ANALYTIC SOLUTIONS AND NUMERICAL SIMULATIONS OF MASS-SPRING AND DAMPER-SPRING SYSTEMS DESCRIBED BY FRACTIONAL DIFFERENTIAL EQUATIONS

J.F. GÓMEZ-AGUILAR

Departamento de Materiales Solares, Instituto de Energas Renovables,
Universidad Nacional Autónoma de México
Priv. Xochicalco s/n. Col. Centro, Temixco, Morelos, México
*E-mail*: jgomez@ier.unam.mx

*Received June 12, 2014*

In this paper the fractional mass-spring and damper-spring models with Caputo derivative is considered. The order of the derivatives is $0 < \gamma \leq 1$. In order that the equations preserve the physical units of the system an auxiliary parameter $\alpha$ is introduced. Different source terms are introduced in the fractional equation. The classic cases are recovered when $\gamma = 1$.

*Key words*: Fractional calculus, Caputo derivative, Mechanical oscillators; Fractional differential equations.

*PACS*: 45.10.Hj, 46.40.Ff, 45.20.D-.

## 1. INTRODUCTION

Fractional Calculus (FC), which involves derivatives and integrals of any order, is the generalization of the ordinary calculus [1]-[4]. FC provides a more accurate model of physical systems and for problems that are not solvable in the ordinary sense. The models described by FC have memory and fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes [5]-[6]. This is the main advantage of FC in comparison with the classical integer-order models, in which such effects are in fact neglected. Applications of fractional calculus in the field of physics, chemistry, finances, and bioengineering have gained considerable popularity and many important results were obtained during the last years [7]-[15]. In [16] the authors discuss the relationship of the fractional Fourier transform to harmonic oscillation. In [17] is discussed the fractional oscillator equation involving fractional time derivatives of the Riemann-Liouville type. Naber in [18], studied the linearly damped oscillator equation, written as a fractional derivative in the Caputo representation. The solution is found analytically and a comparison with the ordinary linearly damped oscillator is made. In [19] was considered the fractional oscillator, being a generalization of the conventional linear oscillator, in the framework of fractional calculus. Tarasov in

[20] considered the fractional oscillator as an open (non-isolated) system with memory, the environment is defined as an infinite set of independent harmonic oscillators coupled to a system. The equations of motion are obtained from the interaction between the system and the environment with power-law spectral density. Recently, in [21] has been proposed a systematic way to construct fractional differential equations for the physical systems. In particular, the mass-spring and spring-damper systems without source terms have been analyzed, the representation consists in analyzing the dimensionality of the ordinary derivative operator and trying to consistently bring it to a fractional derivative operator. In the present work the idea proposed in [21] is applied to the study of the fractional mass-spring and spring-damper systems for different source terms.

The paper is organized as follows. In Section 2, some basic concepts of fractional calculus are considered. In Section 3, the analytic solution of the systems is presented and numerical simulations with different source terms are performed. Conclusions are given in Section 4.

## 2. PRELIMINARIES

The Caputo Fractional Derivative (CFD) for a function $f(t)$ is defined as follows [4]

$$ {}_0^C D_t^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(\eta)}{(t-\eta)^{\gamma-n+1}} d\eta, \tag{1} $$

where $n = 1, 2, \ldots \in N$ and $n-1 < \gamma \leq n$, the order of the fractional derivative is $0 < \gamma \leq 1$.

The Laplace transform to CFD has the form [4]

$$ L[{}_0^C D_t^{\gamma} f(t)] = S^{\gamma} F(S) - \sum_{k=0}^{m-1} S^{\gamma-k-1} f^{(k)}(0). \tag{2} $$

The Mittag-Leffler function is presented in the solution of fractional differential equations. The Mittag-Leffler function is defined by the series expansion as [22]

$$ E_a(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(am+1)}, \qquad (a > 0). \tag{3} $$

When $a = 1$ we have $e^t$, therefore, the Mittag-Leffler function is a generalization of the exponential function.

For an example

$$ {}_0^C D_t^{\gamma} f(t) + {}_0^C D_t^{\beta} f(t) = h(t), \qquad (0 < \gamma < \beta \leq 1). \tag{4} $$

by applying the Laplace transform

$$(s^\gamma + s^\beta)F(s) = C + H(s), \qquad C = [{}_0^C D_t^{\beta-1} f(t) + {}_0^C D_t^{\gamma-1} f(t)]_{t=0},$$
$$F(s) = \frac{C+H(s)}{s^\gamma + s^\beta} = \frac{C+H(s)}{s^\beta(s^{\gamma-\beta}+1)} = (C+H(s))\frac{s^{-\beta}}{s^{\gamma-\beta}+1}, \tag{5}$$

and the inverse transform we obtain the solution

$$f(t) = C \cdot G(t) + \int_0^t G(t-\tau)h(\tau)d\tau, \tag{6}$$

$$C = [{}_0^C D_t^{\beta-1} f(t) + {}_0^C D_t^{\gamma-1} f(t)]_{t=0}, \qquad G(t) = t^{\gamma-1}E_{\gamma-\beta,\gamma}(-t^{\gamma-\beta}). \tag{7}$$

Some common Laplace transforms are

$$\frac{1}{s^\alpha + a} = t^{\alpha-1}E_{\alpha,\alpha}(-at^\alpha),$$
$$\frac{s^\alpha}{s(s^\alpha + a)} = E_\alpha(-at^\alpha), \tag{8}$$
$$\frac{a}{s(s^\alpha + a)} = 1 - E_\alpha(-at^\alpha).$$

## 3. APPLICATION EXAMPLES

To be consistent with dimensionality of the physical equation it is introduced an auxiliary parameter $\alpha_t$ in the following way [21]

$$\frac{d}{dt} \to \frac{1}{\alpha_t^{1-\gamma}} \cdot \frac{d^\gamma}{dt^\gamma}, \qquad n-1 < \gamma \leq n, \tag{9}$$

where $n$ is integer. When $\gamma = 1$ the expression (9) becomes an ordinary derivative, this is true if the parameter $\alpha_t$ has the dimension of time. This expression represents a temporal derivative since its dimension is inverse second. The parameter $\alpha_t$ represents the fractional time components in the system (such components change the time constant of the system) [23]. Following this idea the fractional differential equation for the mass-spring-damper system with source showed in Fig. 1 is given by

$$\frac{m}{\alpha^{2(1-\gamma)}}\frac{d^{2\gamma}x(t)}{dt^{2\gamma}} + \frac{\beta}{\alpha^{1-\gamma}}\frac{d^\gamma x(t)}{dt^\gamma} + kx(t) = v(t), \qquad 0 < \gamma \leq 1 \tag{10}$$

where $m$ is the mass, $\beta$ is the damped coefficient and $k$ is the spring constant.

From equation (10) we obtain the particular cases:

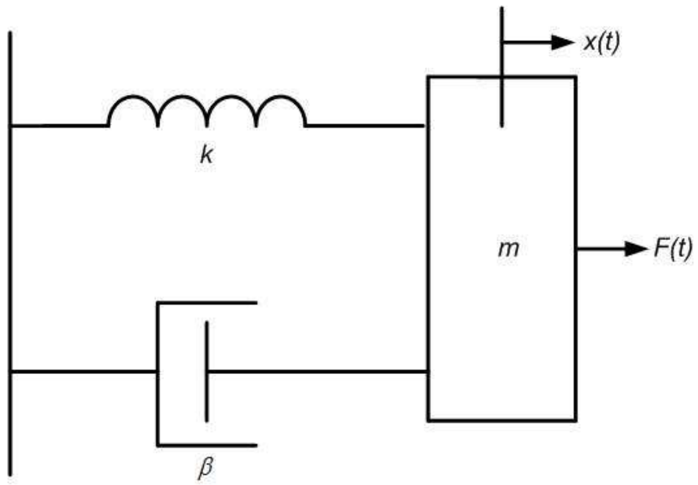


Fig. 1 – Mass-spring-damper system.

1. Mass-spring system, $\beta = 0$

$$\frac{m}{\alpha^{2(1-\gamma)}}\frac{d^{2\gamma}x(t)}{dt^{2\gamma}} + kx(t) = v(t), \qquad 0 < \gamma \leq 1. \tag{11}$$

2. Damper-spring system, $m = 0$

$$\frac{\beta}{\alpha^{1-\gamma}}\frac{d^{\gamma}x(t)}{dt^{\gamma}} + kx(t) = v(t), \qquad 0 < \gamma \leq 1. \tag{12}$$

Now we obtain the analytic solution and numerical simulation of the particular cases 1 and 2 for different source terms.

**Mass-spring system**.

- Constant source, $v(t) = v_0$, $x(0) = x_0$, $(x_0 > 0)$, $\dot{x}(0) = 0$
  The equation (11) may be written as follows

$$\frac{d^{2\gamma}x(t)}{dt^{2\gamma}} = \frac{\eta^2}{k}v_0 - \eta^2 x(t), \tag{13}$$

where

$$\eta^2 = \frac{k\sigma^{2(1-\gamma)}}{m} = \eta_0^2\sigma^{2(1-\gamma)}, \tag{14}$$

is the angular frequency for different values of $\gamma$, and $\eta_0^2 = \frac{k}{m}$ is the fundamental frequency of the system (*i.e.*, when $\gamma = 1$).

The solution for the Eq. (13) is given by

$$x(t) = (x_0 - \frac{v_0}{k})E_{2\gamma}\Big\{-\eta^2 t^{2\gamma}\Big\} + \frac{v_0}{k}, \tag{15}$$

where

$$E_{2\gamma}\Big\{-\eta^2 t^{2\gamma}\Big\} = \sum_{n=0}^{\infty} \frac{\Big(-\eta^2 t^{2\gamma}\Big)^n}{\Gamma(2\gamma n+1)}, \tag{16}$$

is the Mittag-Leffler function.

In the case $\gamma = 1$ from (14) we have $\eta^2 = \eta_0^2 = \frac{k}{m}$, the solution of the Eq. (13) is a periodic function given by

$$x(t) = (x_0 - \frac{v_0}{k})\cos(\eta_0 t) + \frac{v_0}{k}. \tag{17}$$

The expression (17) represents the classical case.

In this case there exists a physical relationship between the auxiliary parameter $\alpha$ and the fractional order time derivative $\gamma$. For the system described by the fractional equation (11), we can write the relationship [21]

$$\gamma = \frac{\alpha}{\sqrt{\frac{m}{k}}} = \sigma\eta_0, \qquad 0 < \alpha \leq \sqrt{\frac{m}{k}}. \tag{18}$$

Taking into account the expression (18), the solution (15) of the equation (11) can be rewritten through $\gamma$ by

$$x(\hat{t}) = (x_0 - \frac{v_0}{k})E_{2\gamma}\Big\{-\gamma^{2(1-\gamma)}\hat{t}^{2\gamma}\Big\} + \frac{v_0}{k}, \tag{19}$$

where $\hat{t} = \eta_0 t$ is a dimensionless parameter. Plots for different values of $\gamma$ are shown in Fig. 2.

- Unit step source, $v(t) = u(t)$, $x(0) = x_0$, $(x_0 > 0)$, $\dot{x}(0) = 0$

The equation (11) may be written as follows

$$\frac{d^{2\gamma}x(t)}{dt^{2\gamma}} = \frac{\eta^2}{k}u(t) - \eta^2 x(t), \tag{20}$$

where $\eta^2$ is given by (14).

The solution for the Eq.(20) is given by

$$x(t) = (x_0 - \frac{1}{k})E_{2\gamma}\Big\{-\eta^2 t^{2\gamma}\Big\} + \frac{1}{k}, \tag{21}$$

where $E_{2\gamma}$ is the Mittag-Leffler function represented by (1).

In the case $\gamma = 1$ from (14) we have $\eta^2 = \eta_0^2 = \frac{k}{m}$. The solution of the Eq. (20) is a periodic function given by

$$x(t) = (x_0 - \frac{1}{k})\cos(\eta_0 t) + \frac{1}{k}. \tag{22}$$

The expression (22) represents the classical case.

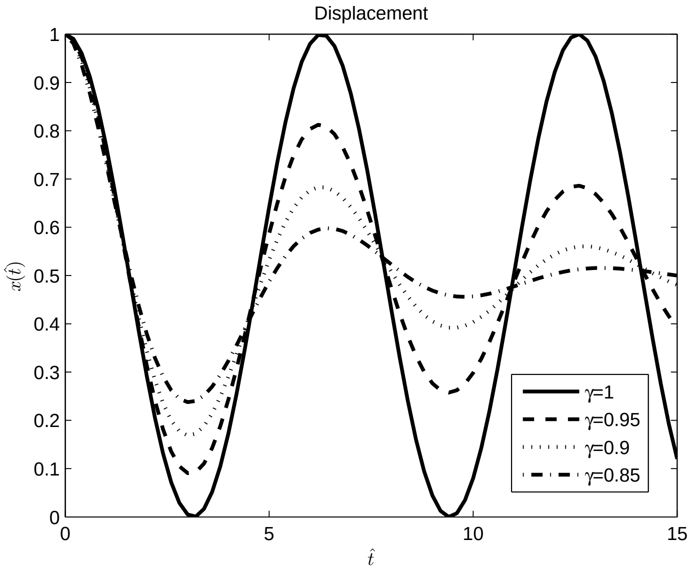


Fig. 2 – Mass-spring system with constant source.

Taking into account the expression (18), the solution (21) of the equation (11) can be rewritten through $\gamma$ by

$$x(\check{t}) = (x_0 - \frac{1}{k})E_{2\gamma}\Big\{-\gamma^{2(1-\gamma)}\check{t}^{2\gamma}\Big\} + \frac{1}{k}, \tag{23}$$

where $\check{t} = \eta_0 t$ is a dimensionless parameter. Plots for different values of $\gamma$ are shown in Fig. 3.

- Periodic source, $v(t) = v_0 \cos(\omega \mathrm{t})$, $x(0) = x_0$, $(x_0 > 0)$, $\dot{x}(0) = 0$
  The equation (11) may be written as follows

$$\frac{d^{2\gamma}x(t)}{dt^{2\gamma}} = \frac{\eta^2}{k}u_0 \cos(\omega \mathrm{t}) - \eta^2 \mathrm{x(t)}, \tag{24}$$

where $\eta^2$ is given by (14).

The solution for the Eq.(24) is given by

$$x(t) = x_0 E_{2\gamma}\Big\{-\eta^2 t^{2\gamma}\Big\} - \frac{v_0}{k}\int_0^t \cos\omega(\mathrm{t-u})\mathrm{E}_{2\gamma}\Big\{-\eta^2 \mathrm{u}^{2\gamma}\Big\}\mathrm{du}, \tag{25}$$

where $E_{2\gamma}$ is the Mittag-Leffler function represented by (1).

In the case $\gamma = 1$ from (14) we have $\eta^2 = \eta_0^2 = \frac{k}{m}$. The solution of the Eq. (24) is a periodic function given by

$$x(t) = x_0 \cos(\eta_0 t) - \frac{v_0}{k}\int_0^t \cos\omega(\mathrm{t-u})\cos(\eta_0 \mathrm{u})\mathrm{du}. \tag{26}$$

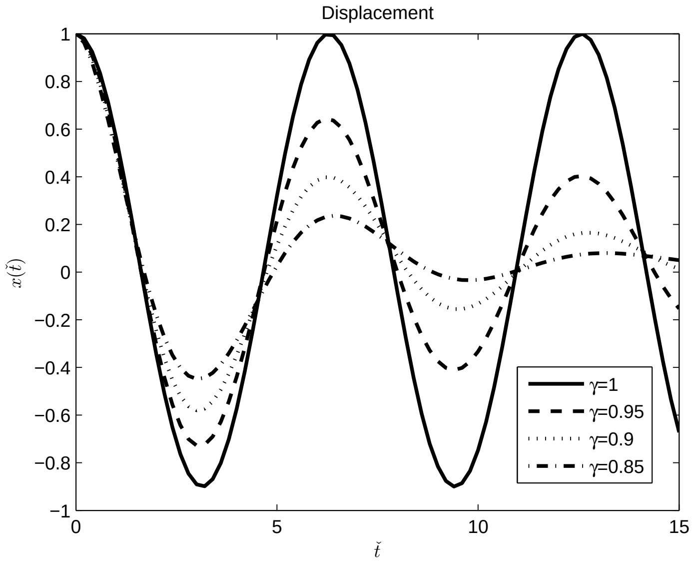


Fig. 3 – Mass-spring system with unit step source.

The expression (26) represents the classical case.

Taking into account the expression (18), the solution (25) of the equation (11) can be rewritten through $\gamma$ by

$$x(\breve{t}) = x_0 E_{2\gamma}\Big\{-\gamma^{2(1-\gamma)}\breve{t}^{2\gamma}\Big\} - \frac{v_0}{k}\int_0^t \mathrm{cos}\omega(\mathrm{t}-\mathrm{u})\mathrm{E}_{2\gamma}\Big\{-\gamma^{2(1-\gamma)}\mathrm{u}^{2\gamma}\Big\}\mathrm{du}, \tag{27}$$

where $\breve{t} = t\eta_0$ is a dimensionless parameter. Plots for different values of $\gamma$ are shown in Fig. 4.

**Damper-spring system**.

- Constant source, $v(t) = v_0$, $x(0) = x_0$, $(x_0 > 0)$, $\dot{x}(0) = 0$
  The equation (12) may be written as follows

$$\frac{d^\gamma x(t)}{dt^\gamma} = \frac{v_0}{\beta}\alpha^{1-\gamma} - \frac{k}{\beta}\alpha^{1-\gamma}x(t), \tag{28}$$

The solution for the Eq.(28) is given by

$$x(t) = (x_0 - \frac{v_0}{k})E_\gamma\Big\{-\frac{k}{\beta}\alpha^{1-\gamma}t^\gamma\Big\} + \frac{v_0}{k}, \tag{29}$$

where

$$E_\gamma\Big\{-\frac{k}{\beta}\alpha^{1-\gamma}t^\gamma\Big\} = \sum_{n=0}^{\infty}\frac{\Big(-\frac{k}{\beta}\alpha^{1-\gamma}t^\gamma\Big)^n}{\Gamma(\gamma n+1)}, \tag{30}$$

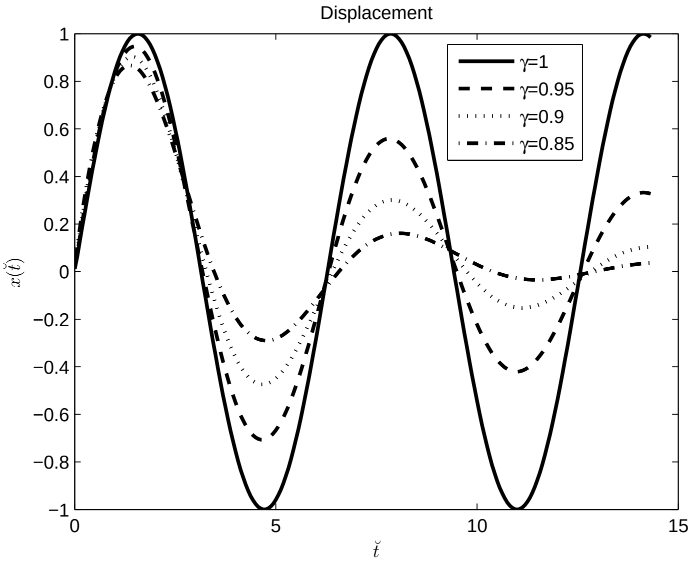


Fig. 4 – Mass-spring system with periodic source.

is the Mittag-Leffler function.

In the case $\gamma = 1$ the expression (29) becomes

$$x(t) = (x_0 - \frac{v_0}{k})\exp^{-\frac{k}{\beta}t} + \frac{v_0}{k}. \tag{31}$$

The expression (31) represents the classical case.

In this case there exists a relationship between the auxiliary parameter $\alpha$ and the fractional order time derivative $\gamma$. For the system described by the fractional equation (12), we can write the relationship [21]

$$\gamma = \frac{k}{\beta}\alpha, \qquad 0 < \alpha \leq \frac{k}{\beta}. \tag{32}$$

Taking into account the expression (32), the solution (29) of the equation (12) can be rewritten through $\gamma$ by

$$x(\hat{t}) = (x_0 - \frac{v_0}{k})E_\gamma\Big\{-\gamma^{1-\gamma}\hat{t}^\gamma\Big\} + \frac{v_0}{k}, \tag{33}$$

where $\hat{t} = \frac{k}{\beta}t$ is a dimensionless parameter. Plots for different values of $\gamma$ are shown in Fig. 5.

- Unit step source, $v(t) = u(t)$, $x(0) = x_0$, $(x_0 > 0)$, $\dot{x}(0) = 0$

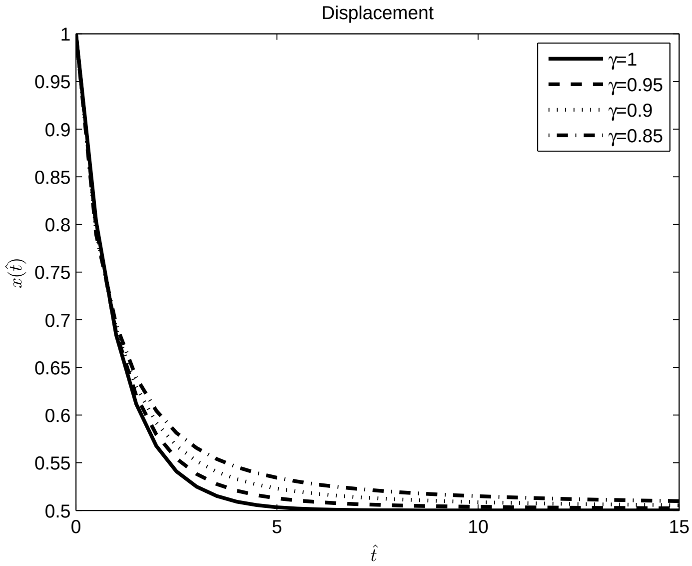


Fig. 5 – Damper-spring system with constant source.

The equation (12) may be written as follows

$$\frac{d^{\gamma}x(t)}{dt^{\gamma}} = \frac{u(t)}{\beta}\alpha^{1-\gamma} - \frac{k}{\beta}\alpha^{1-\gamma}x(t), \tag{34}$$

The solution for the Eq.(34) is given by

$$x(t) = (x_0 - \frac{1}{k})E_{\gamma}\Big\{-\frac{k}{\beta}\alpha^{1-\gamma}t^{\gamma}\Big\} + \frac{1}{k}, \tag{35}$$

where $E_{\gamma}$ is the Mittag-Leffler function represented by (30).

In the case $\gamma = 1$ the expression (35) becomes

$$x(t) = (x_0 - \frac{1}{k})\exp^{-\frac{k}{\beta}t} + \frac{1}{k}. \tag{36}$$

The expression (36) represents the classical case.

Taking into account the expression (32), the solution (35) of the equation (12) can be rewritten through $\gamma$ by

$$x(\check{t}) = (x_0 - \frac{1}{k})E_{\gamma}\Big\{-\gamma^{1-\gamma}\check{t}^{\gamma}\Big\} + \frac{1}{k}, \tag{37}$$

where $\check{t} = \frac{k}{\beta}t$ is a dimensionless parameter. Plots for different values of $\gamma$ are shown in Fig. 6.

- Periodic source, $v(t) = v_0\cos(\omega t)$, $x(0) = x_0$, $(x_0 > 0)$, $\dot{x}(0) = 0$

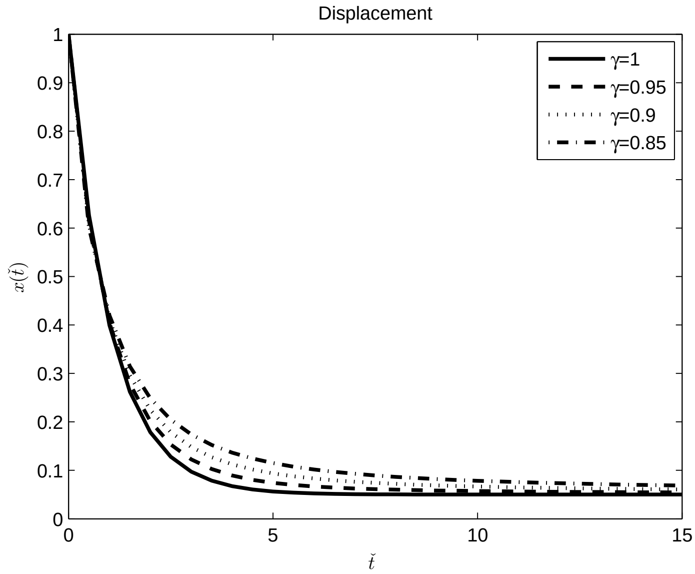


Fig. 6 – Damper-spring system with unit step source.

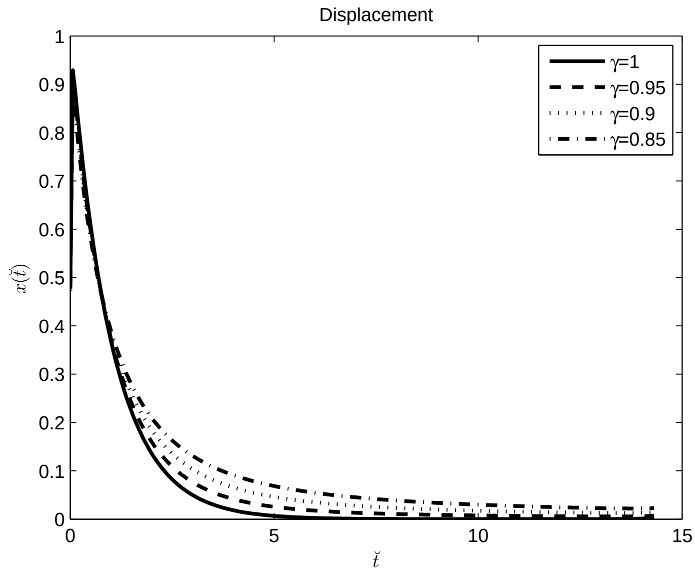


Fig. 7 – Damper-spring system with periodic source.

The equation (12) may be written as follows

$$\frac{d^\gamma x(t)}{dt^\gamma} = \frac{v_0}{\beta}\alpha^{1-\gamma}\cos(\omega \mathrm{t}) - \frac{\mathrm{k}}{\beta}\alpha^{1-\gamma}\mathrm{x}(\mathrm{t}), \tag{38}$$

The solution for the Eq.(38) is given by

$$x(t) = x_0 E_\gamma\Big\{-\frac{k}{\beta}\alpha^{1-\gamma}t^\gamma\Big\} - \frac{v_0}{k}\int_0^t \mathrm{cos}\omega(\mathrm{t}-\mathrm{u})\mathrm{E}_\gamma\Big\{-\frac{\mathrm{k}}{\beta}\alpha^{1-\gamma}\mathrm{u}^\gamma\Big\}\mathrm{du}, \quad (39)$$

where $E_\gamma$ is the Mittag-Leffler function represented by (30).

In the case $\gamma = 1$ the expression (39) becomes

$$x(t) = x_0 \exp^{-\frac{k}{\beta}t} - \frac{v_0}{k}\int_0^t \mathrm{cos}\omega(\mathrm{t}-\mathrm{u})\exp^{-\frac{\mathrm{k}}{\beta}\mathrm{u}}\mathrm{du}. \quad (40)$$

The expression (40) represents the classical case.

Taking into account the expression (32), the solution (39) of the equation (12) can be rewritten through $\gamma$ by

$$x(\breve{t}) = x_0 E_\gamma\Big\{-\gamma^{1-\gamma}\breve{t}^\gamma\Big\} - \frac{v_0}{k}\int_0^t \mathrm{cos}\omega(\mathrm{t}-\mathrm{u})\mathrm{E}_\gamma\Big\{-\gamma^{1-\gamma}\mathrm{u}^\gamma\Big\}\mathrm{du}, \quad (41)$$

where $\breve{t} = \frac{k}{\beta}t$ is a dimensionless parameter. Plots for different values of $\gamma$ are shown in Fig. 7.

## 4. CONCLUSION

In this paper we obtain the analytic expressions and perform the numerical simulations of the mass-spring and damper-spring systems with Caputo derivative. The order of the derivatives is $0 < \gamma \leq 1$. With the purpose of maintaining the physical units of the system an auxiliary parameter $\alpha$ is introduced. The analytical solutions are given in terms of the Mittag-Leffler function depending on the parameter $\gamma$. The source terms involved in the equations are: constant source, unit step source, and periodic source (cosine). The classic cases are recovered when $\gamma = 1$.

In the mass-spring system, the displacement of the fractional oscillator is essentially described by the Mittag-Leffler function $E_{2\gamma}$ represented by (1); if $\gamma$ is less than $1$ the displacement shows the behavior of a damped harmonic oscillator [21]. In the damper-spring system the displacement is essentially described by the Mittag-Leffler function $E_\gamma$ represented by (30); if $\gamma$ is less than $1$ the displacement behaves like one with temporal-decaying amplitude with respect to time $t$. In this case, the fractional exponent shows that the time constant tends to move forward in time as this exponent $\gamma$ is less than 1, that is, the stability occurs in more time than it would take the entire order of exponent (the displacement is slower). This phenomenon indicates the existence of another spring, different from the ideal one shown in Fig. 1 that displys a fractional structure (components that show an intermediate behavior between a conservative system (spring) and a dissipative one (damper)).

In all cases when $\gamma$ is less than 1, the fractional differentiation with respect to time $d^\gamma x(t)/dt^\gamma$ represents a non-local displacement interpreted as an existence of memory effects that correspond to intrinsic dissipation characterized by the exponent $\gamma$ of the fractional derivative and are related to displacement in a fractal space-time geometry. When $\gamma = 1$ (classical case), the displacement shows its natural behavior.

We consider that these results can be useful to understand the behavior of dynamical complex systems, oscillatory processes, and the phenomenon of resonance.

*Acknowledgements.* We would like to thank to Mayra Martínez for interesting discussions. This research was supported by CONACYT.

REFERENCES

1. K. B. Oldham and J. Spanier, *The Fractional Calculus* (Academic Press, New York, 1974).
2. K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley, New York, 1993).
3. S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives, Theory and Applications* (Gordon and Breach, Langhorne, PA, 1993).
4. I. Podlubny, *Fractional Differential Equations* (Academic Press, New York, 1999).
5. M. Caputo and F. Mainardi, *A new dissipation model based on memory mechanism*, Pure and Applied Geophysics, **91**, 134-147 (1971).
6. D. Baleanu, Z. B. Günvenc, and J. A. Tenreiro Machado, *New Trends in Nanotechnology and Fractional Calculus Applications* (Springer, 2010)
7. O. P. Agrawal, J. A. Tenreiro-Machado, and I. Sabatier, *Fractional Derivatives and Their Applications: Nonlinear Dynamics*, **38** (Springer-Verlag, Berlin, 2004).
8. J. A. Tenreiro Machado, *A probabilistic interpretation of the fractional-order differentiation*, Frac. Calc. Appl. Anal. **8**, 73-80 (2003).
9. R. Hilfer, *Applications of Fractional Calculus in Physics* (World Scientific, Singapore, 2000).
10. O. P. Agrawal, *A general formulation and solution scheme for fractional optimal control problems*, Nonlinear Dynamics **38**, 323-337 (2004).
11. R. L. Magin, *Fractional calculus in Bioengineering* (Begell House Publisher, Rodding, 2006).
12. F. Gomez-Aguilar, J. Rosales-Garca, M. Guia-Calderon, and J. Bernal-Alvarado, *Analysis of Equivalent Circuits for Cells: A Fractional Calculus Approach*, Revista de Ingeniera, Investigacin y Tecnologa, UNAM **XIII**, 375-384 (2012).
13. F. Gomez, J. Rosales, and M. Guia, *RLC electrical circuit of non-integer order*, Cent. Eur. J. Phys. **11**, 1361-1365 (2013).
14. Jose Francisco Gomez Aguilar and Dumitru Baleanu, *Solutions of the telegraph equations using a fractional calculus approach*, Proceedings of the Romanian Academy A **15**, 27-34 (2014).
15. F. Gomez, J. Bernal, J. Rosales, and T. Cordova. *Modeling and Simulation of Equivalent Circuits in Description of Biological Systems - A Fractional Calculus Approach*, Journal of Electrical Bioimpedance **3**, 2-11 (2012).
16. M. Alper Kutay and Haldun M. Ozaktas, *The Fractional Fourier Transform and Harmonic Oscillation*, Nonlinear Dynamics **29**, 157-172 (2002).
17. Ya. E. Ryabov and A. Puzenko, *Damped oscillations in view of the fractional oscillator equation*, Physical Review B **66**, 184201 (2002).
18. M. Naber, *Linear Fractionally Damped Oscillator*, International Journal of Differential Equations, **2010**, Article ID 197020 (2010).
19. A.A. Stanislavsky, *Fractional Oscillator*, Physical Review E **70**, 051103 (2004).

20. V.E. Tarasov, *The fractional oscillator as an open system*, Cent. Eur. J. Phys. **10**, 382-389 (2012).
21. J.F. Gómez-Aguilar, J.J. Rosales-García, J.J. Bernal-Alvarado, T. Córdova-Fraga, and R. Guzmán-Cabrera, *Fractional Mechanical Oscillators*, Revista Mexicana de Física **58**, 524-537 (2012).
22. H. J. Haubold, A. M. Mathai, and R. K. Saxena. *Mittag-Leffler functions and their applications*, Journal of Applied Mathematics **2011**, Article ID 298628 (2011).
23. J. F. Gomez-Aguilar, R. Razo-Hernandez, and D. Granados-Lieberman, *A physical interpretation of fractional calculus in observables terms: analysis of the fractional time constant and the transitory response*, Revista Mexicana de Física **60**, 32-38 (2014).