

SOLVING NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS USING EXP-FUNCTION AND $\left(\frac{G'}{G}\right)$ -EXPANSION METHODS

AHMET BEKIR¹, ÖZKAN GÜNER², ALI H. BHRAWY^{3,4}, ANJAN BISWAS^{3,5}

¹Eskisehir Osmangazi University, Art-Science Faculty,
Department of Mathematics-Computer, Eskisehir, Turkey

²Dumlupınar University, School of Applied Sciences,
Department of Management Information Systems, Kutahya, Turkey

³Department of Mathematics, Faculty of Science,
King Abdulaziz University, Jeddah 21589, Saudi Arabia

⁴Department of Mathematics, Faculty of Science,
Beni-Suef University, Beni-Suef 62511, Egypt

⁵Department of Mathematical Sciences,
Delaware State University, Dover, DE 19901-2277, USA
E-mail: biswas.anjan@gmail.com

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This paper presents an approach for solving fractional differential equations by employing the exp-function method and $\left(\frac{G'}{G}\right)$ -expansion method. These methods were applied in two examples to solve non-linear fractional differential equations. The fractional derivatives are described in the modified Riemann–Liouville sense. As a result, many exact analytical solutions are obtained including hyperbolic function solutions and trigonometric function solutions. The results also show that the methods are very effective and convenient for solving nonlinear partial differential equations of fractional order.

Key words: Exact solution; One-dimensional nonlinear fractional wave equation; Time-fractional reaction-diffusion equation.

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1. INTRODUCTION

For the last several decades, fractional calculus has found diverse applications in various scientific and technological fields such as control theory, physics, engineering, biology, computational fluid mechanics, systems identification, control theory, finance and fractional dynamics, signal and image processing, and many other physical processes [1, 2]. Since fractional differential equations (FDEs) are used to describe a large variety of physical phenomena, finding exact solutions to FDEs is an important subject and a hot topic. But these nonlinear fractional differential equations are difficult to get their exact solutions. An effective method for solving such equations is needed.

Li and He [3, 4] proposed a fractional complex transform to convert fractional differential equations into ordinary differential equations (ODEs), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. Then many powerful and efficient methods have been proposed so far including the fractional $\left(\frac{G'}{G}\right)$ -expansion method [5–9], the fractional exp-function method [10–12], the fractional first integral method [13, 14], the fractional sub-equation method [15, 16], the fractional fractional functional variable method [17], the fractional modified trial equation method [18, 19], the fractional simplest equation method [20] and so on. Using these methods, solutions with various forms for given fractional differential equation have been established. More recently, there are a lot of studies on the approximate solutions for nonlinear classical and fractional differential equations, see for instance, [21–30].

There are several different definitions of the concept of a fractional derivative [2]. Some of these are Riemann-Liouville, Grunwald-Letnikov, Caputo, and modified Riemann-Liouville derivative. The most commonly used definitions are the Riemann-Liouville and Caputo derivatives.

The Jumarie's modified Riemann-Liouville derivative [31] of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0)) d\xi, & 0 < \alpha < 1 \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1, n \geq 1. \end{cases} \quad (1)$$

Some important properties of the fractional modified Riemann-Liouville derivative are summarized and useful formulas of them are [32]

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \quad (2)$$

$$D_t^\alpha (f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \quad (3)$$

$$D_t^\alpha f[g(t)] = f'_g[g(t)]D_t^\alpha g(t) = D_g^\alpha [g(t)](g'(t))^\alpha. \quad (4)$$

The above equations play an important role in fractional calculus in the following sections.

The organization of this paper is as follows. In Section 2, we give the description of the exp-function method. In Section 3 we give applications of the exp-function method and in Section 4 we give the description of the $\left(\frac{G'}{G}\right)$ -expansion method. Then in Section 5, we give applications of the $\left(\frac{G'}{G}\right)$ -expansion method. Some conclusions are given in last Section.

2. DESCRIPTION OF THE \exp -FUNCTION METHOD FOR SOLVING FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

We consider the following general nonlinear FDEs of the type

$$P(u, D_t^\alpha u, D_t^\alpha D_t^\alpha u, \dots) = 0, \quad 0 < \alpha < 1 \quad (5)$$

where u is an unknown function, and P is a polynomial of u and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved.

The traveling wave variable is

$$u(x, t) = U(\xi), \quad (6)$$

$$\xi = kx - \frac{ct^\alpha}{\Gamma(1 + \alpha)}, \quad (7)$$

where c and k are nonzero arbitrary constants.

Then we can rewrite Eq. (5) in the following nonlinear ODE;

$$Q(U, U', U'', U''', \dots) = 0. \quad (8)$$

where the prime denotes the derivation with respect to ξ .

According to the \exp -function method, which was developed by He and Wu [33], we assume that the wave solution can be expressed in the following form

$$U(\xi) = \frac{\sum_{n=-c}^d a_n \exp[n\xi]}{\sum_{m=-p}^q b_m \exp[m\xi]} \quad (9)$$

where p, q, c , and d are positive integers which are known to be further determined, a_n and b_m are unknown constants.

We can rewrite Eq. (9) in the following equivalent form

$$U(\xi) = \frac{a_{-c} \exp[-c\xi] + \dots + a_d \exp[d\xi]}{b_{-p} \exp[-p\xi] + \dots + b_q \exp[q\xi]}. \quad (10)$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems.

To determine the value of c and p , we balance the linear term of highest order of equation Eq. (14) with the highest order nonlinear term. Similarly, to determine the value of d and q , we balance the linear term of lowest order of Eq. (8) with lowest order nonlinear term [34–36].

3. APPLICATIONS OF THE PROPOSED METHOD

Example 1:

We solve the nonlinear fractional partial differential equation such as one-dimensional nonlinear fractional wave equation [37]:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + au_{xx} + \beta u + \gamma u^3 = 0, \quad (11)$$

where a , β , and γ are arbitrary constants and α is a parameter describing the order of the fractional time derivative.

For our purpose, we introduce the following transformations;

$$u(x, t) = U(\xi) \quad (12)$$

$$\xi = kx - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \quad (13)$$

where k and c are constants.

Substituting (13) into (11), reduces (11) into an ODE

$$-cU' + ak^2U'' + \beta U + \gamma U^3 = 0, \quad (14)$$

where " U' " = $\frac{dU}{d\xi}$.

Balancing the order of U'' and U^3 in Eq. (14), we obtain

$$U'' = \frac{c_1 \exp[-(c+3p)\xi] + \dots}{c_2 \exp[-4p\xi] + \dots}, \quad (15)$$

and

$$U^3 = \frac{c_3 \exp[-3c\xi] + \dots}{c_4 \exp[-3p\xi] + \dots}, \quad (16)$$

where c_i are determined coefficients only for simplicity. Balancing highest order of Exp-function in Eqs. (15) and (16) we obtain

$$-(c+3p) = -(3c+p), \quad (17)$$

which leads to the result

$$p = c. \quad (18)$$

In the same way to determine values of d and q , we balance the linear term of lowest order in Eq.(14),

$$U'' = \frac{\dots + d_1 \exp[(d+3q)\xi]}{\dots + d_2 \exp[4q\xi]}, \quad (19)$$

and

$$U^3 = \frac{\dots + d_3 \exp[3d\xi]}{\dots + d_4 \exp[3q\xi]}, \quad (20)$$

where d_i are determined coefficients only for simplicity. From (19) and (20), we have

$$3q + d = 4d, \quad (21)$$

and this gives

$$q = d. \quad (22)$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so Eq. (10) reduces to

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \quad (23)$$

Substituting Eq. (23) into Eq. (14), and by the help of *Maple* software, we have

$$\begin{aligned} \frac{1}{A} [R_3 \exp(3\xi) + R_2 \exp(2\xi) + R_1 \exp(\xi) + R_0 \\ + R_{-1} \exp(-\xi) + R_{-2} \exp(-2\xi) + R_{-3} \exp(-3\xi)] = 0, \end{aligned} \quad (24)$$

where

$$\begin{aligned} A &= (b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi))^3, \\ R_3 &= \gamma a_1^3 + \beta a_1 b_1^2, \\ R_2 &= \beta a_0 b_1^2 + c a_0 b_1^2 + 3\gamma a_1^2 a_0 + 2\beta a_1 b_1 b_0 - c a_1 b_1 b_0 \\ &\quad + a k^2 a_0 b_1^2 - a k^2 a_1 b_1 b_0, \\ R_1 &= 2c a_{-1} b_1^2 + \beta a_{-1} b_1^2 + 3\gamma a_1 a_0^2 - c a_1 b_0^2 + \beta a_1 b_0^2 + 3\gamma a_1^2 a_{-1} \\ &\quad + c a_0 b_1 b_0 + a k^2 a_1 b_0^2 + 2\beta a_1 b_1 b_{-1} + 2\beta a_0 b_1 b_0 + 4a k^2 a_{-1} b_1^2 \\ &\quad - 2c a_1 b_1 b_{-1} - 4a k^2 a_1 b_1 b_{-1} - a k^2 a_0 b_1 b_0, \\ R_0 &= 3a k^2 a_1 b_0 b_{-1} + 3a k^2 a_{-1} b_1 b_0 + 6\gamma a_1 a_0 a_{-1} + 2\beta a_1 b_0 b_{-1} \\ &\quad - 6a k^2 a_0 b_1 b_{-1} + \gamma a_0^3 + \beta a_0 b_0^2 + 2\beta a_0 b_1 b_{-1} + 2\beta a_{-1} b_1 b_0 \\ &\quad - 3c a_1 b_0 b_{-1} + 3c a_{-1} b_1 b_0, \\ R_{-1} &= -2c a_1 b_{-1}^2 + 3\gamma a_0^2 a_{-1} + \beta a_{-1} b_0^2 + 3\gamma a_1 a_{-1}^2 + c a_{-1} b_0^2 + \beta a_1 b_{-1}^2 \\ &\quad + 2\beta a_0 b_{-1} b_0 + 2c a_{-1} b_1 b_{-1} + a k^2 a_{-1} b_0^2 + 2\beta a_{-1} b_1 b_{-1} - c a_0 b_{-1} b_0 \\ &\quad + 4a k^2 a_1 b_{-1}^2 - a k^2 a_0 b_{-1} b_0 - 4a k^2 a_{-1} b_1 b_{-1}, \\ R_{-2} &= \beta a_0 b_{-1}^2 + 3\gamma a_0 a_{-1}^2 - c a_0 b_{-1}^2 + c a_{-1} b_{-1} b_0 + 2\beta a_{-1} b_{-1} b_0 \\ &\quad + a k^2 a_0 b_{-1}^2 - a k^2 a_{-1} b_{-1} b_0 \\ R_{-3} &= \gamma a_{-1}^3 + \beta a_{-1} b_{-1}^2. \end{aligned} \quad (25)$$

Solving this system of algebraic equations by using *Maple* software, we get the following results

Case 1:

$$\begin{aligned} a_1 = 0, \quad a_0 = 0, \quad a_{-1} &= \mp \sqrt{\frac{-\beta}{\gamma}} b_{-1}, \\ b_1 = b_1, \quad b_0 = 0, \quad b_{-1} &= b_{-1}, \\ c = -\frac{3\beta}{4}, \quad k &= \mp \sqrt{\frac{\beta}{8a}}, \end{aligned} \quad (26)$$

where b_{-1} is a free parameter. Substituting these results into (23), we get the following exact solution:

$$u_{1,2}(x, t) = \frac{\mp \sqrt{\frac{-\beta}{\gamma}} b_{-1} \exp(-(\mp \sqrt{\frac{\beta}{8a}} x + \frac{3\beta}{4\Gamma(1+\alpha)} t^\alpha))}{b_1 \exp(\mp \sqrt{\frac{\beta}{8a}} x + \frac{3\beta}{4\Gamma(1+\alpha)} t^\alpha) + b_{-1} \exp(-(\mp \sqrt{\frac{\beta}{8a}} x + \frac{3\beta}{4\Gamma(1+\alpha)} t^\alpha))}. \quad (27)$$

Case 2:

$$\begin{aligned} a_1 = 0, \quad a_0 = 0, \quad a_{-1} &= \mp \sqrt{\frac{-\beta}{\gamma}} b_{-1}, \\ b_1 = 0, \quad b_0 = b_0, \quad b_{-1} &= b_{-1}, \\ c = -\frac{3\beta}{2}, \quad k &= \mp \sqrt{\frac{\beta}{2a}} \end{aligned} \quad (28)$$

where b_{-1} is free parameter. Substituting these results into (23), we get the following exact solution:

$$u_{3,4}(x, t) = \frac{\mp \sqrt{\frac{-\beta}{\gamma}} b_{-1} \exp(-(\mp \sqrt{\frac{\beta}{2a}} x + \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha))}{b_0 + b_{-1} \exp(-(\mp \sqrt{\frac{\beta}{2a}} x + \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha))}. \quad (29)$$

Case 3:

$$\begin{aligned} a_1 = 0, \quad a_0 &= \mp \sqrt{\frac{-\beta}{\gamma}} b_0, \quad a_{-1} = 0, \\ b_1 = 0, \quad b_0 &= b_0, \quad b_{-1} = b_{-1}, \\ c = \frac{3\beta}{2}, \quad k &= \mp \sqrt{\frac{\beta}{2a}} \end{aligned} \quad (30)$$

where b_0 is a free parameter. Substituting these results into (23), we get the following exact solution:

$$u_{5,6}(x, t) = \frac{\mp \sqrt{\frac{-\beta}{\gamma}} b_0}{b_0 + b_{-1} \exp(-(\mp \sqrt{\frac{\beta}{2a}} x - \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha))}. \quad (31)$$

Case 4:

$$\begin{aligned} a_1 &= 0, & a_0 &= \mp \sqrt{\frac{-\beta}{\gamma}} b_0, & a_{-1} &= 0, \\ b_1 &= b_1, & b_0 &= b_0, & b_{-1} &= 0, \\ c &= -\frac{3\beta}{2}, & k &= \mp \sqrt{\frac{\beta}{2a}} \end{aligned} \quad (32)$$

where b_0 is a free parameter. Substituting these results into (23), we get the following exact solution:

$$u_{7,8}(x, t) = \frac{\mp \sqrt{\frac{-\beta}{\gamma}} b_0}{b_1 \exp(\mp \sqrt{\frac{\beta}{2a}} x + \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha) + b_0}. \quad (33)$$

Case 5:

$$\begin{aligned} a_1 &= 0, & a_0 &= a_0, & a_{-1} &= \frac{a_0 b_0 \mp a_0^2 \sqrt{-\beta\gamma}}{b_1 \beta}, \\ b_1 &= b_1, & b_0 &= b_0, & b_{-1} &= \frac{\gamma a_0^2 + \beta b_0^2 - b_0^2 \pm a_0 b_0 \sqrt{-\beta\gamma}}{b_1 \beta}, \\ c &= -\frac{3\beta}{2}, & k &= \mp \sqrt{\frac{\beta}{2a}} \end{aligned} \quad (34)$$

where a_0 , b_0 and b_1 are free parameters. Substituting these results into (23), we get the following exact solution:

$$u_{9,10}(x, t) = \frac{a_0 + \frac{a_0 b_0 \mp a_0^2 \sqrt{-\beta\gamma}}{b_1 \beta} \exp(\mp \sqrt{\frac{\beta}{2a}} x + \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha)}{b_1 \exp(\mp \sqrt{\frac{\beta}{2a}} x + \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha) + b_0 + \frac{\gamma a_0^2 + \beta b_0^2 - b_0^2 \pm a_0 b_0 \sqrt{-\beta\gamma}}{b_1 \beta} \exp\left(-(\mp \sqrt{\frac{\beta}{2a}} x + \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha)\right)}. \quad (35)$$

Case 6:

$$\begin{aligned} a_1 &= 0, & a_0 &= \sqrt[4]{\frac{-a_{-1}^2 b_1^2 \beta}{\gamma}}, & a_{-1} &= a_{-1}, \\ b_1 &= b_1, & b_0 &= 0, & b_{-1} &= \frac{\gamma a_{-1}}{\beta} \sqrt{\frac{-\beta}{\gamma}}, \\ c &= -\frac{3\beta}{2}, & k &= \mp \sqrt{\frac{\beta}{2a}} \end{aligned} \quad (36)$$

where a_{-1} and b_1 are free parameters. Substituting these results into (23), we get the

following exact solution:

$$u_{11,12}(x,t) = \frac{\sqrt[4]{\frac{-a^2 - b_1^2 \beta}{\gamma}} + a_{-1} \exp\left(-(\mp \sqrt{\frac{\beta}{2a}} x + \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha)\right)}{b_1 \exp(\mp \sqrt{\frac{\beta}{2a}} x + \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha) + b_{-1} \exp\left(-(\mp \sqrt{\frac{\beta}{2a}} x + \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha)\right)}. \quad (37)$$

Case 7:

$$\begin{aligned} a_1 &= b_1 \sqrt{\frac{-\beta}{\gamma}}, & a_0 &= 0, & a_{-1} &= 0, \\ b_1 &= b_1, & b_0 &= 0, & b_{-1} &= b_{-1}, \\ c &= \frac{3\beta}{4}, & k &= \mp \sqrt{\frac{\beta}{8a}} \end{aligned} \quad (38)$$

where b_1 is a free parameter. Substituting these results into (23), we get the following exact solution:

$$u_{13,14}(x,t) = \frac{b_1 \sqrt{\frac{-\beta}{\gamma}} \exp(\mp \sqrt{\frac{\beta}{8a}} x - \frac{3\beta}{4\Gamma(1+\alpha)} t^\alpha)}{b_1 \exp(\mp \sqrt{\frac{\beta}{8a}} x - \frac{3\beta}{4\Gamma(1+\alpha)} t^\alpha) + b_{-1} \exp(-(\mp \sqrt{\frac{\beta}{8a}} x - \frac{3\beta}{4\Gamma(1+\alpha)} t^\alpha))}. \quad (39)$$

Case 8:

$$\begin{aligned} a_1 &= \mp \sqrt{\frac{-\beta}{\gamma}} b_1, & a_0 &= 0, & a_{-1} &= 0, \\ b_1 &= b_1, & b_0 &= b_0, & b_{-1} &= 0, \\ c &= \frac{3\beta}{2}, & k &= \mp \sqrt{\frac{\beta}{2a}} \end{aligned} \quad (40)$$

where b_1 is a free parameter. Substituting these results into (23), we get the following exact solution:

$$u_{15,16}(x,t) = \frac{\mp \sqrt{\frac{-\beta}{\gamma}} b_1 \exp(\mp \sqrt{\frac{\beta}{2a}} x - \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha)}{b_1 \exp(\mp \sqrt{\frac{\beta}{2a}} x - \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha) + b_0}. \quad (41)$$

Case 9:

$$\begin{aligned} a_1 &= \frac{a_0 b_0 \mp a_0^2 \sqrt{-\beta \gamma}}{b_{-1} \beta}, & a_0 &= a_0, & a_{-1} &= 0, \\ b_1 &= \frac{\gamma a_0^2 + \beta b_0^2 - b_0^2 \pm a_0 b_0 \sqrt{-\beta \gamma}}{b_{-1} \beta}, & b_0 &= b_0, & b_{-1} &= b_{-1}, \\ c &= \frac{3\beta}{2}, & k &= \mp \sqrt{\frac{\beta}{2a}} \end{aligned} \quad (42)$$

where a_0 , b_0 and b_{-1} are free parameters. Substituting these results into (23), we get the following exact solution:

$$u_{17,18}(x, t) = \frac{\frac{a_0 b_0 \mp a_0^2 \sqrt{-\beta} \gamma}{b_{-1} \beta} \exp(\mp \sqrt{\frac{\beta}{2a}} x - \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha) + a_0}{\frac{\gamma a_0^2 + \beta b_0^2 - b_0^2 \pm a_0 b_0 \sqrt{-\beta} \gamma}{b_{-1} \beta} \exp(\mp \sqrt{\frac{\beta}{2a}} x - \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha) + b_0 + b_{-1} \exp(-(\mp \sqrt{\frac{\beta}{2a}} x - \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha))}. \quad (43)$$

Case 10:

$$\begin{aligned} a_1 &= a_1, & a_0 &= \sqrt[4]{\frac{-a_1^2 b_{-1}^2 \beta}{\gamma}}, & a_{-1} &= 0, \\ b_1 &= \frac{\gamma a_1}{\beta} \sqrt{\frac{-\beta}{\gamma}}, & b_0 &= 0, & b_{-1} &= b_{-1}, \\ c &= \frac{3\beta}{2}, & k &= \mp \sqrt{\frac{\beta}{2a}} \end{aligned} \quad (44)$$

where a_1 and b_{-1} are free parameters.

Substituting these results into (23), we get the following exact solution:

$$u_{19,20}(x, t) = \frac{a_1 \exp(\mp \sqrt{\frac{\beta}{2a}} x - \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha) + \sqrt[4]{\frac{-a_1^2 b_{-1}^2 \beta}{\gamma}}}{\frac{\gamma a_1}{\beta} \sqrt{\frac{-\beta}{\gamma}} \exp(\mp \sqrt{\frac{\beta}{2a}} x - \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha) + b_{-1} \exp(\mp \sqrt{\frac{\beta}{2a}} x - \frac{3\beta}{2\Gamma(1+\alpha)} t^\alpha)}. \quad (45)$$

These are the exact solutions of the one-dimensional nonlinear fractional wave equation.

Remark 1: All of the solutions mentioned above have not been reported so far by other authors in the literature, to the best of our knowledge.

Example 2:

We consider the time-fractional reaction-diffusion (RD) problem [38]. This equation was solved by HAM.

$$D_t^\alpha u = u_{xx} + u(1 - u), \quad (46)$$

α is a parameter describing the order of the fractional time derivative.

Similarly we consider:

$$u(x, t) = U(\xi) \quad (47)$$

$$\xi = cx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \quad (48)$$

where c and λ are non zero constants.

Substituting (48) into (46), this equation can reduced into an ODE:

$$\lambda U' + c^2 U'' + U - U^2 = 0, \quad (49)$$

where " U'' " = $\frac{dU}{d\xi}$.

When balancing the order of U'' and U^2 in Eq.(49), we get

$$U'' = \frac{c_1 \exp[-(c+3p)\xi] + \dots}{c_2 \exp[-4p\xi] + \dots}, \quad (50)$$

and

$$U^2 = \frac{c_3 \exp[-2c\xi] + \dots}{c_4 \exp[-2p\xi] + \dots}, \quad (51)$$

where c_i are determined coefficients only for simplicity. Balancing highest order of Exp-function in Eqs.(50) and (51) we obtain

$$-(c+3p) = -(2c+2p), \quad (52)$$

and

$$p = c. \quad (53)$$

In the same way to determine values of d and q , we balance the linear term of lowest order in Eq. (49),

$$U'' = \frac{\dots + d_1 \exp[(d+3q)\xi]}{\dots + d_2 \exp[4q\xi]}, \quad (54)$$

and

$$U^2 = \frac{\dots + d_3 \exp[2d\xi]}{\dots + d_4 \exp[2q\xi]}, \quad (55)$$

where d_i are determined coefficients only for simplicity. From (54) and (55), we have

$$3q + d = 2d + 2q, \quad (56)$$

and this gives

$$q = d. \quad (57)$$

Substituting Eq.(58) into Eq.(49), and by the help of *Maple* software, we have

$$\begin{aligned} \frac{1}{A} [R_3 \exp(3\xi) + R_2 \exp(2\xi) + R_1 \exp(\xi) + R_0 \\ + R_{-1} \exp(-\xi) + R_{-2} \exp(-2\xi) + R_{-3} \exp(-3\xi)] = 0, \quad (58) \end{aligned}$$

where

$$\begin{aligned}
 A &= (b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi))^3, \\
 R_3 &= a_1 b_1^2 - a_1^2 b_1, \\
 R_2 &= -c^2 a_1 b_1 b_0 + \lambda a_1 b_1 b_0 - a_1^2 b_0 + a_0 b_1^2 + 2a_1 b_1 b_0 \\
 &\quad - \lambda a_0 b_1^2 - 2a_1 a_0 b_1 + c^2 a_0 b_1^2, \\
 R_1 &= -\lambda a_0 b_1 b_0 - c^2 a_0 b_1 b_0 + 2\lambda a_1 b_1 b_{-1} - 4c^2 a_1 b_1 b_{-1} \\
 &\quad - a_0^2 b_1 - a_1^2 b_{-1} + a_1 b_0^2 + a_{-1} b_1^2 + 2a_0 b_1 b_0 - 2a_1 a_{-1} b_1 \\
 &\quad + \lambda a_1 b_0^2 + 2a_1 b_1 b_{-1} - 2a_1 a_0 b_0 + 4c^2 a_{-1} b_1^2 - 2\lambda a_{-1} b_1^2 + c^2 a_1 b_0^2, \\
 R_0 &= 3\lambda a_1 b_0 b_{-1} + 3c^2 a_1 b_0 b_{-1} + 3c^2 a_{-1} b_1 b_0 - 6c^2 a_0 b_1 b_{-1} \\
 &\quad + a_0 b_0^2 - a_0^2 b_0 + 2a_1 b_0 b_{-1} + 2a_{-1} b_1 b_0 + 2a_0 b_1 b_{-1} - 3\lambda a_{-1} b_1 b_0 \\
 &\quad - 2a_1 a_0 b_{-1} - 2a_1 a_{-1} b_0 - 2a_0 a_{-1} b_1, \\
 R_{-1} &= -2\lambda a_{-1} b_1 b_{-1} - 4c^2 a_{-1} b_1 b_{-1} - c^2 a_0 b_{-1} b_0 + \lambda a_0 b_{-1} b_0 \\
 &\quad + a_{-1} b_0^2 - a_{-1}^2 b_1 - a_0^2 b_{-1} + a_1 b_{-1}^2 + 2\lambda a_1 b_{-1}^2 - 2a_1 a_{-1} b_{-1} \\
 &\quad + 2a_0 b_{-1} b_0 + 2a_{-1} b_1 b_{-1} - \lambda a_{-1} b_0^2 + c^2 a_{-1} b_0^2 + 4c^2 a_1 b_{-1}^2 - 2a_0 a_{-1} b_0, \\
 R_{-2} &= -\lambda a_{-1} b_0 b_{-1} - c^2 a_{-1} b_0 b_{-1} + a_0 b_{-1}^2 - a_{-1}^2 b_0 + 2a_{-1} b_0 b_{-1} \\
 &\quad + \lambda a_0 b_{-1}^2 - 2a_0 a_{-1} b_{-1} + c^2 a_0 b_{-1}^2, \\
 R_{-3} &= -a_{-1}^2 b_{-1} + a_{-1} b_{-1}^2.
 \end{aligned} \tag{59}$$

Solving this system of algebraic equations by using *Maple* software, we get the following results,

Case 1:

$$\begin{aligned}
 a_1 &= 0, \quad a_0 = 0, \quad a_{-1} = b_{-1}, \\
 b_1 &= \frac{b_0^2}{4b_{-1}}, \quad b_0 = b_0, \quad b_{-1} = b_{-1}, \\
 c &= \mp \frac{1}{\sqrt{6}}, \quad \lambda = \frac{5}{6}.
 \end{aligned} \tag{60}$$

where b_0 and b_{-1} are free parameters. Substituting these results into (23), we get the following exact solution

$$U_{1,2}(\xi) = \frac{b_{-1} \exp\left(-\left(\mp \frac{1}{\sqrt{6}} x - \frac{5t^\alpha}{6\Gamma(1+\alpha)}\right)\right)}{\frac{b_0^2}{4b_{-1}} \exp\left(\mp \frac{1}{\sqrt{6}} x - \frac{5t^\alpha}{6\Gamma(1+\alpha)}\right) + b_0 + b_{-1} \exp\left(-\left(\mp \frac{1}{\sqrt{6}} x - \frac{5t^\alpha}{6\Gamma(1+\alpha)}\right)\right)}. \tag{61}$$

If we take $b_{-1} = 1$ and $b_0 = 2$ Eq. (61) becomes

$$u_{1,2}(x, t) = \frac{\cosh\left(\mp \frac{1}{\sqrt{6}} x - \frac{5t^\alpha}{6\Gamma(1+\alpha)}\right) - \sinh\left(\mp \frac{1}{\sqrt{6}} x - \frac{5t^\alpha}{6\Gamma(1+\alpha)}\right)}{2\left(1 + \cosh\left(\mp \frac{1}{\sqrt{6}} x - \frac{5t^\alpha}{6\Gamma(1+\alpha)}\right)\right)}. \tag{62}$$

which are exact solutions of the time-fractional reaction-diffusion equation.

Case 2:

$$\begin{aligned} a_1 &= 0, & a_0 &= 0, & a_{-1} &= b_{-1}, \\ b_1 &= \frac{b_0^2}{4b_{-1}}, & b_0 &= b_0, & b_{-1} &= b_{-1}, \\ c &= -1, & \lambda &= 5. \end{aligned} \quad (63)$$

where b_0 and b_{-1} are free parameters. Substituting these results into (23), we get the following exact solution

$$U_{3,4}(\xi) = \frac{b_{-1} \exp\left(-\left(-x - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right)\right)}{\frac{b_0^2}{4b_{-1}} \exp\left(-x - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right) + b_0 + b_{-1} \exp\left(-\left(-x - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right)\right)}. \quad (64)$$

If we take $b_{-1} = 1$ and $b_0 = 2$, Eq. (64) becomes

$$u_{3,4}(x, t) = \frac{\cosh\left(-x - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right) - \sinh\left(-x - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right)}{2\left(1 + \cosh\left(-x - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right)\right)}. \quad (65)$$

which are exact solutions of the time-fractional reaction-diffusion equation.

Case 3:

$$\begin{aligned} a_1 &= 0, & a_0 &= a_0, & a_{-1} &= \frac{a_0^2}{4b_1}, \\ b_1 &= b_1, & b_0 &= a_0, & b_{-1} &= \frac{a_0^2}{4b_1}, \\ c &= i, & \lambda &= 5. \end{aligned} \quad (66)$$

where a_0 and b_1 are free parameters and $i^2 = -1$. Substituting these results into (23), we get the following exact solution

$$U_{5,6}(\xi) = \frac{a_0 + \frac{a_0^2}{4b_1} \exp\left(-\left(ix - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right)\right)}{b_1 \exp\left(ix - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right) + a_0 + \frac{a_0^2}{4b_1} \exp\left(-\left(ix - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right)\right)}. \quad (67)$$

If we take $b_1 = 1$ and $a_0 = 2$ Eq. (67) becomes

$$u_{5,6}(x, t) = \frac{2 + \cosh\left(ix - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right) - \sinh\left(ix - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right)}{2\left(1 + \cosh\left(ix - \frac{5t^\alpha}{\Gamma(1+\alpha)}\right)\right)}. \quad (68)$$

which are exact solutions of the time-fractional reaction-diffusion equation.

4. DESCRIPTION OF THE $\left(\frac{G'}{G}\right)$ -EXPANSION METHOD FOR SOLVING FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

Suppose the solution of equation (8) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$U(\xi) = \sum_{i=0}^m a_i \left(\frac{G'}{G}\right)^i, \quad a_m \neq 0, \quad (69)$$

where a_i ($i = 0, 1, 2, \dots, m$) are constants, while $G(\xi)$ satisfies the following second order linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (70)$$

with λ and μ are being constants. Then the positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in equation (8) and by substituting equation (69) into equation (8) and using equation (70) we collect all terms with the same order of $\left(\frac{G'}{G}\right)$. Then by equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for a_i ($i = 0, 1, 2, \dots, m$), λ , μ , c , and k . By solving the equations system and substituting a_i ($i = 0, 1, 2, \dots, m$), λ , μ , c , k and the general solutions of equation (70) into equation (69), we can get a variety of exact solutions of equation (5) [39–42].

5. APPLICATIONS OF THE PROPOSED METHOD

Example 1:

Now we study the one-dimensional nonlinear fractional wave equation and solve it by using the $\left(\frac{G'}{G}\right)$ -expansion method. We use the ansatz (14). By simple calculations, we balance U'' term with U^3 term in (14) and we get

$$m + 2 = 3m, \quad (71)$$

so that

$$m = 1. \quad (72)$$

Therefore, we may choose

$$U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right), \quad a_1 \neq 0. \quad (73)$$

By using Eq. (70), from Eq. (73) we have

$$U'(\xi) = -a_1 \left(\frac{G'}{G}\right)^2 - a_1 \lambda \left(\frac{G'}{G}\right) - a_1 \mu, \quad (74)$$

and

$$U''(\xi) = 2a_1 \left(\frac{G'}{G}\right)^3 + 3a_1\lambda \left(\frac{G'}{G}\right)^2 + (2a_1\mu + a_1\lambda^2) \left(\frac{G'}{G}\right) + a_1\lambda\mu, \quad (75)$$

Substituting Eq. (73)-(75) into Eq. (14), collecting the coefficients of $\left(\frac{G'}{G}\right)^i$ ($i = 0, \dots, 3$) and set it to zero we obtain the system

$$\begin{aligned} \gamma a_1^3 + 2ak^2 a_1 &= 0, \\ 3ak^2 a_1 \lambda + ca_1 + 3\gamma a_0 a_1^2 &= 0, \\ 2ak^2 a_1 \mu + ca_1 \lambda + 3\gamma a_0^2 a_1 + ak^2 a_1 \lambda^2 + \beta a_1 &= 0, \\ ca_1 \mu + \gamma a_0^3 + ak^2 a_1 \lambda \mu + \beta a_0 &= 0. \end{aligned} \quad (76)$$

Solving this system by using *Maple* software gives

$$\begin{aligned} a_0 &= \mp \sqrt{\frac{(2\mu - \lambda^2 \pm \lambda\sqrt{\lambda^2 - 4\mu})\beta}{2\gamma(\lambda^2 - 4\mu)}}, \quad a_1 = \frac{a_0}{\lambda} - \frac{\mu\beta}{\lambda\gamma(\lambda^2 - 4\mu)a_0}, \\ c &= \mp \frac{3\beta}{2\sqrt{\lambda^2 - 4\mu}}, \quad k = \mp \sqrt{\frac{\beta}{2a(\lambda^2 - 4\mu)}}, \end{aligned} \quad (77)$$

where λ and μ are arbitrary constants.

By using Eq. (77), expression (73) can be written as

$$U(\xi) = a_0 + \left(\frac{a_0}{\lambda} - \frac{\mu\beta}{\lambda\gamma(\lambda^2 - 4\mu)a_0}\right) \left(\frac{G'}{G}\right), \quad (78)$$

where $a_0 = \mp \sqrt{\frac{(2\mu - \lambda^2 \pm \lambda\sqrt{\lambda^2 - 4\mu})\beta}{2\gamma(\lambda^2 - 4\mu)}}$.

Substituting general solutions of Eq. (70) into Eq. (78) we have two types of traveling wave solutions of the one-dimensional nonlinear fractional wave equation as follows. When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solution

$$\begin{aligned} U_{1,2}(\xi) &= \frac{a_0}{2} + \frac{\mu\beta}{2(\lambda^2 - 4\mu)\gamma a_0} \\ &+ \left(\frac{a_0\sqrt{\lambda^2 - 4\mu}}{2\lambda} - \frac{\mu\beta}{2\lambda\gamma a_0\sqrt{\lambda^2 - 4\mu}}\right) \left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right), \end{aligned} \quad (79)$$

where $\xi = \mp \sqrt{\frac{\beta}{2a(\lambda^2 - 4\mu)}}x \mp \frac{3\beta}{2\Gamma(1+\alpha)\sqrt{\lambda^2 - 4\mu}}t^\alpha$ and C_1, C_2 are arbitrary constants.

On the other hand, assuming $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, the traveling wave

solution of (79) can be written as

$$u_{3,4}(x, t) = \mp \frac{1}{2} \sqrt{-\frac{\beta}{\gamma}} \left(1 + \tanh\left(\mp \frac{1}{2} \sqrt{\frac{\beta}{2a}} x \mp \frac{3\beta}{4\Gamma(1+\alpha)} t^\alpha\right) \right), \quad (80)$$

assuming $C_1 = 0, C_2 \neq 0, \lambda > 0, \mu = 0$, then we obtain

$$u_{5,6}(x, t) = \mp \frac{1}{2} \sqrt{-\frac{\beta}{\gamma}} \left(1 + \coth\left(\mp \frac{1}{2} \sqrt{\frac{\beta}{2a}} x \mp \frac{3\beta}{4\Gamma(1+\alpha)} t^\alpha\right) \right). \quad (81)$$

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solution

$$U_{7,8}(\xi) = \frac{a_0}{2} + \frac{\mu\beta}{2(\lambda^2 - 4\mu)\gamma a_0} + \left(\frac{a_0 \sqrt{4\mu - \lambda^2}}{2\lambda} + \frac{\mu\beta}{2\lambda\gamma a_0 \sqrt{4\mu - \lambda^2}} \right) \left(\frac{-C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{C_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right), \quad (82)$$

Also, if we assume $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then

$$u_{9,10}(x, t) = \mp \frac{1}{2} \sqrt{-\frac{\beta}{\gamma}} \left(1 + \tanh\left(\mp \frac{1}{2} \sqrt{\frac{\beta}{2a}} x \mp \frac{3\beta}{4\Gamma(1+\alpha)} t^\alpha\right) \right), \quad (83)$$

and when $C_1 = 0, C_2 \neq 0, \lambda > 0, \mu = 0$, the solution of Eq. (82) will be

$$u_{11,12}(x, t) = \mp \frac{1}{2} \sqrt{-\frac{\beta}{\gamma}} \left(1 + \coth\left(\mp \frac{1}{2} \sqrt{\frac{\beta}{2a}} x \mp \frac{3\beta}{4\Gamma(1+\alpha)} t^\alpha\right) \right). \quad (84)$$

So we obtain the solutions $u_{3,4}(x, t)$ and $u_{5,6}(x, t)$.

Remark 2: The established solutions above for this equation are new exact solutions so far in the literature, to the best of our knowledge.

Example 2:

Now we consider the time-fractional RD equation.

$$u(x, t) = U(\xi) \quad (85)$$

$$\xi = cx - \frac{kt^\alpha}{\Gamma(1+\alpha)}, \quad (86)$$

where c and k are non zero constants.

Substituting (86) into (46), this equation can reduced into an ODE

$$kU' + c^2U'' + U - U^2 = 0, \quad (87)$$

where " U' " = $\frac{dU}{d\xi}$.

Balancing the order of U'' and U^2 in Eq.(87), we have

$$m + 2 = 2m, \quad (88)$$

so that

$$m = 2. \quad (89)$$

Suppose that the solutions of (14) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0. \quad (90)$$

By using Eq. (70), from Eq. (87) we have

$$U''(\xi) = 6a_2 \left(\frac{G'}{G}\right)^4 + (2a_1 + 10a_2\lambda) \left(\frac{G'}{G}\right)^3 + (8a_2\mu + 3a_1\lambda + 4a_2\lambda^2) \left(\frac{G'}{G}\right)^2 + (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2) \left(\frac{G'}{G}\right) + 2a_2\mu^2 + a_1\lambda\mu, \quad (91)$$

and

$$U^2(\xi) = a_2^2 \left(\frac{G'}{G}\right)^4 + 2a_1a_2 \left(\frac{G'}{G}\right)^3 + 2a_0a_2 \left(\frac{G'}{G}\right)^2 + a_1^2 \left(\frac{G'}{G}\right)^2 + 2a_0a_1 \left(\frac{G'}{G}\right) + a_0^2. \quad (92)$$

Substituting Eq. (91)-(92) into Eq. (87), collecting the coefficients of $\left(\frac{G'}{G}\right)^i$ ($i = 0, \dots, 4$) and set it to zero we obtain the system

$$\begin{aligned} 6c^2a_2 - a_2^2 &= 0, \\ 2c^2a_1 - 2ka_2 - 2a_1a_2 + 10c^2a_2\lambda &= 0, \\ -a_1^2 - 2ka_2\lambda - 2a_0a_2 + 4c^2a_2\lambda^2 + a_2 - ka_1 + 3c^2a_1\lambda + 8c^2a_2\mu &= 0, \\ c^2a_1\lambda^2 + a_1 - 2a_0a_1 - 2ka_2\mu - ka_1\lambda + 2c^2a_1\mu + 6c^2a_2\lambda\mu &= 0, \\ -a_0^2 + 2c^2a_2\mu^2 - ka_1\mu + c^2a_1\lambda\mu + a_0 &= 0. \end{aligned} \quad (93)$$

Solving this system by *Maple* software gives

$$\begin{aligned} a_0 &= \frac{\lambda^2 - 2\mu \mp \lambda\sqrt{\lambda^2 - 4\mu}}{2(\lambda^2 - 4\mu)}, \quad a_1 = \frac{\lambda \mp \sqrt{\lambda^2 - 4\mu}}{\lambda^2 - 4\mu}, \\ a_2 &= \frac{1}{\lambda^2 - 4\mu}, \quad k = \pm \frac{5}{6\sqrt{\lambda^2 - 4\mu}}, \quad c = \mp \frac{\sqrt{6}}{6\sqrt{\lambda^2 - 4\mu}}, \end{aligned} \quad (94)$$

where λ and μ are arbitrary constants.

By using Eq. (90), expression (92) can be written as

$$U(\xi) = \frac{\lambda^2 - 2\mu \mp \lambda\sqrt{\lambda^2 - 4\mu}}{2(\lambda^2 - 4\mu)} + \left(\frac{\lambda \mp \sqrt{\lambda^2 - 4\mu}}{\lambda^2 - 4\mu}\right) \left(\frac{G'}{G}\right) + \frac{1}{\lambda^2 - 4\mu} \left(\frac{G'}{G}\right)^2,$$

Substituting general solutions of Eq. (70) into above equation we have two types of traveling wave solutions of time-fractional RD equation as follows. When

$$\lambda^2 - 4\mu > 0,$$

$$U_{1,2}(\xi) = \frac{1}{4} \mp \frac{1}{2} \left(\frac{C_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right) + \frac{1}{4} \left(\frac{C_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{C_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + C_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi} \right)^2, \quad (95)$$

$$\text{where } \xi = \mp \frac{1}{\sqrt{\lambda^2 - 4\mu}} \left(\frac{\sqrt{6}}{6} x + \frac{5}{6\Gamma(1+\alpha)} t^\alpha \right).$$

When $\lambda^2 - 4\mu < 0$,

$$U_{3,4}(\xi) = \frac{1}{4} \mp \frac{i}{2} \left(\frac{-C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{C_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right) - \frac{1}{4} \left(\frac{-C_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{C_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + C_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi} \right)^2, \quad (96)$$

$$\text{where } \xi = \mp \frac{1}{\sqrt{\lambda^2 - 4\mu}} \left(\frac{\sqrt{6}}{6} x + \frac{5}{6\Gamma(1+\alpha)} t^\alpha \right).$$

In particular, if $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$, then $U_{1,2}$ is given by

$$u_{1,2}(x, t) = \frac{1}{4} \mp \frac{1}{2} \tanh \left\{ \mp \left(\frac{\sqrt{6}}{12} x + \frac{5}{12\Gamma(1+\alpha)} t^\alpha \right) \right\} + \frac{1}{4} \tanh^2 \left\{ \mp \left(\frac{\sqrt{6}}{12} x + \frac{5}{12\Gamma(1+\alpha)} t^\alpha \right) \right\}, \quad (97)$$

and $U_{3,4}$ becomes

$$u_{3,4}(x, t) = \frac{1}{4} \pm \frac{1}{2} \tanh \left\{ \mp \left(\frac{\sqrt{6}}{12} x + \frac{5}{12\Gamma(1+\alpha)} t^\alpha \right) \right\} + \frac{1}{4} \tanh^2 \left\{ \mp \left(\frac{\sqrt{6}}{12} x + \frac{5}{12\Gamma(1+\alpha)} t^\alpha \right) \right\}. \quad (98)$$

6. CONCLUSIONS

In this paper, analytical solutions for the one-dimensional nonlinear fractional wave and RD equations have been obtained, and the Exp-function and (G'/G) -expansion methods were successfully used to obtain these exact solutions. These solutions include generalized hyperbolic function solutions and generalized trigonometric function solutions, which may be useful to further understand the mechanisms of the complicated nonlinear physical phenomena and FDEs. Moreover, the obtained results show that the proposed methods are quite effective, promising and convenient for solving nonlinear fractional differential equations. So the methods presented in this work can be applied to other fractional partial differential equations. The *Maple* software has been used for programming and computations in this work.

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