RANDOM MATRICES, POINT-GROUP SYMMETRIES, AND MANY-BODY SYSTEMS*

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Whenever a system has a symmetry, Noether's theorem tells us there is an associated conserved quantity. For quantum systems this means the Hamiltonian takes on a block-diagonal form, labeled by "good" quantum numbers. While Noether's theorem tells us this structure exists, it doesn't tell us the ordering of the states, or which quantum numbers tend to be associated with the ground state. Yet in physical systems we know the ground state tends to belong to the most symmetric irrep. Could it be otherwise? To explore this question, I construct random matrices with point-group symmetries and find an order above and beyond Noether's theorem. The pattern persists in detailed simulations of many-body systems, even when all physics besides symmetry is taken out. For atomic nuclei, for example, this means the ground states of even-even nuclides have J=0 not due to pairing as we are taught, but due to underlying mathematical structures.

Key words: random matrices, point-group symmetries.

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One of the most important results in mathematical physics is Noether's theorem. In classical mechanics a (continuous) symmetry leads to a conserved quantity, for example translational invariance leads to conservation of momentum, invariance under displacement in time leads to conservation of energy, and rotational invariance leads to conservation of angular momentum.

In quantum mechanics we can think about symmetries in terms of groups and their irreps. Group irreps divide up a Hilbert space into subspaces; if a Hamiltonian is invariant under a symmetry, meaning it commutes with the generators of a group, then the Hamiltonian becomes block-diagonal in the irreps.

For continuous symmetries, we label the irreps by *quantum numbers*, which in turn arise from the eigenvalues of the Casimir(s) of the symmetries. For discrete symmetries the Hamiltonian is still block-diagonal in the irreps, although lacking a Casimir there may not be a 'natural' quantum number labeling the irreps.

While Noether's theorem is a powerful result, it tells us nothing about the relative ordering of states. In natural systems, however, we observe an ordering so

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Table 1

Ground state quantum numbers for single-species systems, comparing the percentage of ground states of a given J for configuration-interaction (CI) simulations with interaction drawn from the two-body random ensemble ($f_{\rm TBRE}$) against fraction of the states with a given J in the CI model space $f_{\rm space}$. The two cases are 8 fermions in a $j = 21/2$ shell, and the $N = 7$ interacting boson model (IBM).

$(21/2)^8$		
J	$f_{\rm space}$	$f_{\rm TBRE}$
0	0.4	55
1	0.5	0
2	1	7
3	1	0.2
4	2	2

IBM, $N = 7$		
J	$f_{\rm space}$	$f_{\rm TBRE}$
0	11	55
1	(no states)	
2	17	13
3	6	0.08
4	17	4

ubiquitous we take it for granted, namely that the ground state and states lying low in the spectrum tend to belong to 'small' quantum numbers with the most symmetric wave functions. For example, under translational symmetry the ground state has momentum $p = 0$, under rotational symmetry the ground state has $J = 0$, etc.

Of course this arises because of physics: in most Hamiltonians the kinetic energy terms are quadratic in linear momentum p and/or angular momentum J. This in turn occurs because Nature, or physicists, prefer *almost*-local theories, and the first nontrivial non-local terms are quadratic in the gradient.

But surprisingly, even when one removes all trace of 'physics' the pattern remains. In nuclear structure this is seen in the discovery that rotationally invariant but otherwise random two-body Hamiltonians tend to yield ground states with $J = 0$, just like 'realistic' interactions, even though such states are a small fraction of the total space [1–3]. This is illustrated in Table 1, where I carry out many-body calculations with interactions drawn from the two-body random ensemble (TBRE) for two cases, first 8 fermions in a $j = 21/2$ shell, and then 7 bosons in the standard interacting boson model (IBM) [4] with s- and d-bosons. Here $f_{\rm space}$ is the fraction of the many-body space with a given J, while f_{TBRE} is the actual fraction carried out using random interactions. In both cases f_{TBRE} is enhanced for $J = 0$ (although both values being 55% is a coincidence).

This phenomenon is robust, insensitive to details of the model space or of the distribution of matrix elements; for example while textbooks traditionally ascribe the $J = 0$ g.s. to the pairing interaction [5, 6], $J = 0$ still dominates the ground state even when the pairing matrix elements are all set to zero [7].

Over the past decade there have been many proposed 'explanations.' As the distribution of many-body systems with two-body interactions tend to have a Gaussian distribution of states [8], a number of authors have focused on widths [9, 10],

while others have statistically averaged in a single j-shell the coupling of multiple angular momenta [11]. As a recent Letter stated, 'the simple question of symmetry and chaos asks for a simple answer which is still missing [12].'

To investigate this phenomenon, I propose a novel approach. In physics we often delve deeper by stripping away assumptions to see what remains. Previous studies of random two-body interactions used shell-model diagonalizing codes, but instead of carefully calculated matrix elements of two-body interactions they used random numbers, insisting only on rotational invariance. Now I go a step further: I abandon the shell-model framework, take random matrices, often used to investigate statistical properties of complex systems [5, 13], and impose discrete rotational symmetries of the regular polyhedra upon them, and then looking at the widths for different irreps.

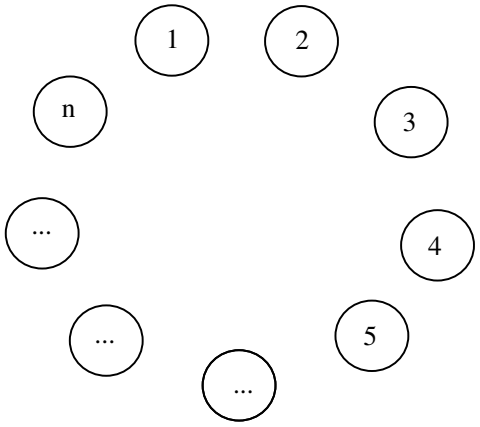

Fig. 1 – Illustration of discrete rotation symmetry in the plane, C_n.

Let's start with C_n symmetry, the symmetry of discrete rotations in a plane. Figure 1 illustrates this with n loci evenly spaced in a circle. The generator of discrete rotations is

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \tag{1}$$

which sends $1 \rightarrow 2$, $2 \rightarrow 3$ and so on. One can easily show by hand that the most

general real Hamiltonian invariant under discrete rotations $\mathbf{H} = \mathbf{THT}^{-1}$ is

$$\mathbf{H} = \begin{pmatrix} a & b & c & d & \dots & c & b \\ b & a & b & c & \dots & d & c \\ c & b & a & b & \dots & e & d \\ \vdots & & & & & & \\ b & c & d & e & \dots & b & a \end{pmatrix} \tag{2}$$

where $a, b, c, \dots$ are random numbers.

One can solve this exactly without diagonalizing by discrete Fourier transforms. One writes the matrix element $H_{i,j}$ in terms of a real function of a single index, $H_{i,j} = F_{i-j}$, so that, in (2), $F_0 = a, F_1 = b, F_2 = c$, etc.. The function $F_{-j} = F_j$ from hermiticity, and by further inspection one gleans $F_{n-j} = F_{j+1}$.

The next step is the Fourier decomposition of F_j:

$$F_j = \frac{1}{n} \sum_{k=0}^{n-1} h_k \exp(i2\pi kj/n). \tag{3}$$

Inverting, and using $F_{n-j} = F_{j+1}$, I get

$$h_k = F_0 + \sum_{j=1}^{n-1} F_n \cos(2\pi kj/n) = F_0 + \sum_{j=1}^{[n/2]} F_n \zeta_{j,n} \cos(2\pi kj/n) \tag{4}$$

where $[x]$ is the floor function and $\zeta_{j,n} = (2 - \delta_{j,n/2+1})$ prevents double-counting when n is even. It is easy to verify that $h_k = h_{n-k}$. The h_k are the eigenvalues of $\mathbf{H}$, and one is justified in using k to label the different irreps.

While (4) gives the eigenvalues from a simple sum, we consequently have no *a priori* way of identifying the ground state energy which irrep it belongs to.

Next I assume there are additional degrees of freedom, as yet unspecified, and replace the scalars $a, b, c, \dots$ by $m \times m$ real symmetric matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ so that

$$\mathbf{H} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \dots & \mathbf{C} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{C} & \dots & \mathbf{D} & \mathbf{C} \\ \mathbf{C} & \mathbf{B} & \mathbf{A} & \mathbf{B} & \dots & \mathbf{E} & \mathbf{D} \\ \vdots & & & & & & \\ \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{E} & \dots & \mathbf{B} & \mathbf{A} \end{pmatrix} \tag{5}$$

a matrix of dimension $mn \times mn$.

This matrix can no longer be immediately solved. What we *can* do, however,

is to put $\mathbf{H}$ into block-diagonal form, that is,

$$\mathbf{H} = \begin{pmatrix} \mathbf{h}_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \mathbf{h}_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \mathbf{h}_2 & 0 & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & \mathbf{h}_n \end{pmatrix} \tag{6}$$

where

$$\mathbf{h}_k = \mathbf{F}_0 + \sum_{n=1}^{[n/2]} \mathbf{F}_j \zeta_{j,n} \cos(2\pi k j/n) \tag{7}$$

Here comes the key step. While we cannot analytically compute the ground state energy of each $\mathbf{h}_k$, we can compute the variance. If we assume the $\mathbf{F}_j$ are independent random matrices, each with the same variance σ_0^2 (assumptions which can be relaxed), then the variance of the kth block is

$$\sigma_k^2 = \sigma_0^2 \left(1 + \sum_{j=2}^{[n/2]+1} \zeta_{j,n}^2 \cos^2(2\pi k(1-j)/n) \right) \tag{8}$$

For $k = 0$, this yields approximately $\sigma_0^2(2n+1)$, while otherwise this will yield approximately $\sigma_0^2(n+1)$ (because the average of $\cos^2 x \approx 1/2$). Thus the matrices with $k = 0$ (and for n even, $k = n/2$) will have the larger widths and the ground state k will be one of those two extremal values.

Now C_n is an abelian group, but we can do the same analysis for other, non-abelian point-symmetry groups based on regular polyhedra. First, consider the tetrahedron, illustrated in Fig. 2. The most general Hamiltonian invariant under any discrete rotation about any of its facets is

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{B} & \mathbf{B} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{B} \\ \mathbf{B} & \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{B} & \mathbf{B} & \mathbf{A} \end{pmatrix} \tag{9}$$

Bringing this into block diagonal form,

$$\begin{pmatrix} \mathbf{A}+3\mathbf{B} & 0 & 0 & 0 \\ 0 & \mathbf{A}-\mathbf{B} & 0 & 0 \\ 0 & 0 & \mathbf{A}-\mathbf{B} & 0 \\ 0 & 0 & 0 & \mathbf{A}-\mathbf{B} \end{pmatrix} \tag{10}$$

That is, there is an irrep of dimension 1 with matrix $\mathbf{A}+3\mathbf{B}$ and an irrep of dimension 3 with matrix $\mathbf{A}-\mathbf{B}$. Again assuming $\mathbf{A}, \mathbf{B}$ are independent but have the same variance the variance of the 1-dimensional irrep (which has the most symme-

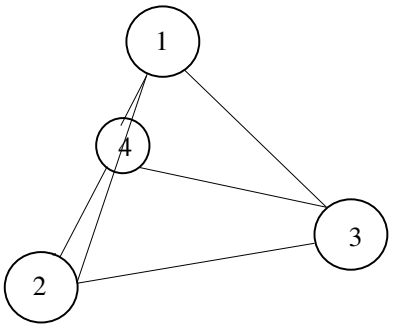

Fig. 2 – Illustration of tetrahedron space symmetry.

tric eigenvector) is $10\sigma_0^2$ while that of the 3-dimensional irrep is $2\sigma_0^2$. All else being equal, the ground state is much more likely belong to the 1-dimensional irrep.

With the basic idea in hand, it is easy to consider other polyhedra. For example for the cube, Fig. 3, the Hamiltonian is of the form

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{B} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{C} \\ \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{B} & \mathbf{C} & \mathbf{D} \\ \mathbf{C} & \mathbf{B} & \mathbf{A} & \mathbf{B} & \mathbf{D} & \mathbf{C} & \mathbf{B} & \mathbf{C} \\ \mathbf{B} & \mathbf{C} & \mathbf{B} & \mathbf{A} & \mathbf{C} & \mathbf{D} & \mathbf{C} & \mathbf{B} \\ \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{C} & \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{B} \\ \mathbf{C} & \mathbf{B} & \mathbf{C} & \mathbf{D} & \mathbf{B} & \mathbf{B} & \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{C} & \mathbf{B} & \mathbf{C} & \mathbf{C} & \mathbf{B} & \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} & \mathbf{C} & \mathbf{B} & \mathbf{B} & \mathbf{C} & \mathbf{B} & \mathbf{A} \end{pmatrix} \tag{11}$$

with two 1-dimensional irreps with matrices $\mathbf{A} \pm 3\mathbf{B} + 3\mathbf{C} \pm \mathbf{D}$ and two 3-dimensional irreps with matrices $\mathbf{A} - \mathbf{C} \pm (\mathbf{B} - \mathbf{D})$. The 1-dimensional irreps have variance $20\sigma_0^2$ while the 3-dimensional irreps have variance $4\sigma_0^2$.

This suggests that in many-body system, subspaces with low-valued quantum numbers will have larger widths. But in realistic, finite many-body calculations, subspaces with different Js have different dimensions. Furthermore, in the above argument each $\mathbf{h}_J$ has independent random matrix elements, which typically has a semi-circular density of states [13], yet for many-body systems with only two-body interactions the density of states tends towards a Gaussian [8].

Nonetheless, one can investigate by removing the issue of dimensionality. Consider a many-body Hamiltonian matrix for angular momentum J, $\mathbf{h}_J$, with dimension N_Jl an example would be the Hamiltonian matrix computed in the configuration-interaction shell model [14–16]. First one must remove the centroid, $E_J = N_J^{-1}\mathrm{tr}\mathbf{h}_J$, and then compute the width $\sigma_J^2 = N_J^{-1}\mathrm{tr}(\mathbf{h}_J)^2 - E_J^2$ for the subspace with angular

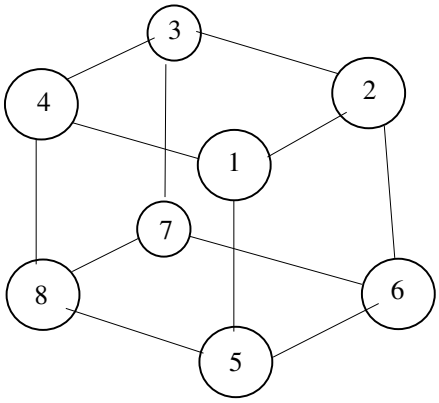

Fig. 3 – Illustration of cubic space symmetry.

momentum J. More helpful to us is the root-mean-squared (rms) matrix element,

$$\gamma_J = \frac{1}{\sqrt{N_J}}\sigma_J. \tag{12}$$

This is just the width in the matrix elements, not in the matrices.

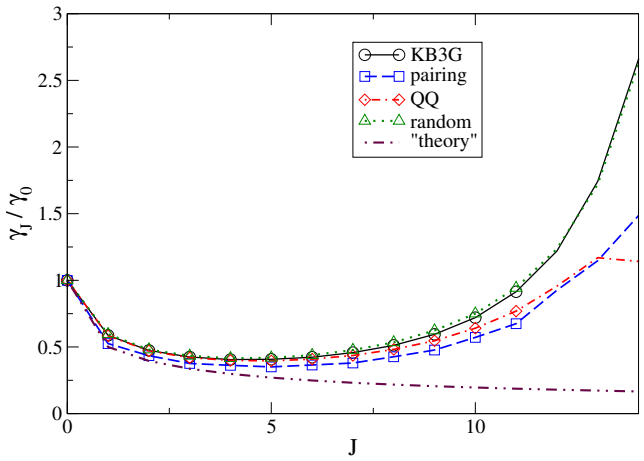

Fig. 4 – Root-mean-squared matrix elements γ_J scaled by γ_0 for different interactions for ^{50}Ca (10 neutrons in pf-shell). Black circles + solid black line, KB3G interaction; blue square+dashed line, pairing interaction; red diamonds+dot-dashed line, $Q\cdot Q$ interaction; green triangles + dotted line, random two-body matrix elements. "Theory" is derived from a simple integral (eqn. 15).

To compare across interactions, and even across systems, I divide by the $J=0$ mean-squared matrix element, γ_0. Fig. 4 shows the mean-squared matrix elements for ^{50}Ca in the pf shell (10 valence neutrons) for the realistic, monopole-modifed Kuo-Brown G-matrix interaction (KB3G) [17], for the pairing interaction, for $Q\cdot Q$,

and for a random two-body interaction (TBRE). Despite the very different forms of the interactions, they show very similar behaviors.

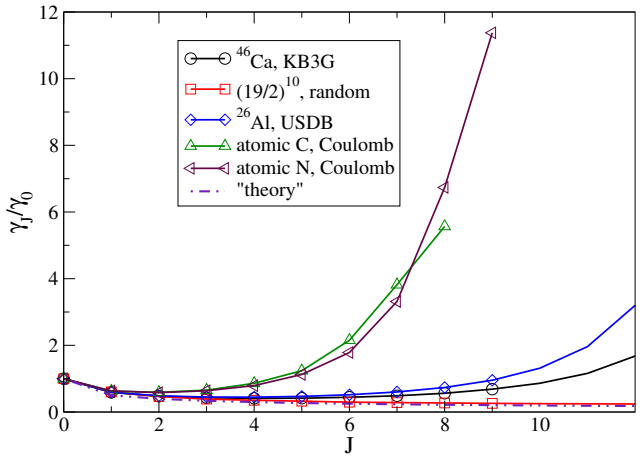

Fig. 5 – Root-mean-squared matrix elements γ_J scaled by γ_0 for different systems, in order of increasing M-scheme dimensionality: 6 neutrons in the pf shell (^{46}Ca) with the KB3G interaction; 10 identical fermions in a $j = 19/2$ shell with a random two-body interaction; 5 protons and 5 neutrons in the sd shell (^{26}Al) with the USDB interaction; and the electronic structure of neutral carbon and nitrogen with the Coulomb interaction. "Theory" is derived from a simple integral (eqn. 15).

This similarity continues even for very different systems, as shown in Fig. 5, for a sample variety of different systems and interactions. All of these were computed in the M-scheme, and the M-scheme dimensions varied from about 4000 (^{46}Ca, or 6 neutrons in the pf shell, with the KB3G interaction [17]), to atomic nitrogen (using the CVB1 model space [18]) with the Coulomb interaction, with an M-scheme dimension of 1.4 million. In between are 10 fermions in a $j = 19/2$ space with a random two-body interaction, ^{26}Al with the universal sd-shell interaction, version B (USDB) [19], and atomic carbon with the Coulomb interaction. All these systems, and many others not shown, show similar trends. The main difference between them is not determined by the dimensionality but by the maximum J in the system, where the graphs turn up.

It turns out one can attempt a semi-analytic analysis similar to what I did with the regular space groups. Consider wave functions that can be written in the form $\psi_l(\vec{v})Y_{lm}(\theta,\phi)$ where all of the rotational information is bound up in the spherical harmonic Y_{lm} and $\vec{v}$ refers to internal degrees of freedom [20]. Using the angles from spherical coordinates, the Hamiltonian is of the form $H(\theta'\phi',\theta\phi)$, but imposing rotational invariance means $\hat{H}$ can only depend on the angle ω *between* θ',ϕ' and θ,ϕ as given by $\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. Then $H(\theta'\phi',\theta\phi) = F(\omega)$, where $F(\omega) = F(\omega+2\pi)$ is a periodic function and, using Hermiticity (and assum-

ing H is real) $F(\omega) = F(2\pi - \omega)$ is symmetric with respect to $\omega = \pi$. Expanding

$$F(\omega) = \sum_J h_J \frac{2J+1}{4\pi} P_J(\cos\omega)$$

$$= \sum_J h_J \sum_{M=-1}^{J} Y_{JM}(\theta',\phi') Y^*_{JM}(\theta,\phi) \tag{13}$$

so clearly the h_J are again the eigenvalues, with Y_{JM} as eigenfunctions and with the eigenvalues independent of N, as one expects.

Once more I assume $\mathbf{F}(\omega)$ to be a matrix-valued function, and thus $\mathbf{h}_J$ to be a symmetric matrix given by

$$\mathbf{h}_J = 2\pi \int_0^\pi P_J(\cos\omega)\mathbf{F}(\omega)\sin\omega d\omega. \tag{14}$$

One should think of $\mathbf{h}_J$ as the J-projection of a Hamiltonian in a shell-model configuration space. As before, let $\bar{\sigma}$ be the variance of the matrix elements $\mathbf{F}$ independent of ω. Then I estimate the variance of the matrix elements of $\mathbf{h}_J$

$$\sigma_J^2 = 4\pi^2\bar{\sigma}^2 \int_0^\pi P_J^2(\cos\omega)\sin^2\omega d\omega \tag{15}$$

Eqn. (15) can be computed numerically, and leaving off the factor $4\pi^2\bar{\sigma}^2$, the values are 1.571, 0.393, 0.245, 0.178, 0.139 for $J = 0, 1, 2, 3, 4$ respectively. Taking the square roots, one gets the γ_J.

In Figs. 4 and 5 I've included the above estimates of γ_J as "theory." This provides a limiting case with no maximum J.

This analysis suggests the predominance of angular-momentum zero ground states is primarily a function of the width of the angular-momentum-projected many-body Hamiltonian; furthermore, the width is largely decoupled from the microphysics, instead depending only on the projection integrand (15) and on the dimensionality of subspaces with good quantum numbers. The simplicity and decoupling from the microphysics may be why the phenomenon is so robust and so universal.

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