'3'}\Psi^{\alpha,\mathbf{K}}_{1'23'4} + \ldots\end{aligned} \tag{1}$$

In total, there are six terms coming from permutations. The ε_i are kinetic energies plus mean field corrections; v_{1234} are the matrix elements of the two body interaction, and f_i is a Fermi-Dirac distribution of the uncorrelated nucleons accounting for phase space blocking. Repeated indexes are summed over and index numbers comprise momenta and spins. The above equation considers, therefore, *one* quartet in a gas of uncorrelated nucleons at temperature T. The analogous two body equation can be used to determine the critical temperature T_c for the onset of superfluidity or supraconductivity where T_c has to be determined so that the eigenvalue comes at two times the chemical potential μ. This is the famous Thouless criterion of BCS theory. In analogy with pairing, one has to find the critical temperature T_c^α so that the eigenvalue of the four body equation (1) comes at 4μ for a quartet at rest. The in medium four body equation is very difficult to solve. Nonetheless, the solution has been found employing the Faddeev-Yakubovsky equations and using the Malfliet-Tjohn bare nucleon-nucleon interaction which yields realistic nucleon-nucleon phase shifts and properties of an isolated α particle [3].

To simplify the problem, we made in addition a very easy to handle variational ansatz of the four body wave function in (1). It consists of a mean field ansatz for the α particle projected on good total momentum. In momentum space this is (the singlet spin-isospin part is suppressed)

$$\Psi_{1234} \propto \delta(\mathbf{K}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\varphi(\mathbf{k}_1)\varphi(\mathbf{k}_2)\varphi(\mathbf{k}_3)\varphi(\mathbf{k}_4) \tag{2}$$

Inserting this ansatz into (1), one obtains a nonlinear HF-type of equation for the S-wave function $\varphi(\mathbf{k})$. Of course, for quartet condensation, we choose $\mathbf{K}=0$. With the mean field ansatz (2), one cannot use a bare force. We adjusted an effective separable force with two parameters which are chosen to reproduce the binding energy and radius of the free α particle. The full Faddeev-Yakubovsky solution of (1) is shown for symmetric matter in Fig. 1 (crosses).

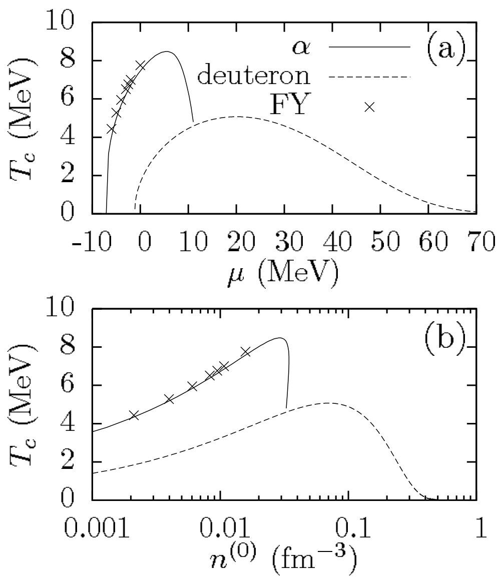

Fig. 1 – Critical temperatures for α particle and deuteron condensation in symmetric nuclear matter as a function of μ (a) and density $n^{(0)}$ (b).

We see that the ansatz (2) which very much eases the otherwise difficult solution of (1) works very well (continuous line). Also shown is the critical temperature for deuteron condensation. The striking feature is that α particle condensation abruptly breaks down already at very low density which approximately coincides with the point where the α's start to overlap appreciatively (this fact was already found in [2] using a somewhat different variational ansatz for Ψ_{1234}). On the other hand deuteron condensation goes on up to very high densities and the limit is only triggered by the range of the effective force (which has been readjusted to reproduce the deuteron properties). This is so for symmetric nuclear matter. For strong asymmetry, deuteron condensation breaks down earlier than α condensation because the α particle according to its much stronger binding is less sensitive to asymmetry [4]. The above features describe the afore mentioned phenomenon that α condensation only exists in the BEC phase, *i.e.*, at low density, whereas deuteron continuously goes from negative to positive chemical potentials where for the latter the deuterons turn into large size Cooper pairs. More on this can be found in [2] [3]. We should mention that our calculation of T_c^α is only reliable rather close to the break down point, *i.e.*, to the maximum. For lower densities, the T_c^α should join the one for condensation of ideal bosons (α's). To describe this feature, one should extend our theory to the so-called Nozières Schmitt-Rink (NSR) theory [5] for pairing, see also [6], to α particle condensation. This, however, has not been worked out so far and remains a task for the future.

At zero temperature, there are many α's which go into the condensate phase. For this, we have to set up an approach analogous to the nonlinear BCS theory. Equation (1) corresponds to the linearized version and only describes *one* α particle in an otherwise uncorrelated gas (at finite T) of fermions. In finite nuclei, there may exist such a situation even at zero temperature. This is the case of ^{212}Po which can to a certain extent be viewed as an α particle sitting on top of the doubly magic core of ^{208}Pb which can be well described by a HF-mean field approach, *i.e.* a Fermi gas in a container, see [7].

After having considered the linearized version of the equation for the quartet order parameter at the critical temperature, let us now try to write down, in analogy to the BCS case, the fully non-linear system of equations for the quartet order parameter. To clearly see the analogy to the BCS case, let us repeat the latter equations in a slightly unusual way. The pairing order parameter allowing for non-zero c.o.m. momentum of the pairs, obeys the equation

$$(\varepsilon_{k_1}+\varepsilon_{k_2})\kappa_{\mathbf{k}_1\mathbf{k}_2}+(1-n_{k_1}-n_{k_2})\Delta_{\mathbf{k}_1\mathbf{k}_2}=2\mu\kappa_{\mathbf{k}_1\mathbf{k}_2} \tag{3}$$

where $\Delta_{\mathbf{k}_1\mathbf{k}_2}=\sum v_{k_1k_2k_{1'}k_{2'}}\kappa_{\mathbf{k}_{1'}\mathbf{k}_{2'}}$ and $\kappa_{\mathbf{k}_1\mathbf{k}_2}=\langle \mathrm{BCS}|c_{\mathbf{k}_1}c_{\mathbf{k}_2}|\mathrm{BCS}\rangle=u_{\mathbf{k}_1}v_{\mathbf{k}_2}$ is the usual pairing tensor (with spin indices suppressed) and $n_k=v_k^2=1-u_k^2$ are the BCS occupation numbers. The ε_k's are, as before, the kinetic energies, eventually including a mean field correction. The occupation numbers can be obtained from the Dyson equation for the single particle Green's function (the latter is derived from the Gorkov equation in eliminating the anomalous Green's function)

$$G_k^\omega=G_k^0+G_k^0M_k^\omega G_k^\omega \quad \text{with} \quad M_k=\frac{\Delta_k\Delta_k^*}{\omega+\varepsilon_k} \tag{4}$$

the BCS mass operator where Δ_k is the diagonal part of the gap for cases where the pairs are at rest. From the single particle Green's function, obviously we can calculate the occupation numbers closing, thus, the typical BCS self-consistency cycle. Inspired by the BCS case, we then write for the quartet order parameter, see (1)

$$(4\mu-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)\kappa_{1234} = (1-n_1-n_2)v_{121'2'}\kappa_{1'2'34} + (1-n_1-n_3)v_{131'3'}\kappa_{1'23'4}+\dots \quad (5)$$

with $\kappa_{1234} = \langle c_1c_2c_3c_4\rangle$.

Again, we have to close the equation with the Dyson equation for the single particle Green's function to obtain the occupation numbers. However, the single particle mass operator now contains the quartet order parameter

$$M_1^\alpha = \sum_{234} \frac{\Delta_{1234}^{(13)}[\bar{n}_2^0\bar{n}_3^0\bar{n}_4^0 + n_2^0n_3^0n_4^0]\Delta_{1234}^{(13)*}}{\omega+\varepsilon_2+\varepsilon_3+\varepsilon_4}\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4) \quad (6)$$

with $\bar{n}^0 = 1-n^0$ and n_i^0 being the uncorrelated Fermi function, *i.e.*, the Fermi step and $\Delta_{1234}^{(13)} = \Delta_{1234} - [(k_3 \leftrightarrow k_4)-(k_2 \leftrightarrow k_3)]$ where $\Delta_{1234} = \sum v_{123'4'}\kappa_{3'4'34}$.

Let us give some hints how to derive such a mass operator. The first observation is that the following non-linear destruction operator

$$q_\nu = u_{k_1}^\nu c_{k_1} - \frac{1}{3!}\sum v_{k_2k_3k_4}^\nu c_{k_2}^+c_{k_3}^+c_{k_4}^+ \quad (7)$$

kills the following quartet coherent state

$$|Z\rangle = e^{\frac{1}{4!}\sum_{k_1k_2k_3k_4}Z_{k_1k_2k_3k_4}c_{k_1}^+c_{k_2}^+c_{k_3}^+c_{k_4}^+}|\text{vac}\rangle \quad \text{with} \quad Z_{k_1k_2k_3k_4} = \sum_\nu (u^{-1})_{k_1}^\nu v_{k_2k_3k_4}^\nu \quad (8)$$

Minimizing the average single particle energy

$$E_\nu = \langle Z|\{q_\nu,[H-\mu\hat{N},q_\nu^+]\}|Z\rangle / \langle Z|\{q_\nu,q_\nu^+\}|Z\rangle, \quad (9)$$

where $\{.,.\}$ is the anti-commutator, with respect to the amplitudes u^ν, v^ν, leads to a two by two eigenvalue problem for those amplitudes. Eliminating the v^ν amplitudes leads to an effective single particle equation where the energy dependent potential can be identified with the above mass operator performing some lowest order approximation for the three hole propagator.

Before trying to solve the system of equations, let us discuss the differences between the pairing and the quartet case. The first thing which strikes is that the three 'holes' only have total momentum $\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4 = -\mathbf{k}_1$ and, therefore, we have a remaining sum over momenta. In the pairing case with only one 'hole', there is no sum. Furthermore in the pairing case the hole propagator has no phase space factor because 'forward' and 'backward' going parts add up to one: $\bar{n}^0+n^0 = 1$. In the quartet case there are three hole propagators and the corresponding sum of phase space factors does NOT add up to one, *i.e.* $\bar{n}_1^0\bar{n}_2^0\bar{n}_3^0 + n_1^0n_2^0n_3^0 \neq 1$!! This makes a dramatic difference with the pairing case. In order to understand this, let us compare the level density of a single hole with the one of three holes:

$$g^{1h}(\omega) = \sum_k [\bar{n}_k^0 + n_k^0]\delta(\omega + \varepsilon_k) = \sum_k \delta(\omega + \varepsilon_k) \tag{10}$$

$$g^{3h}(\omega) = \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} [\bar{n}_{k_1}^0 \bar{n}_{k_2}^0 \bar{n}_{k_3}^0 + n_{k_1}^0 n_{k_2}^0 n_{k_3}^0]\delta(\omega + \varepsilon_{k_1} + \varepsilon_{k_2} + \varepsilon_{k_3}) \tag{11}$$

The 3h level density is shown in Fig.2.

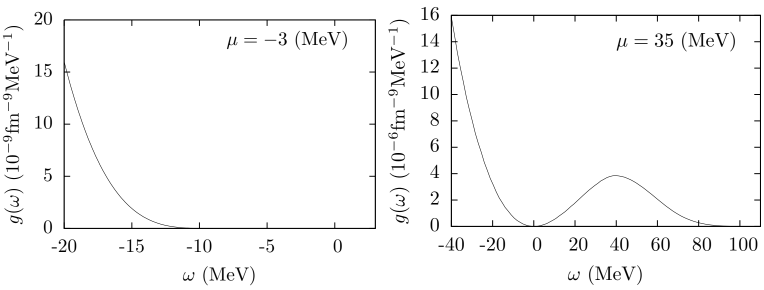

Fig. 2 – 3h level densities for negative and positive chemical potentials, respectively. Note that on the horizontal axis, the origin corresponds to $\mu = 0$.

We see that for positive μ there is a striking difference with the 1h level density. At positive μ, $g^{1h}(\omega = \mu)$ is obviously finite (not shown), whereas $g^{3h}(\omega = 3\mu)$ goes through zero (this remains true for any value of positive μ, even small). This is because phase space constraints and energy conservation cannot be fulfilled simultaneously at the Fermi surface in the latter case as is easily verified. That is, in the quartet case, exactly at the point where the correlations should be built up, namely at the Fermi-level, there is no level density! As a consequence, no quartet condensation is possible for positive μ. On the contrary for negative μ, $n^0 = 0$ and, thus, the phase space factor in the case of g^{3h} is also equal to one and then there is no qualitative difference with the $1h$ case. This explains in a natural way why quartet condensation is not possible at positive chemical potential.

In conclusion for positive μ only pairing survives whereas quartetting breaks down and only exists in the BEC phase with negative μ. The situation may be different when other light clusters are present, *i.e.* in a mixed gas of, *e.g.*, nucleons, tritons (^{3}He), deuterons. Then a nucleon with momentum $\mathbf{k}$ may, eventually, directly pair with, *e.g.*, a triton of momentum $-\mathbf{k}$ (or the other way round), rather similar to the standard pairing situation besides the fact that now two fermions with different masses have to pair up. Similar considerations hold for the pairing of two deuterons (pairing of 'bosons'). In compact star physics such situations may exist when the star is cooling down. The extension of our theory to this scenario is a task for the future.

The full solution of the nonlinear set of equations (6) and (7) is again very much eased in taking for the order parameter the factorization ansatz (2). The most interesting result is that the occupation numbers, *e.g.*, for μ around zero is far from being close to saturation. At $k = 0$ it is only approximately $n_{k=0} \sim 0.30$. This scenario is analogous to pairing in the BEC regime. More results can be found in [8].

3. REMARKS ON FINITE NUCLEI

As we know from pairing, a direct observation of condensation phenomena only is possible in finite nuclei. Of course, in such small systems, there cannot exist a condensation in the macroscopic sense. Nevertheless, as we know very well, only a handful of Cooper pairs suffices to show clear signatures of pairing in nuclei. For α particle condensation it is the same story. We also only can expect that there exists about a handful of α particles, essentially in lighter $n\alpha$ nuclei, in a gaseous phase at low density. It is indeed surprising that such states at low density with about $\rho = \rho_0/3 - \rho_0/4$ with ρ_0 the density at saturation, indeed exist as excited quite long lived states in those nuclei. The most famous example is the Hoyle state in ^{12}C at 7.65 MeV, just about 300 keV above the 3α threshold. We will not dwell much on the successful theoretical description of this state (and others, *e.g.*, in ^{16}O) with the THSR wave function [1], since this has been presented in the recent literature a great number of times [9]. Let us only make a couple of remarks. The THSR wave function is schematically written for a finite number of quartets as

$$\Psi_{n\alpha}^{\mathrm{THSR}} = \mathcal{A}[\Phi_\alpha \Phi_\alpha .... \Phi_\alpha] \tag{12}$$

where the single α wave function Φ_α depends on four spatial coordinates, the spin-isospin part being suppressed. This wave function is fully antisymmetric due to the anti-symmetrizer $\mathcal{A}$ and is analogous to the number projected BCS wave function $\Psi^{\mathrm{BCS}} = \mathcal{A}[\phi_{\mathrm{pair}}\phi_{\mathrm{pair}}.....\phi_{\mathrm{pair}}]$ where ϕ_{pair} is the Cooper pair wave function depending on two spatial coordinates. The calculus with the α condensate wave function is very much facilitated with a variational ansatz where Φ_α is split into a product of a c.o.m. Gaussian with a large width parameter B times another, intrinsic, Gaussian depending only on the relative coordinates of the α particle and having a width parameter b which corresponds to the size of an isolated α particle. The first important remark to be made is that this THSR wave function contains two important limits: if $B = b$ then it corresponds to a pure harmonic oscillator Slater determinant. If $B \gg b$, then the α's are so distant from one another that the Pauli principle among the different α's can be neglected and, thus, the anti-symmetrizer be dropped. The THSR wave function is then a pure product state of α particles, *i.e.*, a condensate state of ideal bosons. Reality is, of course, in between those limits and one important task is to find out whether reality is closer to a Slater determinant or to a Bose condensate. That this question must be carefully investigated, can also be deduced from the fact that a number projected BCS wave function always leads to a non trivial pairing solution. For example for ^{208}Pb, one obtains a non trivial BCS solution in spite of the fact that ^{208}Pb is certainly not superfluid. Only when the original particle number breaking BCS theory with $|\mathrm{BCS}\rangle = e^{\sum z_{kk'} c_k^+ c_{-k}^+}|\mathrm{vac}\rangle$ has a non-trivial solution, we can speak of a superfluid nucleus. For ^{208}Pb BCS has no solution. One way to analyze whether the THSR approach leads primarily to an α condensate or to a Slater determinant, is to investigate the bosonic occupation numbers. It has been found that in the Hoyle state the α's are condensed to 70% [9], the remaining part being scattered out of the condensate due to the action of the Pauli principle.

4. CONCLUSIONS

As we have seen, the existence of α gas and α condensate states in nuclear systems where the α's play practically the role of elementary bosons, is fascinating. Nuclear physics is at the forefront of this kind of physics. In the future, experiments with cold atoms trapping four (or more) different kinds of fermions may also open wide perspectives in the field of cluster physics. For more reading on α cluster states, we recommend to consult our review article [9].

REFERENCES

1. Tohsaki A, Horiuchi H, Schuck P, and Röpke G 2001 *Phys. Rev. Lett.* **87**, 192501
2. Röpke G, Schnell A, Schuck P, Nozières P 1998 *Phys. Rev. Lett.* **80**, 3177
3. Sogo T, Lazauskas R, Roepke G, Schuck P 2009 *Phys. Rev.* C **79**, 051301
4. Sogo T, Roepke G, Schuck P 2010 *Phys. Rev.* C **82**, 034322
5. Nozières P, Schmitt-Rink S 1985 *J. Low Temp. Phys.* **59**, 195
6. Jin M, Urban M, Schuck P 2010 *Phys. Rev. C* **82**, 024911
7. Roepke G *et al.*, arXiv:1407.0510.
8. Sogo T, Roepke G, Schuck P 2010 *Phys. Rev.* C **81**, 064310
9. Yamada T, Funaki Y, Horiuchi H, Roepke G, Schuck P, Tohsaki A 2011 arXiv:1103.3940, "Clusters in Nuclei (Lecture Notes in Physics)-Vol.2-", ed. by C. Beck, Springer Verlag, Berlin