INTEGRABLE RICHARDSON-GAUDIN MODELS IN MESOSCOPIC PHYSICS*

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The exact solution of the SU(2) pairing Hamiltonian with non-degenerate single particle orbits was introduced by Richardson in the early sixties. The exact solution passed almost unnoticed till was recovered in the last decade in an effort to describe the disappearance of superconductivity in ultra-small superconducting grains. Since then it has been extended to several families integrable models, called the Richardson-Gaudin (RG) models. In particular, the rational family of integrable RG models has been widely applied to mesoscopic systems like small grains, quantum dots and nuclear systems where finite size effects play an important role. We will first introduce these families of integrable models and then we will describe the first applications of the hyperbolic family to spinless cold fermionic atoms in two dimensional lattices and to heavy nuclei.

Key words: integrable systems, Richardson-Gaudin models, rational models, hyperbolic models, pairing interaction.

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1. INTRODUCTION

The work of Bardeen, Cooper and Schrieffer (BCS) of 1957 [1] gave the first microscopic description of the superconducting phenomenon assuming a quantum pairing Hamiltonian and a variational wave function based on a coherent state of pairs. The following year, Bohr, Mottelson and Pines [2] noted that similar physics may underlie the large gaps seen in the spectra of even-even atomic nuclei, emphasizing however that finite-size effects would be critical for a proper description of such systems. The program to include number conservation in the BCS theory within nuclear structure [3] started at roughly the same time at which Richardson [4] showed that for a pure pairing Hamiltonian it is possible to *exactly* solve the Schrödinger equation by following closely Cooper's original idea. Years later and from a different perspective, Gaudin introduced an integrable spin model having striking similarities with the Richardson exact solution [5]. In spite of the fact that exact solvability is linked to quantum integrability, he couldn't find the explicit relation between both models. Years later, we were able to find this relation through a generalization of the

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Gaudin integrals of motion giving rise to three classes of pairing-like models that we called the RG integrable models, all of which were integrable and all of which could be solved exactly for both fermion and boson systems [6]. During the last decade, the rational family of the RG models was extensively used to describe ultra-small superconduction grains, heavy nuclei, quantum dots, ultra-cold atomic gases, etc [7]. More recently, we have found two physical realizations of the hyperbolic family, one for p-wave polarized atomic gases in two dimensional lattices [8, 9], and the other as a potentially useful realistic pairing Hamiltonian for heavy nuclei. In this contribution we will first introduce the RG integrable models and then we will briefly describe the exact solution for $p_x + ip_y$ superfluids as a first realization of the hyperbolic RG model. Finally, we will present preliminary results showing how the integrable hyperbolic Hamiltonian could reproduce Gogny Hartree-Fock-Bogoliubov (HFB) gaps in heavy nuclei.

2. RICHARDSON'S EXACT SOLUTION OF THE PAIRING MODEL

We begin our discussion of Richardson's solution of the pairing model by assuming a system of N fermions moving in a set of L single-particle states l, each having a total degeneracy Ω_l, and with an additional internal quantum number m that labels the states within the l subspace. If the quantum number l represents angular momentum, the degeneracy of a single-particle level l is $\Omega_l = 2l+1$ and $-l \leq m \leq l$. In general, however, l could label different quantum numbers. The operators on which the pairing Hamiltonian is based are

$$\widehat{n}_l = \sum_m a^\dagger_{lm} a_{lm} \ , \quad P^\dagger_l = \sum_m a^\dagger_{lm} a^\dagger_{l\overline{m}} = (P_l)^\dagger \ , \tag{1}$$

where $a^\dagger_{lm}\,(a_{lm})$ creates (annihilates) a particle in the state (lm) and the state $(l\overline{m})$ is the corresponding time-reversed state or conjugate state in case of broken time reversal system. The number operator $\widehat{n}_l$, the pair creation operator $P^\dagger_l$ and the pair annihilation operator P_l close the commutation algebra

$$\left[\widehat{n}_l, A^\dagger_{l'}\right] = 2\delta_{ll'} \ , \quad \left[A_l, A^\dagger_{l'}\right] = 2\delta_{ll'}\left(\Omega_l - 2\widehat{n}_l\right) \ . \tag{2}$$

The reduced BCS model also known as the constant pairing model solved by Richardson is

$$H_P = \sum_l \varepsilon_l \widehat{n}_l + \frac{g}{2}\sum_{ll'} P^\dagger_l P_{l'}. \tag{3}$$

The approximation leading to the Richardson Hamiltonian must be supplemented by a cut-off restricting the number of l states in the single-particle space.

In condensed-matter problems this cut-off is naturally provided by the Debye frequency of the phonons. In nuclear physics, the choice of cut-off depends on the specific nucleus and on the set of active or valence orbits in which the pairing correlations develop.

A generic state of M correlated fermion pairs and ν unpaired particles can be written as

$$|n_1, n_1, \cdots, n_L, \nu\rangle = \frac{1}{\sqrt{\mathcal{N}}} \left(P_1^\dagger\right)^{n_1} \left(P_2^\dagger\right)^{n_2} \cdots \left(P_L^\dagger\right)^{n_L} |\nu\rangle \,, \tag{4}$$

where $\mathcal{N}$ is a normalization constant. The number of pairs n_l in level l is constrained by the Pauli principle to be $0 \leq 2n_l + \nu_l \leq \Omega_l$, where ν_l denotes the number of unpaired particles in that level. The unpaired state $|\nu\rangle = |\nu_1, \nu_2 \cdots \nu_L\rangle$, with $\nu = \sum_l \nu_l$, is defined such that

$$P_l \, |\nu\rangle = 0 \,, \;\; \widehat{n}_l \, |\nu\rangle = \nu_l \, |\nu\rangle \,. \tag{5}$$

A state with ν unpaired particles is said to have seniority ν. The total number of collective (or Cooper) pairs is $M = \sum_l n_l$ and the total number of particles is $N = 2M + \nu$.

The dimension of the Hamiltonian matrix in the Hilbert space of Eq. (4) grows exponentially with the number of pairs and it quickly exceeds the limits of large-scale diagonalizing. For the pairing Hamiltonian (3) to be exactly solvable implies that the exponential complexity of the problem should be reduced to an algebraic problem. Indeed, Richardson showed that the exact unnormalised eigenstates of the Hamiltonian of Eq. (3) can be written as

$$|\Psi\rangle = B_1^\dagger B_2^\dagger \cdots B_M^\dagger \, |\nu\rangle \;\;, \tag{6}$$

where the collective pair operators B_α have the form appropriate to the solution of the one-pair problem,

$$B_\alpha^\dagger = \sum_l \frac{1}{2\varepsilon_l - E_\alpha} P_l^\dagger. \tag{7}$$

In the one-pair problem, the quantities E_α that enter Eq. (7) are the eigenvalues of the pairing Hamiltonian, *i.e.*, the *pair energies*. Richardson proposed to use the M pair energies E_α in the many-body wave function of Eq. (6) as parameters which are then chosen to fulfill the eigenvalue equation $H_P \, |\Psi\rangle = E \, |\Psi\rangle$.

Richardson showed that the ansatz (6) is an eigenstate of the Hamiltonian (3) if the pair energies E_α are a particular solution of the set of M nonlinear coupled equations

$$1 - 4g \sum_{l} \frac{d_l}{2\varepsilon_l - E_\alpha} + 4g \sum_{\beta(\neq\alpha)} \frac{1}{E_\alpha - E_\beta} = 0 \,, \tag{8}$$

where $d_l = \frac{\nu_l}{2} - \frac{\Omega_l}{4}$ is related to the effective pair degeneracy of single-particle level l.

The energy eigenvalue associated with a given solution for the pair energies is

$$E = \sum_{l} \varepsilon_l \nu_l + \sum_{\alpha} E_\alpha. \tag{9}$$

There are as many independent solutions of the equation (8) as the dimension of the Hilbert space. Therefore, the ansatz (6) provides the complete set of eigenstates of the Richardson Hamiltonian. Unlike the case of a single pair where the pair energy E corresponds to the energy eigenvalue and therefore it is real, for the many-body case the pair energies could be real or they could appear in pairs of complex conjugate values. Upon inspection of the pair wave function (7), we conclude that only pairs with complex pair energies represent truly correlated pairs.

3. RICHARDSON-GAUDIN INTEGRABLE MODELS

Classical integrability is a crucial concept for the study of dynamics of classical system. A classical system with *M* degrees of freedom is integrable if it possesses *M* independent integrals of motion that fulfill the Poisson brackets algebra. The important consequence of classical integrability is that the evolution of the system can be obtained by effective *integration* in the action-angle variables. A natural extension of the concept of integrability to quantum systems would require that a system with *M* quantum degrees of freedom should have *M* independent hermitian operators that commute among themselves, the integrals of motion. However, quantum integrability suffers from the serious drawback of the impossibility to proof the independence of two commuting hermitian operators. In fact, as shown by von Neumann in 1931 [10], two commuting hermitian operators could be expressed as a function of a third hermitian operator. Precisely, this difficulty makes quantum integrability a still debated concept and an active research field. In spite of this drawback, we will use the generalization of the definition of classical integrability to quantum systems, employing as an independent confirmation the study of the statics of level spacings which, as shown by Berry and Tabor [11], should be described by a Poisson distribution if the system is quantum integrable. The RG models are based on the SU(2) algebra with three generators S^z, S^+ and S^-. The three elements have the following commutation algebra:

$$\left[S_l^z, S_{l'}^{\pm}\right] = \pm\delta_{ll'} S_l^{\pm} \ , \quad \left[S_l^+, S_{l'}^-\right] = 2\delta_{ll'} S_l^z \ . \tag{10}$$

It is important to note that the SU(2) generators may have different representations in terms of fermionic creation and annihilations operators as well as the inherent angular momentum representation. Each representation gives rise to different physical problems. We are interested here to make contact with the pair representation associated with pair superfluidity and to the Richardson's exact solution. In doing so, we note that it could be established a relation with the three operators of the Richardson Hamiltonian (1) such that they fulfilled the commutation relations of SU(2).

$$S_l^z = 2\widehat{n}_l - \frac{\Omega_l}{2}, \quad S_l^+ = \frac{1}{2} P_l^{\dagger} = (S_l)^{\dagger} \tag{11}$$

Taking into account that the SU(2) has one quantum degree of freedom, our task will be to find a set of L hermitian operators constructed in terms of the SU(2) generators such that they commute with the total z component $S^z = \sum_l S_l^z$. The reason for this restriction is that in the pair representation $S^z = 2\widehat{N} - \frac{1}{2}\sum_l \Omega_l$, and therefore the operators will preserve the total number of particles. Let us write the most general linear a quadratic operators R_l

$$R_l = S_l^z + 2g \sum_{l'(\neq l)} \left[\frac{X_{ll'}}{2} \left(S_l^+ S_{l'}^- + S_l^- S_{l'}^+ \right) + Y_{ll'} S_l^z S_{l'}^z \right] . \tag{12}$$

Where the X and Y matrices are yet arbitrary. Using this freedom to impose integrability, it is straightforward to show that the conditions for these operators to commute among themselves $[R_l, R_l']$ are

$$Y_{ij} X_{jk} + Y_{ki} X_{jk} + X_{ki} X_{ij} = 0 \ . \tag{13}$$

These set of conditions are precisely the same found by Gaudin [5] in his integrable spin model known as the Gaudin magnet. There two generic solutions for the Gaudin conditions (13) giving rise to two families of integrable models

I. The Rational Model

$$X_{ij} = Y_{ij} = \frac{1}{\eta_i - \eta_j} \tag{14}$$

II. The Hyperbolic Model

$$X_{ij} = \frac{1}{\sinh(\eta_i - \eta_j)} \ , \ Y_{ij} = \coth(\eta_i - \eta_j) \tag{15}$$

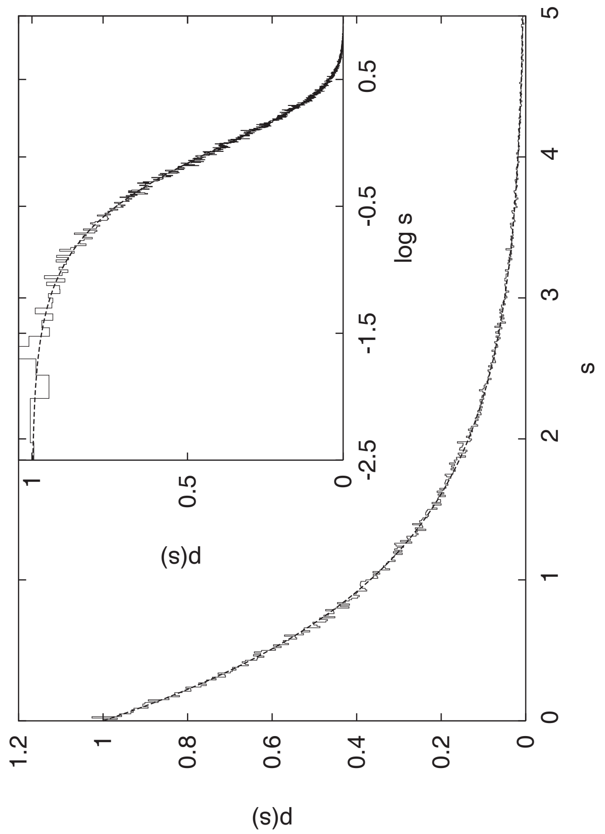

Fig. 1 – Nearest neighbor spacing distribution, $p(s)$, for 200 ensemble members each one with $L = 13$, $M = 6$ and randomly chosen parameters. The dashed curve correspond to the Poisson limit.

Each model defines a completely integrable family where the L operators R_l play the role of the integrals of motion. Any function of the integrals of motion define an integrable Hamiltonian. In particular one and two-body Hamiltonians arise as linear combinations of the integrals of motions $H = \sum_l \varepsilon_l R_l$. Both families are exactly solvable in the sense that one can formally write the form of the complete set of eigenstates common to the set of integrals of motion (see [6] for details). Moreover, the rational family has the same set of eigenstates represented by the Richardson ansatz (6) for the solution of the solution of the pairing model.

Within the rational family, if we choose the $\varepsilon_l = \eta_l$ the integrable Hamiltonian reduces to the Richardson Hamiltonian (3). In fact, this was precisely the way in which it was shown that the reduced BCS Hamiltonian (3) was integrable [12] even before the advent of the RG models and without having the knowledge of the Richardson exact solution.

One of the most important characteristics of the RG models is that they posses a large number of free parameters that could be adjusted to define a realistic pairing interaction adapted to the physical system of interest. Alternative, they could be chosen randomly still preserving quantum integrability. As mentioned before, Berry and Tabor showed in the semi-classical limit that the spectral distribution of quantum integrable Hamiltonians should be of the Poisson type. Though the level spacing distribution of all quantum integrable systems with more than one degree of freedom approximately follow a Poisson distribution, the numerical tests are limited by size of the Hilbert space amenable to an exact diagonalizing. Typically, just a few thousand of energy levels could be use to construct the level spacing histogram. In order to verify the correctness of the Berry and Tabor conjecture we have studied an ensemble of 200 Hamiltonians derived from the rational model selecting the free parameters randomly [13]. We used the case of $(L, M) = (13, 6)$ with a Hilbert space dimension $D = 1716$. In the Poisson limit, characteristic of a regular system, the nearest neighbor spacing distribution behaves as $p(s) = \exp(-s)$. Fig. 1 compares the $p(s)$ distribution of our ensemble with the expected Poisson limit. It can be seen that the histogram and the theoretical curve match perfectly. Actually, figure 1 constitutes the most precise numerical verification of the Berry and Tabor theoretical proof due to the fact that we were able to accumulate statistics by using an ensemble of random integrable Hamiltonians which wouldn't be possible for any other integrable model.

The rational model has been extensively exploited in the last decade in applications to nuclear structure, cold atomic gases, quantum dots, ultrasmall superconducting grains, quantum optical models, *etc.*. We will not continue describing these applications which are summarize in two recent reviews [7, 14]. Instead we will present in the next section the first physical applications of the hyperbolic model.

4. THE HYPERBOLIC MODEL

We start with the integrals of motion of the hyperbolic RG model [6], which can be written in a compact form [14] by making the replacements $\sinh(x) = \frac{\eta-1}{2\sqrt{\eta}}$ and $\coth(x) = \frac{\eta-1}{\eta+1}$ as

$$R_i = S_i^z + 2g \sum_{j \neq i} \left[\frac{\sqrt{\eta_i \eta_j}}{\eta_i - \eta_j} \left(S_i^+ S_j^- + S_i^- S_j^+ \right] \right) + \frac{\eta_i + \eta_j}{\eta_i - \eta_j} S_i^z S_j^z \right], \qquad (16)$$

where S_i^z, $S_i^{\pm}$, are the three generators of the $SU(2)_i$ algebra of mode i, $i = 1, \cdots, L$, with spin representation s_i such that $\langle S_i^2 \rangle = s_i(s_i + 1)$. We assume that there are L copies of the $SU(2)$ algebra or equivalently L modes. Therefore, the L operators R_i contain L free parameters η_i plus the strength of the quadratic term g. The integrals of motion (16) commute with the z component of the total spin $S^z = \sum_{i=1}^{L} S_i^z$.

It is worthwhile to verify that the set of operators R_i commute among themselves, conforming a complete set of integrals of motion. Therefore, they have a complete set of common eigenstates which are parametrized by the ansatz

$$|\Psi\rangle = \prod_{\alpha=1}^{M} S_\alpha^+ |\nu\rangle \,,\; S_\alpha^+ = \sum_i \frac{\sqrt{\eta_i}}{\eta_i - E_\alpha} S_i^+ \,, \tag{17}$$

where the E_α are the pair energies or pairons which are to be determined such that the ansatz (17) satisfies the eigenvalue equations $R_i |\Psi\rangle = r_i |\Psi\rangle$.

In the pairing representations each $SU(2)$ copy is associated with a single particle level i and M is the number of active pairs. The vacuum $|\nu\rangle$ is defined by a set of seniorities, $|\nu\rangle$ = $|\nu_1, \nu_2, \ldots, \nu_l\rangle$, where the *seniority* ν_i is the number of unpaired particles in level i with single particle degeneracy Ω_i, such that $s_i = (\Omega_i - 2\nu_i)/4$.

Although any function of the integrals of motion generates an exactly solvable Hamiltonian, we will restrict ourselves in this presentation to the simple linear combination $H = \sum_i \eta_i R_i$ that after some algebraic manipulations reduces to

$$H = \sum_i \eta_i S_i^z - G \sum_{i,j} \sqrt{\eta_i \eta_j} S_i^+ S_j^- \,. \tag{18}$$

This separable Hamiltonian has the eigenvectors (17) and the eigenvalues $E = \sum_i < \nu | S_i^z | \nu > + \sum_\alpha E_\alpha$, where the pairons E_α are a solution of the set of non-linear Richardson equations

$$\sum_i \frac{s_i}{\eta_i - E_\alpha} - \sum_{\alpha'(\neq\alpha)} \frac{1}{E_{\alpha'} - E_\alpha} = \frac{Q}{E_\alpha}, \tag{19}$$

with $Q = \frac{1}{4\gamma} - \frac{L_c}{4} + \frac{M-1}{2}$, $L_c = 2\sum_i s_i$, and M is the number of pairons.

Each particular solution of Eq. (19) defines a unique eigenstate. For the remaining discussion we will assume that $\langle \nu | H_h | \nu \rangle = 0$, which amounts to a simple shift in the energy scale, without loss of generality.

5. THE $p_x + ip_y$ PAIRING HAMILTONIAN

In recent years p-wave paired superfluids have attracted a lot of attention, in part due to their exotic properties [15]. Of particular interest is the chiral two-dimensional (2D) $p_x + ip_y$ superfluid of spinless fermions, that supports a topological phase with zero energy Majorana modes [16] and, unlike the s-wave superfluid, it has a quantum phase transition in the crossover from BCS to BEC whose properties are not yet well understood. Therefore, the derivation of an exactly solvable model could be essential for the understanding of this exotic superfluid.

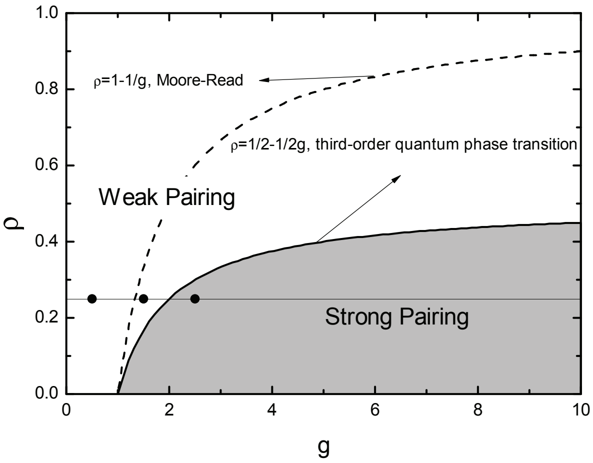

Fig. 2 – Phase diagram of the $p_x + ip_y$ model in terms of the density ρ and the pairing strength g. The three circles at quarter filling indicate the configurations studied in the following figure.

In two spatial dimensions, one can define a representation of the $SU(2)$ algebra in terms of creation (annihilation) spinless fermions operators in momentum space, $c_k^\dagger$ (c_k). Each pair of states $(k, -k)$ is associated to a single-particle level η_k, where the index k now refers to the momentum in 2D (in order to avoid double counting we select $k_x > 0$ to label the levels). Furthermore, one can include a phase factor in the definition of $SU(2)$ generators:

$$S_k^z = \frac{1}{2}\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k} - 1\right), S_k^+ = \frac{k_x + ik_y}{|k|} c_k^\dagger c_{-k}^\dagger, S_k^- = \frac{k_x - ik_y}{|k|} c_{-k} c_k. \quad (20)$$

By taking $\eta_k = k^2$, one obtains the exactly solvable $p_x + ip_y$ model first introduced Ibañez *et al.* [17]:

$$H_{p_x+ip_y} = \sum_{k,k_x>0} \frac{k^2}{2}\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}\right) - G \sum_{\substack{k,k_x>0,\\ k',k'_x>0}} (k_x + ik_y)(k'_x - ik'_y) c_k^\dagger c_{-k}^\dagger c_{-k'} c_{k'}. \quad (21)$$

Coming back to the Richardson equations (19) that solves the Hamiltonian (21), we recognize two special cases: *case (i)* all pairons are real and negative if $\frac{1}{G} \leq L - 2M + 1$; we will see that the boundary coincides with the phase transition line. *case (ii)* all pairons converge to zero for $\frac{1}{G} = L - M + 1$; this situation determines the so called *Moore-Read* line [17, 18], with interesting properties associated with the fractional quantum Hall effect. Between these two regimes, a fraction of the pairons can converge to zero at integer values of G^{-1}. The phase diagram of the $p_x + ip_y$ Hamiltonian (21) depicted in Fig. 3 is characterized by the density $\rho = M/L$ and the scaled pairing strength $g = GL$. The transition between the strong pairing region (BEC) with all pairons real and negative and the weak pairing region (BCS) takes place when one pairon change sign implying that one of the bound molecules in the BEC gets unbounded. For this reason we characterized the transition as a confinement-deconfinement quantum phase transition.

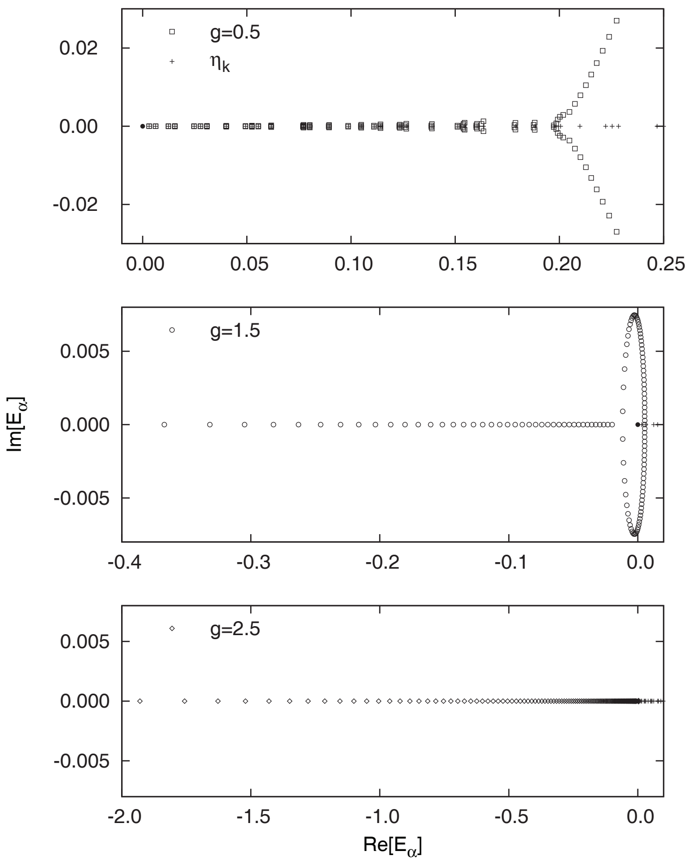

Fig. 3 – Pairon distribution for $L = 504$ at quarter filling, $\rho = \frac{M}{L} = 0.25$, for $g = GL = 0.5, g = 1.0$, and $g = 2.5$.

In order to get a more quantitative picture of the pairon distribution in the three regions of the quantum phase diagram, we plot in Fig. 2 the pairon distributions for three representative values of the coupling strength, $g = 0.5, 1.5, 2.5$, at quarter filling for a disk of radius 18 corresponding to a total pair degeneracy $L = 504$. The positions of these points in the quantum phase diagram of Fig. 3 is indicated by the three filled circles. In the weak coupling BCS region part of the pairons stick to the lower part of the real positive axis, while the remaining pairons form an arc in the complex plane. Approaching the Moore-Read line it looks like the arc is going to close around the origin, but just at the Moore-Read line all pairons collapse to zero, and then a first real negative pairon emerges. In the intermediate weak pairing region a successive series of collapses ensues, at integer values of Q, each time producing one more real negative pairon and reducing the size of the arc around the origin. When the last pairon turns real and negative, the system enters the strong pairing phase. From then on the most negative pairon diverges proportional to the interaction strength G, while the least negative pairon converges to a finite value.

In order characterize the quantum phase transition, we study the energy density derivatives as describe by the BCS theory which is exact in the thermodynamic limit. As can be seen in Fig. 3, the third derivative shows a discontinuity confirming that the phase transition is third order in the Ehrenfest classification.

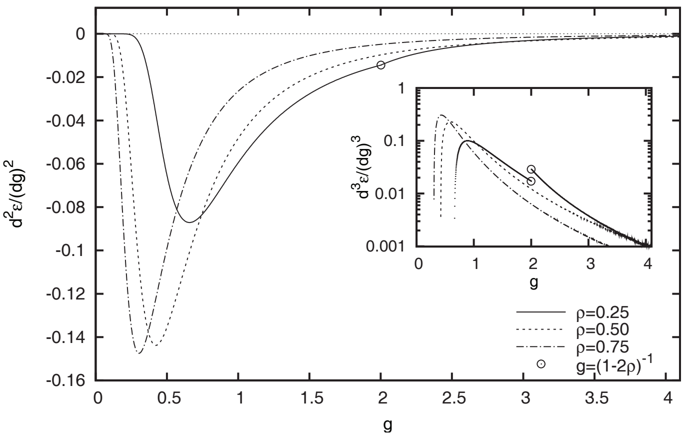

Fig. 4 – Higher order derivatives of the energy density as a function of g for various densities. The open circles mark the transition point at $g = (1 - 2\rho)^{-1}$.

6. THE INTEGRABLE NUCLEAR PAIRING HAMILTONIAN

Let us come back to the separable pairing Hamiltonian (18) to note that if we interpret the parameters η_k as single particle energies corresponding to a nuclear mean field potential, the pairing interaction has the unphysical behavior of increasing the strength with energy. In order to reverse this unwanted effect we define $\eta_k = 2(\varepsilon_k - \alpha)$, where the free parameter α plays the role of an energy cut-off and ε_k is the single particle energy in the mean field level k. Making use of the pair representation of the $SU(2)$, $S_k^+ = c_k^\dagger c_{\overline{k}}^\dagger$, $S_k^z = \frac{1}{2}(c_k^\dagger c_k + c_{\overline{k}}^\dagger c_{\overline{k}} - 1)$, the exactly solvable pairing Hamiltonian (18) reduces to

$$H = \sum_{k>0} \varepsilon_k \left(c_k^\dagger c_k + c_{\overline{k}}^\dagger c_{\overline{k}} \right) - G \sum_{kk',>0} \sqrt{(\alpha - \varepsilon_k)(\alpha - \varepsilon_{k'})} c_k^\dagger c_{\overline{k}}^\dagger c_{\overline{k'}} c_{k'} \qquad (22)$$

Our aim is to compare at the BCS level of approximation the results coming from the integrable Hamiltonian (22) with those from a Gogny HFB calculation. As a first step in ascertain the quality of the Hamiltonian (22) to reproduce the superfluid features of heavy nuclei we compare the pairing tensor $u_k v_k$ and the pairing gaps Δ_k with those of a self-consistent mean field Gogny calculation in the canonical basis. The pairing gaps and pairing tensor of the integrable pairing Hamiltonian in the BCS approximation are

$$\Delta_k^{Exact} = G\sqrt{\alpha - \varepsilon_k} \sum_{k'>0} \sqrt{\alpha - \varepsilon_{k'}} < c_{\overline{k'}} c_{k'} >= \Delta \sqrt{\alpha - \varepsilon_k} \qquad (23)$$

$$u_k v_k = \frac{\Delta \sqrt{\alpha - \varepsilon_k}}{2\sqrt{(\varepsilon_k - \mu)^2 + (\alpha - \varepsilon_k)\Delta^2}}. \qquad (24)$$

Note that the gaps Δ_k and the pairing tensor $u_k v_k$ depend on a single gap parameter Δ and have a square root dependence on the single particle energy. Hence, the model has a highly restricted form for both magnitudes that we will test against the Gogny gaps $\Delta_k^G = \sum_{k'} V_{k\overline{k},\overline{k'}k'} u_{k'}^G v_{k'}^G$ and pairing tensor $u_k^G v_k^G$, where $V_{k\overline{k},\overline{k'}k'}$ are the matrix elements of the Gogny force in the canonical basis and $(u^G v^G)$ is the HFB eigenvector. We take the single particle energies ε_k of the integrable Hamiltonian from the HF energies of the Gogny HFB calculations and we set up a fitting procedure for the two model parameters α and G. We performed the first application to ^{238}U obtaining the values $G = 1.99 \times 10^{-3}$ MeV and $\alpha = 25$ MeV for the proton system. The number of resulting active orbits is $L = 148$ with $M = 46$ proton pairs. The corresponding dimension of the Hilbert space is $D = 4.83 \times 10^{38}$, well beyond the limits of large scale diagonalizing. However, the exact solution reduces to solve a problem of 46 non-linear coupled equations.

In figure 5 we plot the pairing tensor and the gaps for protons in ^{238}U. In spite of the significant dispersion of the Gogny gaps due to the details of the Gogny force in the canonical basis, it is clear that the integrable gaps follow correctly the global trend. It is interesting to note that a constant pairing interaction, extensively used in the past and also exactly solvable within the rational family of RG models, would give a non reliable constant gap (horizontal line).

These preliminary results suggest that the hyperbolic model could be extremely useful in nuclear structure calculations as a realistic exactly solvable benchmark to test approximations beyond HFB. On a more ambitious respect, it might be possible to fit the pairing strength G as a function of N and Z to the whole table of nuclides

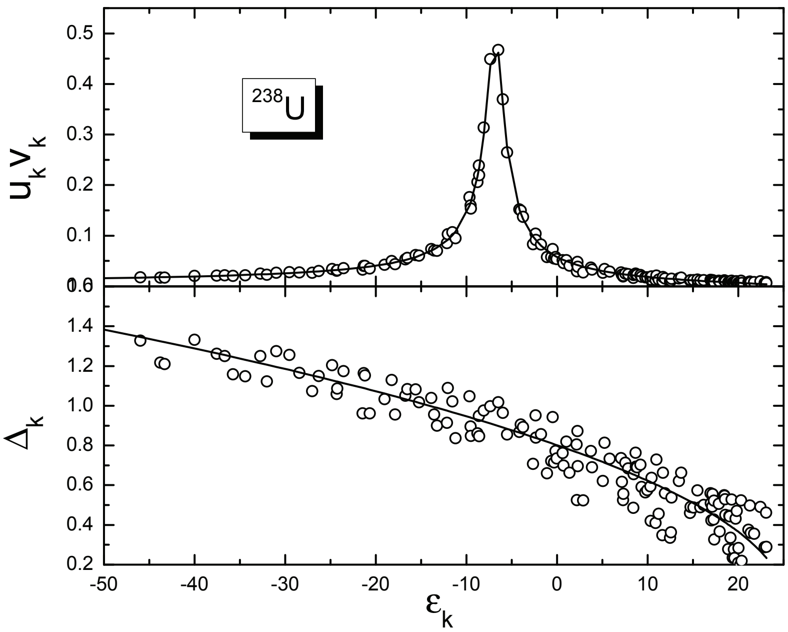

Fig. 5 – Pairing tensor and gaps for protons in ^{238}U. Open circles are Gogny HFB calculations in the canonical basis while the continuous lines correspond to the hyperbolic Hamiltonian.

and to set up a program of self-consistent Hartree-Fock plus exact pairing. Work along these lines is in progress.

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