A NOVEL SPECTRAL APPROXIMATION FOR THE TWO-DIMENSIONAL FRACTIONAL SUB-DIFFUSION PROBLEMS

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This paper reports a new numerical method that enables easy and convenient discretization of a two-dimensional sub-diffusion equation with fractional derivatives of any order. The suggested method is based on Jacobi tau spectral procedure together with the Jacobi operational matrix for fractional derivatives, described in the Caputo sense. Such approach has the advantage of reducing the problem to the solution of a system of algebraic equations, which may then be solved by any standard numerical technique. The validity and effectiveness of the method are demonstrated by solving two numerical examples, which are presented in the form of tables and graphs to make more easier comparisons with the exact solutions and the results obtained by other methods.

Key words: Two-dimensional fractional diffusion equations; Tau method; Shifted Jacobi polynomials; Operational matrix; Caputo derivative.

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1. INTRODUCTION

Fractional calculus goes back to the beginning of the theory of differential calculus and deals with the generalization of standard integrals and derivatives to an arbitrary real-valued order [1–3]. This subject is one of the most interdisciplinary fields of mathematics and has gained considerable popularity and importance due to its attractive applications as a new modeling tool in a variety of scientific and engineering fields, such as viscoelastic systems [4], transport phenomena [5], control [6], market dynamics [7], fractional kinetics [8], pharmacokinetics [9], advection-dispersion systems in earth science [10], and tumor growth [11]. Modeling successes for these phenomena have recently been reported using fractional derivatives for both ordinary and partial differential equations. In particular, it has been reported that certain kinds of dynamic natural phenomena, such as nonlocal diffusion modeled in fractal dimensions, cannot be properly described in terms of traditional integer order differential equations [3, 12, 13]. Fractional diffusion equation is a class of important fractional partial differential equations, which has been widely applied in modeling of anomalous diffusive systems, unification of diffusion and wave propagation phenomenon, description of fractional random walk, etc. [14, 15]. Fractional kinetic equations have proved particularly useful in the context of anomalous sub-diffusion [16]. For recent articles for solving various versions of such equations, see Refs. [17–22].

Fractional sub-diffusion equation is a subclass of anomalous diffusive systems, which is obtained by replacing the time derivative in ordinary diffusion by a fractional derivative of order ν with $0<\nu<1$. The mean square displacement of the particles from the original starting site is no longer linear in time but verifies a generalized Fick's second law. Sub-diffusive motion is characterized by an asymptotic long time behavior of the mean square displacement of the form

$$\left\langle x^2(t)\right\rangle \sim \frac{2K_\nu}{\Gamma(1+\nu)}t^\nu, \quad t\to\infty,$$

where ν $(0<\nu<1)$ is the anomalous diffusion exponent and K_ν is the generalized diffusion coefficient. It turns out that the probability density function $u(\mathbf{x},t)$ that describes anomalous sub-diffusive particle follows the fractional diffusion equation [16, 23]:

$$u_t = {}_0D_t^{1-\nu}\Delta u, \qquad t>0,$$

where Δ is the Laplacian and ${}_0D_t^{1-\nu}$ denotes the Caputo fractional derivative operator.

Several numerical methods have been proposed in the last few years for solving such equation. In Refs. ([24]-[33]), the researchers developed finite difference method. Particularly, Cui [29] and Gao *et al.* [31] concentrated on the compact difference schemes for promoting the spatial accuracy. Zhuang *et al.* [32] proposed explicit and implicit Euler approximations for the variable-order fractional advection-diffusion equation with a nonlinear source term and they discussed an implicit numerical method for the nonlinear fractional reaction sub-diffusion process in Ref. [34]. Zhang and Sun [35] used alternating direction implicit (ADI) schemes for a two-dimensional time-fractional sub-diffusion equation. Lin and Xu [36] proposed a finite difference method in the time direction and a Legendre spectral method in the space direction for the time-fractional diffusion equation. Li and Xu [37] extended their previous work and proposed a spectral method in both temporal and spatial discretizations for a time-space fractional partial differential equation based on a weak formulation and a detailed error analysis was carried out. Spectral methods offer spectral accuracy, meaning that the error decreases exponentially fast as the number of degrees of freedom increases. However, the application of spectral methods to irregular domains is not straightforward. Spectral methods have emerged as competitive alternatives to these methods [37–41].

In this paper, our main goal is to propose a new spectral algorithm in both temporal and spatial discretizations to provide an accurate numerical solution of the two-dimensional fractional sub-diffusion equations. This algorithm is based principally on shifted Jacobi tau spectral method combined with the generalized shifted Jacobi operational matrix of derivatives. To the best of our knowledge, there are no results on the spectral tau method for solving the two-dimensional time- or space-fractional partial differential equations.

The outline of this article is as follows: In Section 2, we present some fractional calculus preliminaries. In Section 3, the shifted Jacobi polynomials with their properties and their operational matrix to fractional differentiation are presented. In Section 4, by using tau spectral method, we construct and develop an algorithm for the solution of the two-dimensional fractional sub-diffusion equations with Dirichlet conditions. The accuracy and efficiency of of the proposed spectral algorithm are investigated with two illustrative numerical experiments in Section 5. The last section consists of some obtained conclusions.

2. FRACTIONAL DERIVATIVES

A complication associated with fractional derivatives is that there are several definitions of exactly what a fractional derivative means [3]. The two most commonly used definitions are based on Riemann-Liouville and Caputo senses. In order to simplify the basic definitions we consider the time interval $[0,\tau]$ instead of $[a,\tau]$ and omit $a=0$ as an index in the differential operator. If we consider the function $f(t)\in C^n[0,\tau]$, then the Riemann-Liouville definition for fractional derivative is given by:

$$ {}_0^{RL}D_t^{\nu} f(t)=\begin{cases}\frac{1}{\Gamma(n-\nu)}\frac{d^n}{dt^n}\int\limits_0^t \frac{f(\tau)}{(t-\tau)^{\nu+1-n}}d\tau, & n-1<\nu<n,\\ f^n(t), & \nu=n.\end{cases} \tag{1}$$

where ν is the order of the derivative. An alternative definition, known as the *Caputo fractional derivative*, is defined as

$$ {}_0D_t^{\nu} f(t)=\begin{cases}\frac{1}{\Gamma(n-\nu)}\int\limits_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\nu+1-n}}d\tau, & n-1<\nu<n,\\ f^n(t), & \nu=n.\end{cases} \tag{2}$$

These two definitions are not generally equivalent but they are related as

$$ {}_0^{RL}D_t^{\nu} f(t) = {}_0D_t^{\nu} f(t) + \sum_{\mu=0}^{n-1} \frac{t^{\mu-\nu} f^{\mu}(0)}{\Gamma(\mu+1-\nu)}. \tag{3} $$

One easily realizes the equivalence between the two for $f^{\mu}(0) = 0$, $\mu = 0, \ldots, n-1$. The Caputo fractional derivative, nowadays the most popular fractional operator among engineers and applied scientists, was obtained by reformulating the classical definition of Riemann-Liouville fractional derivative in order to make possible the solution of fractional initial value problems with standard initial conditions. For the Riemann-Liouville definition, such conditions must be imposed on fractional derivative that is often not available. For this reason we shall focus on the Caputo definition in this work. For the Caputo derivative, we have the following some basic properties that are needed in this paper [3].

Lemma 2.1. *For $\nu > 0$, $\gamma > -1$ and constant C:*

$$ \begin{aligned} &{}_0D_t^{\nu} C = 0, \\ &{}_0D_t^{\nu}\big(f(t)+g(t)\big) = {}_0D_t^{\nu} f(t) + {}_0D_t^{\nu} g(t), \\ &{}_0D_t^{\nu} t^{\gamma} = \begin{cases} 0, & \nu > \gamma, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\nu)} t^{\gamma-\nu}, & 0 < \nu \leqslant \gamma. \end{cases} \end{aligned} \tag{4} $$

3. OPERATIONAL MATRICES OF SHIFTED JACOBI POLYNOMIALS

Denote $P_i^{(\theta,\vartheta)}(z)$; $\theta > -1$, $\vartheta > -1$ as the i-th order Jacobi polynomial defined on $\Omega = [-1,1]$. As all classic orthogonal polynomials, $P_i^{(\theta,\vartheta)}(z)$ constitute an orthogonal system with respect to the weight function $\omega^{(\theta,\vartheta)}(z) = (1+z)^{\vartheta}(1-z)^{\theta}$, *i.e.*,

$$ \int_{-1}^{1} P_j^{(\theta,\vartheta)}(z)\, P_k^{(\theta,\vartheta)}(z)\, \omega^{(\theta,\vartheta)}(z) dt = h_k^{(\theta,\vartheta)} \delta_{jk}, \tag{5} $$

where δ_{jk} is the Kronecker delta and

$$ h_k^{(\theta,\vartheta)} = \frac{2^{\theta+\vartheta+1}\Gamma(k+\theta+1)\Gamma(k+\vartheta+1)}{(2k+\theta+\vartheta+1)k!\Gamma(k+\theta+\vartheta+1)}. $$

In order to use Jacobi polynomials on the interval $[0,L]$, we define the so called shifted Jacobi polynomials by introducing the change of variable $z = (\frac{2x}{L} - 1)$. Let the shifted Jacobi polynomials $P_j^{(\theta,\vartheta)}\left(\frac{2x}{L} - 1\right)$ be denoted by $P_{L,j}^{(\theta,\vartheta)}(x)$. $P_{L,j}^{(\theta,\vartheta)}(x)$ are constituting an orthogonal system with respect to the weight function $\omega_L^{(\theta,\vartheta)}(x) = x^{\vartheta}(L-x)^{\theta}$ over $[0,L]$ with the orthogonality property:

$$ \int_0^L P_{L,j}^{(\theta,\vartheta)}(x) P_{L,k}^{(\theta,\vartheta)}(x) \omega_L^{(\theta,\vartheta)}(x) dx = h_{L,k}^{(\theta,\vartheta)} \delta_{jk}, \tag{6} $$

where $h_{L,k}^{(\theta,\vartheta)} = \left(\frac{L}{2}\right)^{\theta+\vartheta+1} h_k^{(\theta,\vartheta)}$.
The explicit analytic form of $P_j^{(\theta,\vartheta)}(x)$ is given by

$$P_{L,j}^{(\theta,\vartheta)}(x) = \sum_{k=0}^{j} \frac{(-1)^{(j+k)}\Gamma(j+\vartheta+1)\Gamma(j+k+\theta+\vartheta+1)}{\Gamma(k+\vartheta+1)\Gamma(j+\theta+\vartheta+1)(j-k)!k!L^k} x^k. \tag{7}$$

The endpoint values of the shifted Jacobi polynomial are given as

$$P_{L,i}^{(\theta,\vartheta)}(0) = (-1)^i \frac{\Gamma(i+\vartheta+1)}{\Gamma(\vartheta+1)i!}, \qquad P_{L,i}^{(\theta,\vartheta)}(L) = \frac{\Gamma(i+\theta+1)}{\Gamma(\theta+1)i!}.$$

It is worth recalling important special cases of the shifted Jacobi polynomials, *e.g.*, the shifted Chebyshev polynomials

$$T_{L,j}(x) = \frac{j!\,\Gamma(\frac{1}{2})}{\Gamma(j+\frac{1}{2})} P_{L,j}^{(-\frac{1}{2},-\frac{1}{2})}(x),$$

and the shifted Legendre polynomials

$$P_{L,j}(x) = P_{L,j}^{(0,0)}(x),$$

both defined on $x \in [0,L]$. Indeed Jacobi polynomials and other kinds of polynomials are extensively used for solving different types of both differential equations and partial differential equations see, for example, Refs. [42]-[48].

Assume $u(x)$ is a square integrable function with respect to the Jacobi weight function $\omega_L^{(\theta,\vartheta)}(x)$, then it can be expressed in terms of shifted Jacobi polynomials as

$$u(x) = \sum_{j=0}^{\infty} c_j P_{L,j}^{(\theta,\vartheta)}(x),$$

where the coefficients c_j are given by

$$c_j = \frac{1}{h_{L,j}^{(\theta,\vartheta)}} \int_0^L \omega_L^{(\theta,\vartheta)}(x)\; u(x) P_{L,j}^{(\theta,\vartheta)}(x)dx, \qquad j = 0,1,\cdots. \tag{8}$$

In practice, only the first $(M+1)$-terms of $P_{L,j}^{(\theta,\vartheta)}(x)$ are considered. Then we have

$$u(x) \simeq u_M(x) = \sum_{j=0}^{M} c_j P_{L,j}^{(\theta,\vartheta)}(x) \equiv \mathbf{C}^T \Phi_{L,M}(x), \tag{9}$$

where

$$\mathbf{C}^T \equiv [c_0, c_1, \cdots, c_M],$$

and

$$\Phi_{L,M}(x) \equiv [P_{L,0}^{(\theta,\vartheta)}(x), P_{L,1}^{(\theta,\vartheta)}(x), \cdots, P_{L,M}^{(\theta,\vartheta)}(x)]^T. \tag{10}$$

Notice that the problem under consideration is two-dimensional, which can be described more easily by using Kronecker products of matrices [49]. For the readers' convenience, we first recall that the Kronecker product $A \otimes B$ of the $n \times m$ matrix A and the $p \times q$ matrix B

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{pmatrix}$$

is the $np \times mq$ matrix having the following block structure:

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nm}B \end{pmatrix}$$

Consequently, a function of three independent variables $u(x,y,t)$ that is infinitely differentiable in $\mathcal{I} = [0,\ell] \times [0,h] \times [0,\tau]$ may be expanded in terms of the triple shifted Jacobi polynomials as

$$\begin{aligned} u_{N,M,\widetilde{M}}(x,y,t) &= \sum_{i=0}^{M} \sum_{j=0}^{\widetilde{M}} \sum_{k=0}^{N} a_{i,j,k} P_{\ell,i}^{(\theta,\vartheta)}(x) P_{h,j}^{(\theta,\vartheta)}(y) P_{\tau,k}^{(\theta,\vartheta)}(t) \\ &= \Phi_{\tau,N}^{T}(t) \mathbf{A} \Phi_{\ell,M}(x) \otimes \Phi_{h,\widetilde{M}}(y) \end{aligned} \tag{11}$$

where the shifted Jacobi vectors $\Phi_{\tau,N}(t)$, $\Phi_{\ell,M}(x)$ and $\Phi_{h,\widetilde{M}}(y)$ are defined on (10); also the coefficient matrix $\mathbf{A}$ is given in a block form as follows

$$\begin{aligned} \mathbf{A} &= [A_0, A_1, \cdots, A_M], \\ A_i &= [A_{i,0}, A_{i,1}, \cdots, A_{i,\widetilde{M}}], \\ A_{i,j} &= [a_{i,j,0}, a_{i,j,1}, \cdots, a_{i,j,N}]^T. \end{aligned} \tag{12}$$

where

$$\begin{aligned} a_{ijk} = \frac{1}{h_{\tau,i}^{(\theta,\vartheta)}} \frac{1}{h_{\ell,j}^{(\theta,\vartheta)}} \frac{1}{h_{h,k}^{(\theta,\vartheta)}} \int_0^h \int_0^\ell \int_0^\tau & u(x,y,t) P_{\tau,i}^{(\theta,\vartheta)}(t) P_{\ell,j}^{(\theta,\vartheta)}(x) P_{h,k}^{(\theta,\vartheta)}(y) \\ & \times W^{(\theta,\vartheta)}(x,y,t)\, dt\, dx\, dy, \\ i = 0,1,...,N, \quad j = 0,1,...,M, \quad k = 0,1,...,\widetilde{M}. & \end{aligned} \tag{13}$$

and

$$W^{(\theta,\vartheta)}(x,y,t) = \omega_\tau^{(\theta,\vartheta)}(t)\, \omega_\ell^{(\theta,\vartheta)}(x)\, \omega_h^{(\theta,\vartheta)}(y)$$

The fractional derivative of order ν of the shifted Jacobi vector $\Phi_{L,M}(t)$ can be expressed as

$$D^{\nu}\Phi_{L,M}(x) = \mathbf{D}^{(\nu)}\Phi_{L,M}(x), \tag{14}$$

where $\mathbf{D}^{(\nu)}$ is the $(M+1)\times(M+1)$ operational matrix of fractional derivative of order ν in the Caputo sense and is defined as follows:

$$\mathbf{D}^{(\nu)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \Omega_\nu(\lceil\nu\rceil,0,\theta,\vartheta) & \Omega_\nu(\lceil\nu\rceil,1,\theta,\vartheta) & \cdots & \Omega_\nu(\lceil\nu\rceil,M,\theta,\vartheta) \\ \vdots & \vdots & \cdots & \vdots \\ \Omega_\nu(i,0,\theta,\vartheta) & \Omega_\nu(i,1,\theta,\vartheta) & \cdots & \Omega_\nu(i,M,\theta,\vartheta) \\ \vdots & \vdots & \cdots & \vdots \\ \Omega_\nu(M,0,\theta,\vartheta) & \Omega_\nu(M,1,\theta,\vartheta) & \cdots & \Omega_\nu(M,M,\theta,\vartheta) \end{pmatrix}, \tag{15}$$

where

$$\Omega_\nu(i,j,\theta,\vartheta) = \sum_{k=\lceil\nu\rceil}^{i} \frac{(-1)^{i-k}\,\Gamma(i+\vartheta+1)\,\Gamma(i+k+\theta+\vartheta+1)\,\Gamma(\theta+1)\,L^{-\nu}}{\Gamma(k+\vartheta+1)\,\Gamma(i+\theta+\vartheta+1)\,(i-k)!\,\Gamma(k-\nu+1)}$$
$$\times \sum_{s=0}^{j} \frac{(-1)^{j-s}\,\Gamma(j+s+\theta+\vartheta+1)\,\Gamma(s+k+\vartheta-\nu+1)\,(2j+\theta+\vartheta+1)\,j!}{\Gamma(j+\theta+1)\,\Gamma(s+\vartheta+1)\,(j-s)!\,s!\,\Gamma(s+k+\theta+\vartheta-\nu+2)}. \tag{16}$$

Note that in $\mathbf{D}^{(\nu)}$, the first $\lceil\nu\rceil$ rows, are all zeros. (For the proof, see [50]).

4. THE SHIFTED JACOBI TAU METHOD

In this section, the spectral tau method together with the shifted Jacobi operational matrix of derivatives are applied to solve the two-dimensional time fractional diffusion equation.

Consider the two-dimensional problem of sub-diffusion equation:

$$\frac{\partial u(x,y,t)}{\partial t} = {}_0D_t^{1-\nu}\left[K_1\frac{\partial^2 u(x,y,t)}{\partial x^2} + K_2\frac{\partial^2 u(x,y,t)}{\partial y^2}\right] + q(x,y,t), \quad (x,y)\in\Omega,\ 0<t\leqslant\tau \tag{17}$$

with the initial-boundary conditions

$$\begin{aligned} &u(x,y,0) = h(x,y), && (x,y)\in\Omega, && \\ &u(x,0,t) = f^1(x,t), && u(0,y,t) = \mathrm{g}^1(y,t), && 0<t\leqslant\tau, \\ &u(x,h,t) = f^2(x,t), && u(\ell,y,t) = \mathrm{g}^2(y,t), && 0<t\leqslant\tau. \end{aligned} \tag{18}$$

where K_1 and K_2 are positive constants, and $\Omega = [0,\ell] \times [0,h]$ is a finite rectangular domain. It is worthy to mention here that, since the pure spectral tau method would not use any interpolation operator, we approximated the solution and known functions in terms of shifted Jacobi polynomials in order to use orthogonality of such polynomials for computing the spectral tau coefficients.

In order to use the pure spectral tau method with the shifted Jacobi operational matrix for solving this problem, we assume that $q(x,y,t)$, $h(x,y)$, $\mathrm{g}^r(y,t)$ and $f^r(x,t)$, $r=1,2$ are continuous, smooth functions. Therefore, $u(x,y,t)$, $q(x,y,t)$, $h(x,y)$, $\mathrm{g}^r(y,t)$ and $f^r(x,t)$ can be expanded by the shifted Jacobi polynomials as

$$\begin{aligned}
&u_{N,M,\widetilde{M}}(x,y,t) = \Phi^T_{\tau,N}(t)\mathbf{A}\Phi_{\ell,M}(x)\otimes\Phi_{h,\widetilde{M}}(y),\\
&q_{N,M,\widetilde{M}}(x,y,t) = \Phi^T_{\tau,N}(t)\mathbf{Q}\Phi_{\ell,M}(x)\otimes\Phi_{h,\widetilde{M}}(y),\\
&h_{N,M,\widetilde{M}}(x,y) = \Phi^T_{\tau,N}(0)\mathbf{H}\Phi_{\ell,M}(x)\otimes\Phi_{h,\widetilde{M}}(y),\\
&\mathrm{g}^1_{N,M,\widetilde{M}}(y,t) = \Phi^T_{\tau,N}(t)\mathbf{G}^1\Phi_{\ell,M}(0)\otimes\Phi_{h,\widetilde{M}}(y),\\
&\mathrm{g}^2_{N,M,\widetilde{M}}(y,t) = \Phi^T_{\tau,N}(t)\mathbf{G}^2\Phi_{\ell,M}(\ell)\otimes\Phi_{h,\widetilde{M}}(y),\\
&f^1_{N,M,\widetilde{M}}(x,t) = \Phi^T_{\tau,N}(t)\mathbf{F}^1\Phi_{\ell,M}(x)\otimes\Phi_{h,\widetilde{M}}(0),\\
&f^2_{N,M,\widetilde{M}}(x,t) = \Phi^T_{\tau,N}(t)\mathbf{F}^2\Phi_{\ell,M}(x)\otimes\Phi_{h,\widetilde{M}}(h),
\end{aligned} \tag{19}$$

where $\mathbf{A}$ is the unknown coefficients $(N+1)\times(M+1)\times(\widetilde{M}+1)$ matrix, $\mathbf{Q}$, $\mathbf{H}$, $\mathbf{F}^r$ and $\mathbf{G}^r$, $r=1,2$ are known matrices that can be written in a block form as follows

$$\begin{aligned}
&\mathbf{Q} = [Q_0,Q_1,\cdots,Q_M], && \mathbf{H} = [H_0,H_1,\cdots,H_M],\\
&Q_i = [Q_{i,0},Q_{i,1},\cdots,Q_{i,\widetilde{M}}], && H_i = [H_{i,0},H_{i,1},\cdots,H_{i,\widetilde{M}}],\\
&Q_{i,j} = [\mathrm{q}_{i,j,0},\mathrm{q}_{i,j,1},\cdots,\mathrm{q}_{i,j,N}]^T, && H_{i,j} = [\mathrm{h}_{i,j,0},\mathrm{h}_{i,j,1},\cdots,\mathrm{h}_{i,j,N}]^T,
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&\mathbf{F}^r = [F^r_0,F^r_1,\cdots,F^r_M], && \mathbf{G}^r = [G^r_0,G^r_1,\cdots,G^r_M],\\
&F^r_i = [F^r_{i,0},F^r_{i,1},\cdots,F^r_{i,\widetilde{M}}], && G^r_i = [G^r_{i,0},G^r_{i,1},\cdots,G^r_{i,\widetilde{M}}],\\
&F^r_{i,j} = [\mathrm{f}^r_{i,j,0},\mathrm{f}^r_{i,j,1},\cdots,\mathrm{f}^r_{i,j,N}]^T, && G^r_{i,j} = [\mathrm{g}^r_{i,j,0},\mathrm{g}^r_{i,j,1},\cdots,\mathrm{g}^r_{i,j,N}]^T.
\end{aligned} \tag{21}$$

The coefficients $\mathrm{q}_{i,j,k}$, $\mathrm{h}_{i,j,k}$, $\mathrm{f}_{i,j,k}$ and $\mathrm{g}_{i,j,k}$ can be evaluated exactly by

$$\begin{aligned}
\mathrm{q}_{ijk} = \frac{1}{h^{(\theta,\vartheta)}_{\tau,i}}\frac{1}{h^{(\theta,\vartheta)}_{\ell,j}}\frac{1}{h^{(\theta,\vartheta)}_{h,k}}\int_0^h\int_0^\ell\int_0^\tau & q(x,y,t)P^{(\theta,\vartheta)}_{\tau,i}(t)P^{(\theta,\vartheta)}_{\ell,j}(x)P^{(\theta,\vartheta)}_{h,k}(y)\\
&\times W^{(\theta,\vartheta)}(x,y,t)dt\,dx\,dy,\\
i=0,1,...,N,\quad j=0,1,...,M,&\quad k=0,1,...,\widetilde{M},
\end{aligned}$$

$$\mathrm{h}_{ijk} = \frac{1}{h_{\tau,i}^{(\theta,\vartheta)}} \frac{1}{h_{\ell,j}^{(\theta,\vartheta)}} \frac{1}{h_{h,k}^{(\theta,\vartheta)}} \int_0^h \int_0^\ell \int_0^\tau h(x,y) P_{\tau,i}^{(\theta,\vartheta)}(t) P_{\ell,j}^{(\theta,\vartheta)}(x) P_{h,k}^{(\theta,\vartheta)}(y) \times W^{(\theta,\vartheta)}(x,y,t) dt\, dx\, dy,$$

$$i = 0,1,...,N, \quad j = 0,1,...,M, \quad k = 0,1,...,\widetilde{M},$$

$$\mathrm{f}_{ijk}^r = \frac{1}{h_{\tau,i}^{(\theta,\vartheta)}} \frac{1}{h_{\ell,j}^{(\theta,\vartheta)}} \frac{1}{h_{h,k}^{(\theta,\vartheta)}} \int_0^h \int_0^\ell \int_0^\tau \mathrm{f}^r(x,t) P_{\tau,i}^{(\theta,\vartheta)}(t) P_{\ell,j}^{(\theta,\vartheta)}(x) P_{h,k}^{(\theta,\vartheta)}(y) \times W^{(\theta,\vartheta)}(x,y,t) dt\, dx\, dy,$$

$$r = 1,2, \quad i = 0,1,...,N, \quad j = 0,1,...,M, \quad k = 0,1,...,\widetilde{M},$$

$$\mathrm{g}_{ijk}^r = \frac{1}{h_{\tau,i}^{(\theta,\vartheta)}} \frac{1}{h_{\ell,j}^{(\theta,\vartheta)}} \frac{1}{h_{h,k}^{(\theta,\vartheta)}} \int_0^h \int_0^\ell \int_0^\tau \mathrm{g}^r(y,t) P_{\tau,i}^{(\theta,\vartheta)}(t) P_{\ell,j}^{(\theta,\vartheta)}(x) P_{h,k}^{(\theta,\vartheta)}(y) \times W^{(\theta,\vartheta)}(x,y,t) dt\, dx\, dy,$$

$$r = 1,2, \quad i = 0,1,...,N, \quad j = 0,1,...,M, \quad k = 0,1,...,\widetilde{M}.$$

Now, using Eqs. (14) and (19), then it is easy to write

$$\begin{aligned}
\frac{\partial u(x,y,t)}{\partial t} &= \Phi_{\tau,N}^T(t) \mathbf{D}_1^T \mathbf{A} \Phi_{\ell,M}(x) \otimes \Phi_{h,\widetilde{M}}(y), \\
{}_0D_t^{1-\nu} \frac{\partial^2 u(x,y,t)}{\partial x^2} &= \Phi_{\tau,N}^T(t) \mathbf{D}_{1-\nu}^T \mathbf{A} \big(\mathbf{D}_2 \Phi_{\ell,M}(x)\big) \otimes \Phi_{h,\widetilde{M}}(y), \qquad (22) \\
{}_0D_t^{1-\nu} \frac{\partial^2 u(x,y,t)}{\partial y^2} &= \Phi_{\tau,N}^T(t) \mathbf{D}_{1-\nu}^T \mathbf{A} \Phi_{\ell,M}(x) \otimes \big(\mathbf{D}_2 \Phi_{h,\widetilde{M}}(y)\big).
\end{aligned}$$

Employing Eqs. (19) and (22) in Eq. (17) we get

$$\begin{aligned}
\Phi_{\tau,N}^T(t) \mathbf{D}_1^T \mathbf{A} \Phi_{\ell,M}(x) \otimes \Phi_{h,\widetilde{M}}(y) =& K_1 \Phi_{\tau,N}^T(t) \mathbf{D}_{1-\nu}^T \mathbf{A} (\mathbf{D}_2 \Phi_{\ell,M}(x)) \otimes \Phi_{h,\widetilde{M}}(y) \\
&+ K_2 \Phi_{\tau,N}^T(t) \mathbf{D}_{1-\nu}^T \mathbf{A} \Phi_{\ell,M}(x) \otimes \big(\mathbf{D}_2 \Phi_{h,\widetilde{M}}(y)\big) \qquad (23) \\
&+ \Phi_{\tau,N}^T(t) \mathbf{Q} \Phi_{\ell,M}(x) \otimes \Phi_{h,\widetilde{M}}(y).
\end{aligned}$$

Due to the orthogonality property,

$$\big(\Phi_{\tau,N}(\mathbf{x}), \Phi_{\tau,M}^T(\mathbf{x})\big)_{\omega_L^{(\theta,\vartheta)}(x)} = I_\tau^{(N,M)} = (h_{\tau,j}^{(\theta,\vartheta)} \delta_{i,j})_{0 \le i \le N,\, 0 \le j \le M}, \qquad (24)$$

we generate $N \times (M-1) \times (\widetilde{M}-1)$ linear algebraic equations in the unknown expansion coefficients,

$$\begin{aligned} &I_{\tau}^{(N-1,N)}\mathbf{D}_1^T\mathbf{A}\big(I_{\ell}^{(M,M-2)}\otimes I_h^{(\widetilde{M},\widetilde{M}-2)}\big)\\ &\quad = K_1 I_{\tau}^{(N-1,M)}\mathbf{D}_{1-\nu}^T\mathbf{A}\big(\big(\mathbf{D}_2 I_{\ell}^{(M,M-2)}\big)\otimes I_h^{(\widetilde{M},\widetilde{M}-2)}\big)\\ &\quad + K_2 I_{\tau}^{(N-1,M)}\mathbf{D}_{1-\nu}^T\mathbf{A}\big(I_{\ell}^{(M,M-2)}\otimes \big(\mathbf{D}_2 I_h^{(\widetilde{M},\widetilde{M}-2)}\big)\big)\\ &\quad + I_{\tau}^{(N-1,M)}\mathbf{Q}\big(I_{\ell}^{(M,M-2)}\otimes I_h^{(\widetilde{M},\widetilde{M}-2)}\big), \end{aligned} \tag{25}$$

and the rest of linear algebraic equations are obtained from the initial and boundary conditions (18), as

$$\begin{aligned} &\Phi_{\tau,N}^T(0)\mathbf{A}I_{\ell}^{(M,M)}\otimes I_h^{(\widetilde{M},\widetilde{M})} = \Phi_{\tau,N}^T(0)\mathbf{H}I_{\ell}^{(M,M)}\otimes I_h^{(\widetilde{M},\widetilde{M})},\\ &I_{\tau}^{(N-1,N)}\mathbf{A}\Phi_{\ell,M}(0)\otimes I_h^{(\widetilde{M},\widetilde{M}-1)} = I_{\tau}^{(N-1,N)}\mathbf{G}^1\Phi_{\ell,M}(0)\otimes I_h^{(\widetilde{M},\widetilde{M}-1)},\\ &I_{\tau}^{(N-1,N)}\mathbf{A}\Phi_{\ell,M}(\ell)\otimes I_h^{(\widetilde{M},\widetilde{M}-1)} = I_{\tau}^{(N-1,N)}\mathbf{G}^2\Phi_{\ell,M}(\ell)\otimes I_h^{(\widetilde{M},\widetilde{M}-1)},\\ &I_{\tau}^{(N-1,N)}\mathbf{A}I_{\ell}^{(M,M-1)}\otimes \Phi_{h,\widetilde{M}}(0) = I_{\tau}^{(N-1,N)}\mathbf{F}^1 I_{\ell}^{(M,M-1)}\otimes \Phi_{h,\widetilde{M}}(0),\\ &I_{\tau}^{(N-1,N)}\mathbf{A}I_{\ell}^{(M,M-1)}\otimes \Phi_{h,\widetilde{M}}(h) = I_{\tau}^{(N-1,N)}\mathbf{F}^2 I_{\ell}^{(M,M-1)}\otimes \Phi_{h,\widetilde{M}}(h). \end{aligned} \tag{26}$$

The number of the unknown coefficients a_{ijk} is equal to $(N+1)\times(M+1)\times(\widetilde{M}+1)$ and can be obtained from Eqs. (25) and (26). Consequently $u_{N,M,\widetilde{M}}(x,y,t)$ given in Eq. (19) can be calculated.

Remark 4.1. *The cases $\theta=\vartheta=0$ and $\theta=\vartheta=-1/2$ lead to the Legendre and Chebyshev tau methods combined with operational matrices respectively, which are special cases from Jacobi tau method and used most frequently in practice.*

5. NUMERICAL RESULTS AND COMPARISONS

In this section, we present two numerical examples to demonstrate the accuracy and applicability of the proposed method. We also compare the results given from our scheme and those reported in the literature such as compact alternating direction implicit (ADI) scheme [30] and two ADI finite difference schemes, called L_1-ADI and BD-ADI schemes [35, 51]. Note that here we fix the parameters of the shifted Jacobian polynomials as $\theta=0$ and $\vartheta=0$, although one can use any values of these parameters.

Example 1. *Consider the following problem [30]:*

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2} = {}_0D_t^{1-\nu}\Delta u + f(x,y,t), \quad (x,y)\in\Omega,\ t\in J,\\ &u(x,y,t) = u_D(x,y,t), \quad (x,y)\in\partial\Omega,\ t\in J,\\ &u(x,y,0)=0, \quad (x,y)\in\Omega. \end{aligned} \tag{27}$$

The spatial domain is $\Omega=(0,1)\times(0,1)$ with $\partial\Omega$ being its boundary, $J=(0,1]$, and the analytic solution is given by $u(x,y,t)=t\cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)$ with corresponding right hand side function $f(x,y,t)$, and $u_D(x,y,t)$ denotes the trace of $u(x,y,t)$ on $\partial\Omega$.

Cui [30] used the ADI method to split this problem into two separate one-dimensional problems where a Grönwald-Letnikov approximation is used for the Riemann-Liouville time derivative, and the second order spatial derivatives are approximated by the compact finite differences to obtain a fully discrete implicit scheme.

In order to show that the presented method is more accurate than the compact finite difference method, a comparison between our results and those obtained in [30] are given in Table 1 for different values of ν. In addition, Fig. 1 plots the absolute error function $|u(x,y,t)-u_{10,10,10}(x,y,t)$ for different choices of t with $\nu=0.7$.

Example 2. *We consider another problem [35, 41, 51]:*

$$\begin{aligned} {}_0D_t^{\nu}u &= \Delta u+q(x,y,t), \quad (x,y)\in\Omega,\ t\in J,\\ u(x,y,t) &= u_D(x,y,t), \quad (x,y)\in\partial\Omega,\ t\in J,\\ u(x,y,0) &= 0, \quad (x,y)\in\Omega, \end{aligned} \tag{28}$$

in the domain $\Omega=(0,\pi)\times(0,\pi)$ *and let* $J=(0,1]$.

The solution of this problem is $u(x,y,t)=t^2 sin(x)sin(y)$. It can be checked that the corresponding known function is given by

$$q(x,y,t)=2\sin(x)\sin(y)\left(\frac{2^{2-\nu}}{\Gamma(3-\nu)}+t^2\right).$$

Zhang and Sun [35] constructed two ADI finite difference schemes, called L_1-ADI and BD-ADI schemes. Wang [51] extended the results reported in Ref. [35], by establishing a maximum norm error estimate for the ADI discretizations.

In Tables 2 and 3 we make a comparison of the maximum absolute errors of the presented algorithm at various values of ν with L_1-ADI and BD-ADI schemes [35, 51], which shows that the proposed method provides an accurate approximation and yields algebraic convergence rates. In Fig. 2 we plot the absolute errors (AE) as functions of the polynomial degrees $N=M=\widetilde{M}=10$ at $t=0.1,\ 0.3,\ 0.7,\ 0.9$ with $\nu=0.9$.

Table 1

Comparing maximum absolute errors of the present method and ADI scheme [30] at $\nu = 0.25,\ 0.5$ for Example 1.

SJT method			ADI scheme [30]		
N	$\nu = 0.25$	$\nu = 0.5$	h^{-1}	$\nu = 0.25$	$\nu = 0.5$
4	8.266×10^{-4}	8.259×10^{-4}	4	1.520×10^{-2}	2.090×10^{-2}
6	2.253×10^{-6}	2.252×10^{-6}	8	3.921×10^{-4}	2.900×10^{-3}
8	5.782×10^{-9}	5.128×10^{-9}	16	4.922×10^{-5}	1.91×10^{-4}
10	4.294×10^{-9}	4.310×10^{-9}	32	2.616×10^{-5}	9.59×10^{-5}

Table 2

Comparing maximum absolute errors of the proposed method and Refs. [35, 51] at $\nu = 1/2,\ 2/3$ for Example 2.

ν	SJT method		L_1-ADI scheme [35]		BD-ADI scheme [51]	
	N	MAE	N	MAE	N	MAE
1/2	4	4.582×10^{-3}	10	3.491×10^{-3}	20	1.509×10^{-3}
	6	4.748×10^{-5}	20	1.990×10^{-3}	80	2.573×10^{-4}
	8	3.159×10^{-7}	40	1.082×10^{-3}	320	4.185×10^{-5}
	10	1.438×10^{-9}	80	5.713×10^{-4}	640	1.640×10^{-5}
2/3	4	4.539×10^{-3}	10	1.433×10^{-3}	20	3.662×10^{-4}
	6	4.724×10^{-5}	20	5.932×10^{-4}	80	4.779×10^{-5}
	8	3.150×10^{-7}	40	2.424×10^{-4}	320	5.961×10^{-6}
	10	1.436×10^{-9}	80	9.887×10^{-5}	640	2.055×10^{-6}

Table 3

Comparing maximum absolute errors of the proposed method and Refs. [35, 51] at $\nu = 1/3,\ 3/4$ for Example 2.

SJT method			BD-ADI scheme [35]		L_1-ADI scheme [51]	
N	$\nu = 1/3$	$\nu = 3/4$	N	$\nu = 1/3$	N	$\nu = 3/4$
4	4.623×10^{-3}	4.518×10^{-3}	10	3.868×10^{-3}	20	2.300×10^{-3}
6	4.772×10^{-5}	4711×10^{-5}	20	1.643×10^{-3}	80	4.444×10^{-4}
8	3.167×10^{-7}	3.145×10^{-7}	40	6.795×10^{-4}	320	8.264×10^{-5}
10	1.441×10^{-9}	1.435×10^{-9}	80	2.767×10^{-4}	640	3.555×10^{-5}

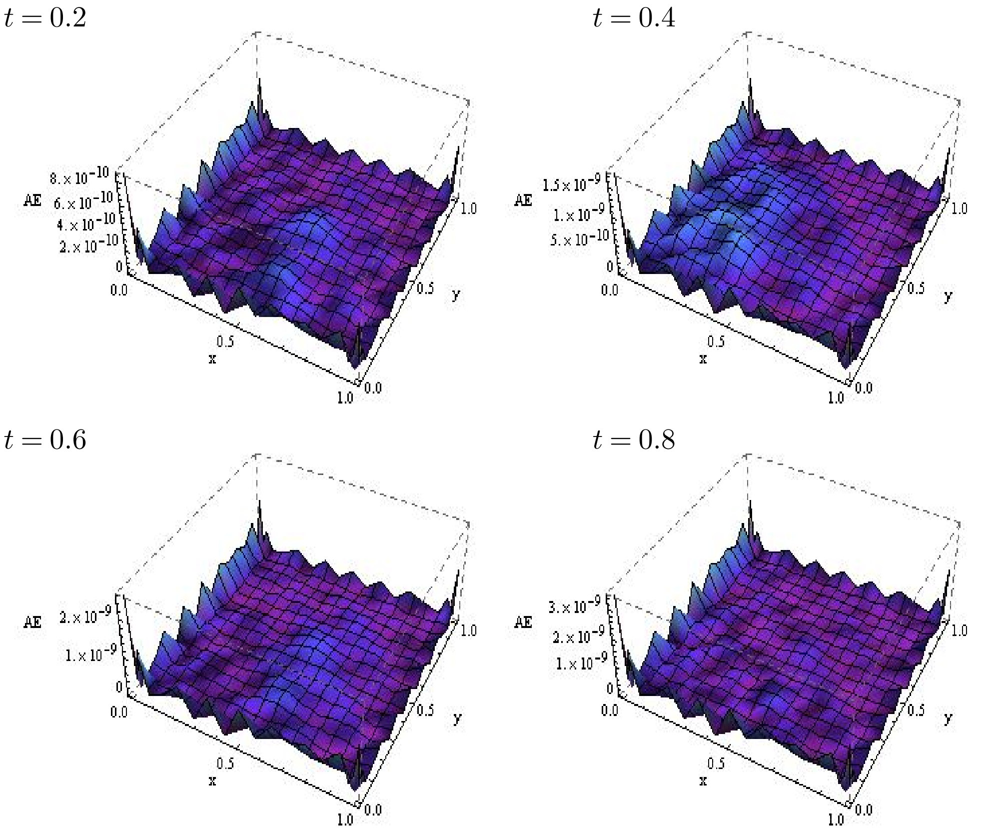

Fig. 1 – The space-time graph of the AE at various choices of t with $\nu = 0.7$ and $N = M = \widetilde{M} = 10$ for Example 1.

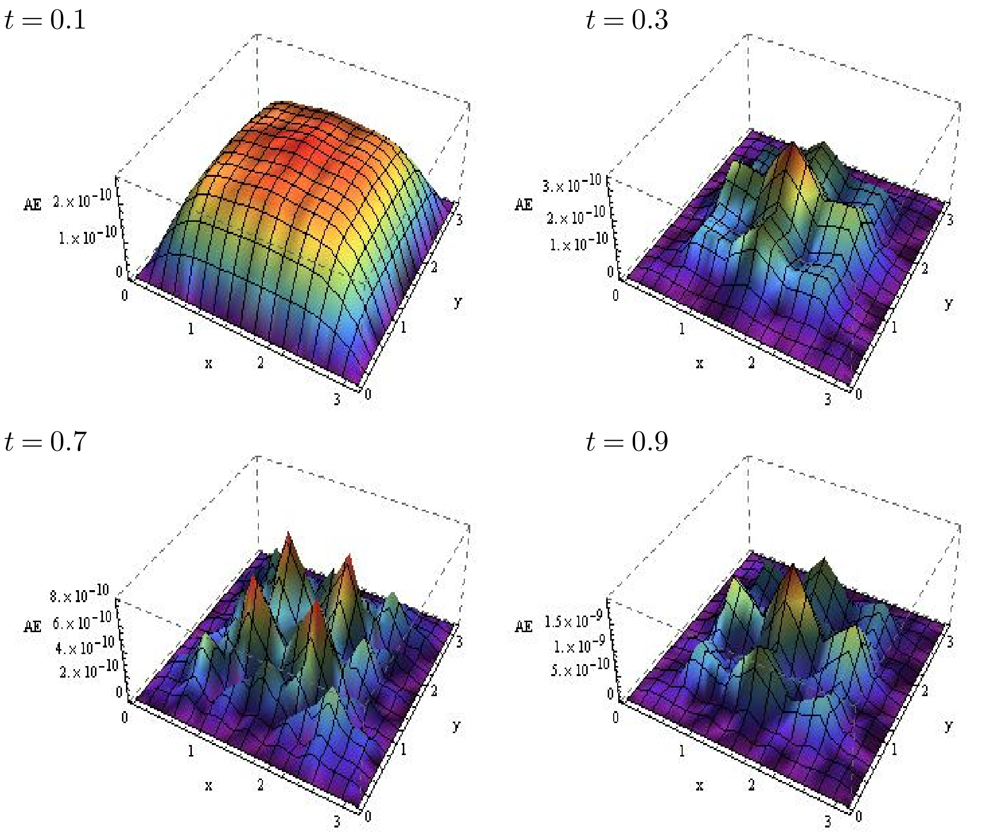

Fig. 2 – The space-time graph of the AE at various choices of t with $\nu = 0.9$ and $N = M = \widetilde{M} = 10$ for Example 2.

6. CONCLUSION

We have presented a new space-time spectral algorithm based on shifted Jacobi tau technique combined with the associated operational matrices of fractional derivatives. This algorithm was employed for solving the two-dimensional time fractional diffusion equation. The time fractional derivatives were given in the Caputo sense.

The proposed algorithm has the advantage of transforming the initial problem into the solution of a system of algebraic equations thus greatly simplifying the problem. In addition, some of known spectral tau approximations can be derived as special cases from our algorithm if we suitably choose the corresponding special cases of Jacobi parameters θ and ϑ. We have presented numerical results in order to demonstrate the effectiveness and the high accuracy of the proposed spectral method. Although we have concentrated on applying our algorithm to solve two-dimensional time fractional diffusion equation, we do claim that such algorithm can be applied to solve similar, however more complicated problems in the three-dimensional case.

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