INTEGRATION OF COMPLEX-VALUED KLEIN-GORDON EQUATION IN Φ-4 FIELD THEORY

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Received July 29, 2014

This paper studies complex-valued Klein-Gordon equation that arises in Φ-4 field theory. Three integration tools are availed of in order to obtain soliton and other solutions to the governing equation that is considered with cubic and power law nonlinearity. The three algorithms that are studied in this paper are G'/G-expansion scheme, Kudryashov's method, and the sine-cosine approach. The corresponding constraint conditions of the solutions are also given. These three distinct integration mechanisms lead to several solutions of the equation that will be of great asset in the physics of the problem.

Key words: Solitons; integrability.

PACS: 02.30.Jr; 05.45.Yv; 02.30.Ik..

1. INTRODUCTION

The theory of nonlinear evolution equation (NLEEs) plays an important role in theoretical and engineering physics. There are several forms of NLEEs that are studied in this context [1]-[38]. A few of them are the nonlinear Schrödinger's equation, Boussinesq equation, sine-Gordon equation and many others. Such equations typically appear in nonlinear optics, fluid dynamics, theoretical physics, electrical engineering and others. This paper will focus on one such NLEE. It is the complex-valued Klein-Gordon equation (cKGE) that appears in Φ-4 field theory. The integrability aspect of this important equation will be addressed in this paper. In this paper, cKGE will be considered with cubic and power laws of nonlinearity.

There are three integration tools that will be adopted in this paper. These are G'/G-expansion scheme, Kudryashov's method, and sine-cosine method. These integration mechanisms will lead to several forms of solution of cKGE. A few of these are solitons, singular solitons, shock waves, and singular periodic solutions. There are several constraint conditions that will naturally fall out of these solution structures. These constraint conditions must remain valid for these various form of solutions to exist.

2. GOVERNING EQUATION

The dimensionless form of complex-valued Klein-Gordon equation (KGE) that is addressed in this paper, is given by [5, 15]

$$q_{tt} - k^2 q_{xx} = aq + bF\left(|q|^2\right)q, \tag{1}$$

In Eq. (1), dependent variable $q(x,t)$ is a complex-valued function. Here a and b are real-valued constants. Eq. (1) is typically studied in the context of $\Phi - 4$ field theory that appears in particle physics and field theory. Moreover, complex-valued KGE is a special case of Higgs equation that was introduced in the study of interaction of scalar nucleons and mesons in particle physics. Therefore, $q(x,t)$ represents the complex scalar nucleon field.

2.1. MATHEMATICAL ANALYSIS

In order to solve Eq. (1), we use the following wave transformation

$$q(x,t) = U(z)e^{i\Phi(x,t)} \tag{2}$$

where $U(z)$ represents the shape of the pulse and

$$z = x - vt, \tag{3}$$

$$\Phi(x,t) = -\kappa x + \omega t + \theta. \tag{4}$$

In Eq. (2), the function $\Phi(x,t)$ is the phase component of the soliton. Then, in Eq. (4), κ is the soliton frequency, while ω is the wave number of the soliton and θ is the phase constant. Finally in Eq. (3), v is the velocity of the soliton. By replacing Eq. (2) into Eq. (1) and separating the real and imaginary parts of the result, we have

$$v = \frac{\kappa k^2}{\omega}, \tag{5}$$

and

$$(v^2 - k^2)U_{zz} - \left(a + \omega^2 - \kappa^2 k^2\right)U - bUF\left(U^2\right) = 0. \tag{6}$$

Equation (6) will now be analyzed in the next three sections where the integration of cKGE will be analyzed. The functional F will be considered for two types of nonlinearity. These are cubic law and power law.

3. G'/G-EXPANSION METHOD

Recently, a new method has been proposed by Wang *et al.* [19] called the G'/G-expansion method to study traveling wave solutions of nonlinear evolution equations. This useful method is developed successfully by many authors, see Refs. [8, 23-27] and the references therein. The G'/G-expansion method [8] is based on the assumptions that the travelling wave solutions can be expressed by a polynomial in G'/G such that $G = G(z)$ satisfies a second order linear ordinary differential equation (ODE).

3.1. AN OVERVIEW OF THE SCHEME

We now summarize the G'/G-expansion method, established in 2011 [8], the details of which can be found in [23-25] among many others.

We assume that the given NLEE for $u(x,t)$ is in the form

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t \partial x}, \frac{\partial^2 u}{\partial t^2}, ...\right) = 0, \tag{7}$$

where P is a polynomial. The essence of the G'/G-expansion method can be presented in the following steps:

Step-1: To find the travelling wave solutions of Eq. (7), we introduce the wave variable

$$u(x,t) = U(z), \quad z = x - vt. \tag{8}$$

Substituting Eq. (8) into Eq. (7), we obtain the following ODE

$$Q\left(U, \frac{dU}{dz}, \frac{d^2U}{dz^2}, ...\right) = 0. \tag{9}$$

Step-2: Eq. (9) is then integrated as long as all terms contain derivatives where integration constants are considered zero.

Step-3: Introduce the solution $U(z)$ of Eq. (9) in the finite series form

$$U(z) = \sum_{l=0}^{N} a_l \left(\frac{G'(z)}{G(z)}\right)^l \tag{10}$$

where a_l are real constants with $a_N \neq 0$ and N is a positive integer to be determined. The function $G(z)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(z) + \lambda G'(z) + \mu G(z) = 0, \tag{11}$$

where λ and μ are real constants to be determined.

Step-4: Determining N, can be accomplished by balancing the linear term of highest order derivatives with the highest order nonlinear term in Eq. (9).

Step-5: Substituting the general solution of (10) together with (11) into Eq. (9) yields an algebraic equation involving powers of G'/G. Equating the coefficients of each power of G'/G to zero gives a system of algebraic equations for $a_l,\ \lambda,\ \mu$, and v. Then, we solve the system with the aid of a computer algebra system, such as *Maple*, to determine these constants. Next, depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, we obtain solutions of Eq. (9). So, we can obtain exact solutions of the given Eq. (7).

3.2. APPLICATION TO cKGE

This subsection will focus on the application of this integration algorithm to cKGE with cubic and power laws of nonlinearities. Accordingly, the study will be split into the following two subsections.

3.2.1. Cubic law nonlinearity

For cubic nonlinearity, $F(s) = s$. Balancing U_{zz} with U^3 in Eq. (6) gives $N = 1$. Therefore, the solution of Eq. (6) can be written in the form

$$U(z) = a_0 + a_1 \left(\frac{G'(z)}{G(z)}\right), \quad a_1 \neq 0, \tag{12}$$

where $G(z)$ satisfies the second-order linear ordinary differential equation

$$G''(z) + \lambda G'(z) + \mu G(z) = 0, \tag{13}$$

where λ and μ are real constants to be determined. From Eqs. (12) and (13) we drive

$$U_{zz} = 2a_1 \left(\frac{G'}{G}\right)^3 + 3a_1\lambda \left(\frac{G'}{G}\right)^2 + \left(2a_1\mu + a_1\lambda^2\right)\left(\frac{G'}{G}\right) + a_1\lambda\mu. \tag{14}$$

Substituting Eqs. (12) and (14) into Eq. (6), collecting all terms with the same powers of G'/G and setting each coefficient to zero, we obtain a system of algebraic equations for a_0, a_1, v, ω, κ, λ, and μ as follows.

$\left(\frac{G'}{G}\right)^3$ coeff.:

$$2(v^2 - k^2)a_1 - ba_1^3 = 0, \tag{15}$$

$\left(\frac{G'}{G}\right)^2$ coeff.:

$$3(v^2 - k^2)a_1\lambda - 3ba_0a_1^2 = 0,$$

$\left(\frac{G'}{G}\right)^1$ coeff.:

$$(v^2-k^2)(2a_1\mu+a_1\lambda^2)-3ba_0^2a_1-\left(a+\omega^2-\kappa^2k^2\right)a_1=0,$$

$\left(\frac{G'}{G}\right)^0$ coeff.:

$$(v^2-k^2)a_1\lambda\mu-ba_0^3-\left(a+\omega^2-\kappa^2k^2\right)a_0=0.$$

With the aid of *Maple*, we shall find the special solution of the above system

$$\begin{aligned} a_0&=\pm\lambda\sqrt{\frac{v^2-k^2}{2b}},\\ a_1&=\pm\sqrt{\frac{2(v^2-k^2)}{b}},\\ \mu&=\frac{2\omega^2+2a+v^2\lambda^2-k^2\lambda^2-2\kappa^2k^2}{4(v^2-k^2)}, \end{aligned} \tag{16}$$

where $\lambda,\ \kappa,\ \omega,\ v$ are arbitrary constants.

Substituting the solution set (16) into Eq. (12), the solution formulas of Eq. (6) can be written as

$$U(z)=\pm\sqrt{\frac{2(v^2-k^2)}{b}}\left\{\frac{\lambda}{2}+\frac{G'(z)}{G(z)}\right\}. \tag{17}$$

Substituting the general solutions of second order linear ODE into Eq. (17) gives two types of traveling wave solutions.

Case-I: When $\left(v^2-k^2\right)\left(a+\omega^2-\kappa^2k^2\right)<0$, we obtain the hyperbolic function travelling wave solution

$$\begin{aligned} q(x,t)&=\pm\sqrt{\frac{\kappa^2k^2-a-\omega^2}{b}}\\ &\times\frac{C_1\sinh\left(\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(k^2-v^2)}}\,(x-vt)\right)+C_2\cosh\left(\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(k^2-v^2)}}\,(x-vt)\right)}{C_1\cosh\left(\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(k^2-v^2)}}\,(x-vt)\right)+C_2\sinh\left(\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(k^2-v^2)}}\,(x-vt)\right)}\\ &\times e^{i(-\kappa x+\omega t+\theta)}, \end{aligned} \tag{18}$$

where v is given by (5).

On the other hand, assuming $C_1\neq 0$ and $C_2=0$, the topological 1-soliton solution of the complex-valued Klein-Gordon equation with cubic law nonlinearity can be written as.

$$q(x,t)=\pm\sqrt{\frac{\kappa^2k^2-a-\omega^2}{b}}\tanh\left[\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(k^2-v^2)}}\,(x-vt)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (19)$$

where v is given by (5).

Next, assuming $C_1=0$ and $C_2\neq 0$, then we obtain the singular 1-soliton solution of the cKGE with cubic law nonlinearity

$$q(x,t)=\pm\sqrt{\frac{\kappa^2k^2-a-\omega^2}{b}}\coth\left[\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(k^2-v^2)}}\,(x-vt)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (20)$$

where v is given by (5).

Case-II: When $\left(v^2-k^2\right)\left(a+\omega^2-\kappa^2k^2\right)>0$, we obtain the hyperbolic function traveling wave solution

$$q(x,t)=\pm\sqrt{\frac{a+\omega^2-\kappa^2k^2}{b}}$$
$$\times\frac{-C_1\sin\left(\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(v^2-k^2)}}\,(x-vt)\right)+C_2\cos\left(\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(v^2-k^2)}}\,(x-vt)\right)}{C_1\cos\left(\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(k^2-v^2)}}\,(x-vt)\right)+C_2\sin\left(\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(v^2-k^2)}}\,(x-vt)\right)}$$
$$\times e^{i(-\kappa x+\omega t+\theta)}, \quad (21)$$

where v is given by (5).

Now, for $C_1\neq 0$ and $C_2=0$,

$$q(x,t)=\pm\sqrt{\frac{a+\omega^2-\kappa^2k^2}{b}}\tan\left[\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(v^2-k^2)}}\,(x-vt)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (22)$$

and when $C_1=0,\ \ C_2\neq 0$, the solution of the complex-valued Klein-Gordon equation with cubic law nonlinearity will be

$$q(x,t)=\pm\sqrt{\frac{a+\omega^2-\kappa^2k^2}{b}}\cot\left[\sqrt{\frac{a+\omega^2-\kappa^2k^2}{2(v^2-k^2)}}\,(x-vt)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (23)$$

where v is given by (5).

3.2.2. Power law nonlinearity

Power law nonlinearity arises when $F(s)=s^n$, where the parameter n is referred to as the nonlinearity parameter. Balancing U'' with U^{2n+1} in Eq. (6) gives $N=1/n$. In order to obtain an analytic solution, we utilize the transformation

$$U=V^{\frac{1}{2n}}, \quad (24)$$

in Eq. (6) to find

$$\left(v^2-k^2\right)\left\{(1-2n)(V')^2+2nVV''\right\}-4\left(a+\omega^2-\kappa^2k^2\right)n^2V^2-4bn^2V^3=0. \tag{25}$$

Balancing the order of VV'' and V^3 in Eq. (25), we have $N=2$. Therefore, one may choose

$$V(z)=a_0+a_1\left(\frac{G'(z)}{G(z)}\right)+a_2\left(\frac{G'(z)}{G(z)}\right)^2, \quad a_2\neq 0, \tag{26}$$

where $G(z)$ satisfies the second-order linear ordinary differential equation

$$G''(z)+\lambda G'(z)+\mu G(z)=0, \tag{27}$$

where λ and μ are real constants to be determined. From Eqs. (26) and (27) we drive

$$V_z=-2a_2\left(\frac{G'}{G}\right)^3-(a_1+2a_2\lambda)\left(\frac{G'}{G}\right)^2 -(a_1\lambda+2a_2\mu)\left(\frac{G'}{G}\right)-a_1\mu, \tag{28}$$

$$\begin{aligned}V_{zz}=6a_2\left(\frac{G'}{G}\right)^4&+(2a_1+10a_2\lambda)\left(\frac{G'}{G}\right)^3\\ &+\left(4a_2\lambda^2+8a_2\mu+3a_1\lambda\right)\left(\frac{G'}{G}\right)^2\\ &+\left(6a_2\lambda\mu+2a_1\mu+a_1\lambda^2\right)\left(\frac{G'}{G}\right)+a_1\lambda\mu+2a_2\mu^2.\end{aligned} \tag{29}$$

Substituting Eq. (26) in Eq. (25) and equating all the coefficients of powers of G'/G to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we have

$$\begin{aligned}a_0&=\frac{\mu(v^2-k^2)(1+n)}{bn^2},\\ a_1&=\frac{\lambda(v^2-k^2)(1+n)}{bn^2},\\ a_2&=\frac{(v^2-k^2)(1+n)}{bn^2},\\ \omega&=\pm\frac{\sqrt{4n^2(\kappa^2k^2-a)+(\lambda^2-4\mu)(v^2-k^2)}}{2n},\end{aligned} \tag{30}$$

where v, λ, μ, κ and θ are arbitrary constants.

Substituting the solution set (30) into Eq. (26), the solution formulae of Eq.

(25) can be written as

$$V(z)=\frac{\left(v^2-k^2\right)(1+n)}{bn^2}\left\{\mu+\lambda\left(\frac{G'(z)}{G(z)}\right)+\left(\frac{G'(z)}{G(z)}\right)^2\right\}. \tag{31}$$

Substituting the general solutions of second order linear ODE in (31) gives two types of travelling wave solutions.

Case-I: When $\Delta=\lambda^2-4\mu>0$, we obtain the hyperbolic function travelling wave solution

$$q(x,t)=\Bigg[\frac{(v^2-k^2)(1+n)(4\mu-\lambda^2)}{4bn^2} \times\left\{1-\left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)\right)}\right)^2\right\}\Bigg]^{\frac{1}{2n}} \times e^{i\left(-\kappa x\pm\left\{\frac{\sqrt{4n^2(\kappa^2k^2-a)+(\lambda^2-4\mu)(v^2-k^2)}}{2n}\right\}t+\theta\right)}, \tag{32}$$

where v is given by (5).

Now, assuming $C_1\neq 0$ and $C_2=0$ the travelling wave solution of the complex-valued Klein-Gordon equation with power law nonlinearity can be written as.

$$q(x,t)=\left[\frac{(v^2-k^2)(1+n)(4\mu-\lambda^2)}{4bn^2}\operatorname{sech}^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)\right)\right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x\pm\left\{\frac{\sqrt{4n^2(\kappa^2k^2-a)+(\lambda^2-4\mu)(v^2-k^2)}}{2n}\right\}t+\theta\right)}, \tag{33}$$

where v is given by (5).

Again, assuming $C_1=0$ and $C_2\neq 0$, then we obtain

$$q(x,t)=\left[\frac{(v^2-k^2)(1+n)(\lambda^2-4\mu)}{4bn^2}\operatorname{csch}^2\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)\right)\right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x\pm\left\{\frac{\sqrt{4n^2(\kappa^2k^2-a)+(\lambda^2-4\mu)(v^2-k^2)}}{2n}\right\}t+\theta\right)}, \tag{34}$$

where v is given by (5).

Case-II: When $\Delta=\lambda^2-4\mu<0$, we obtain the hyperbolic function travelling wave solution

$$q_{14}(x,t) = \Big[\frac{(v^2-k^2)(1+n)(4\mu-\lambda^2)}{4bn^2} \times \left(1+\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)\right)}\right)^2\right)\Big]^{\frac{1}{2n}} \times e^{i\left(-\kappa x \pm\left\{\frac{\sqrt{4n^2(\kappa^2k^2-a)+(\lambda^2-4\mu)(v^2-k^2)}}{2n}\right\}t+\theta\right)}, \quad (35)$$

where v is given by (5).

Assuming $C_1 \neq 0$ and $C_2 = 0$, then

$$q(x,t) = \left[\frac{(v^2-k^2)(1+n)(4\mu-\lambda^2)}{4bn^2}\sec^2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)\right)\right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x \pm\left\{\frac{\sqrt{4n^2(\kappa^2k^2-a)+(\lambda^2-4\mu)(v^2-k^2)}}{2n}\right\}t+\theta\right)}, \quad (36)$$

where v is given by (5).

When $C_1 = 0, \;\; C_2 \neq 0$ the solution of the cKGE with power law nonlinearity will be

$$q(x,t) = \left[\frac{(v^2-k^2)(1+n)(4\mu-\lambda^2)}{4bn^2}\csc^2\left(\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)\right)\right]^{\frac{1}{2n}} \times e^{i\left(-\kappa x \pm\left\{\frac{\sqrt{4n^2(\kappa^2k^2-a)+(\lambda^2-4\mu)(v^2-k^2)}}{2n}\right\}t+\theta\right)}, \quad (37)$$

where v is given by (5).

4. KUDRYASHOV'S METHOD

The modification of truncated expansion method, which is a direct and effective algebraic method for computing exact traveling wave solutions, was first proposed by Kudryashov [11]. The modification of truncated expansion method that is known as the Kudryashov method is one of the most effective methods for finding the exact solution of high order NLEEs [10]. The most complete description of this method was given in [12]. The successful application of this method to NLEEs was performed in works [10–14].

4.1. REVIEW OF THE ALGORITHM

Let us present the algorithm of modification of truncated expansion method (Kudryashov method) for finding exact solutions of nonlinear PDEs. We consider the nonlinear PDE in the following form:

$$P_1\left(u, u_t, u_x, u_{xx}, ...\right) = 0. \quad (38)$$

Using travelling wave $u(x,t) = U(z),\ z = x - vt$ carries Eq. (38) into the following ordinary differential equation (ODE):

$$P_2\left(U, -vU_z, U_z, U_{zz}, ...\right) = 0. \quad (39)$$

The Kudryashov method contains the following steps.

Step-1: We look for exact solution of Eq. (39) in the form

$$U = \sum_{l=0}^{N} a_l \left(G(z)\right)^l \quad (40)$$

where $a_l(l = 0,\ 1, ...,\ N)$ are constants to be determined later, such that $a_N \neq 0$, while $G(z)$ has the form

$$G(z) = \frac{1}{1 + K\exp(z)} \quad (41)$$

a solution to the Riccati equation

$$G'(z) = G^2(z) - G(z) \quad (42)$$

where K is an arbitrary constant.

Step-2: We determine the positive integer N in Eq. (40) by considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq. (39).

Step-3: We substitute Eq. (40) into Eq. (39), and calculate all the necessary derivatives $U_z,\ U_{zz}, ...$ of the unknown function $U(z)$ as follows:

$$U_z = \sum_{l=1}^{N} a_l l(G-1)G^l, \quad (43)$$

$$U_{zz} = \sum_{l=1}^{N} a_l l\left\{(1+l)G^2 - (2l+1)G + l\right\}G^l, \quad (44)$$

and so on. Substituting Eqs. (40), (43) and (44) into Eq. (39), we obtain the polynomial

$$E_2[G(z)] = 0. \quad (45)$$

Step-4: Collecting all the terms of the same powers of the function $G(z)$ in the polynomial (45) and equating them to zero, we obtain a system of algebraic equations which can be solved by computer programs such as *Maple* and *Mathematica* to get the unknown parameters a_l and v. Consequently, we obtain the exact solutions of Eq. (38).

4.2. APPLICATION TO cKGE

This section will apply Kudryashov's technique to cKGE. There are two types of the nonlinear function F will be considered. These are the cubic law and power law nonlinearity. The details are enlisted in the next two subsections.

4.2.1. Cubic law nonlinearity

In this subsection, we will apply the Kudryashov method to obtain the exact solution of the complex-valued Klein-Gordon equation with cubic law nonlinearity. The exponent order of Eq. (6) is $N = 1$. So we look for solution of Eq. (6) in the following form

$$U(z) = a_0 + a_1 G(z). \tag{46}$$

Substituting Eq. (46) into Eq. (6), we obtain the system of algebraic equations in the following form

$$G^3 : 2(v^2 - k^2)a_1 - ba_1^3 = 0, \tag{47}$$

$$G^2 : -3(v^2 - k^2)a_1 - 3ba_0a_1^2 = 0, \tag{48}$$

$$G^1 : -3ba_0^2a_1 - \left(a + \omega^2 - \kappa^2 k^2\right) a_1 + (v^2 - k^2)a_1 = 0, \tag{49}$$

$$G^0 : -ba_0^3 - \left(a + \omega^2 - \kappa^2 k^2\right) a_0 = 0. \tag{50}$$

Solving these under-determined algebraic equations, we arrive at the following results:

$$\begin{aligned} a_0 &= \pm\sqrt{\frac{v^2 - k^2}{2b}}, \\ a_1 &= \mp\sqrt{\frac{2(v^2 - k^2)}{b}}, \\ \omega &= \pm\sqrt{\frac{k^2 - v^2 - 2a + 2\kappa^2 k^2}{2}}, \end{aligned} \tag{51}$$

where v, κ, and θ are arbitrary constants.

Substituting Eq. (51) into (46), we obtain exact solution of the cKGE with cubic nonlinearity as:

$$q(x,t) = \mp\sqrt{\frac{2(v^2 - k^2)}{b}} \tanh\left[\frac{1}{2}(x - vt)\right] e^{i\left(-\kappa x \pm \left\{\sqrt{\frac{k^2 - v^2 - 2a + 2\kappa^2 k^2}{2}}\right\} t + \theta\right)}, \tag{52}$$

and

$$q(x,t)=\mp\sqrt{\frac{2(v^2-k^2)}{b}}\coth\left[\frac{1}{2}(x-vt)\right]e^{i\left(-\kappa x\pm\left\{\sqrt{\frac{k^2-v^2-2a+2\kappa^2k^2}{2}}\right\}t+\theta\right)}, \quad (53)$$

where v is given by (5).

4.2.2. Power law nonlinearity

Balancing the order of VV'' and V^3 in Eq. (25), we have $N=2$. So, the Kudryashov method suggests the use of the finite expansion

$$V(z)=a_0+a_1G(z)+a_2G^2(z). \quad (54)$$

Substituting Eq. (54) into Eq. (25) and equating all the coefficients of powers of $G(z)$ to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we have

$$\begin{aligned} a_0 &= 0, \\ a_1 &= \frac{(k^2-v^2)(1+n)}{bn^2}, \\ a_2 &= -\frac{(k^2-v^2)(1+n)}{bn^2}, \\ \omega &= \pm\frac{\sqrt{4n^2(\kappa^2k^2-a)+v^2-k^2}}{2n}, \end{aligned} \quad (55)$$

where v, κ, and θ are arbitrary constants.

Using values of parameters (55) we have following solution of Eq. (54):

$$V(z)=\frac{(v^2-k^2)(1+n)}{bn^2}\left\{G^2(z)-G(z)\right\}. \quad (56)$$

Combining (56) with (41), we obtain the exact solution to ODE (25) in the form

$$V(z)=\frac{(k^2-v^2)(1+n)}{bn^2}\frac{Ke^z}{(1+Ke^z)^2}. \quad (57)$$

By using (24), we have the exact solution of the complex-valued Klein-Gordon equation with power law nonlinearity in the form

$$q(x,t)=\left\{\frac{(k^2-v^2)(1+n)}{bn^2}\frac{Ke^{x-vt}}{(1+Ke^{x-vt})^2}\right\}^{\frac{1}{2n}} \times e^{i\left(-\kappa x\pm\left\{\frac{\sqrt{4n^2(\kappa^2k^2-a)+v^2-k^2}}{2n}\right\}t+\theta\right)}, \quad (58)$$

where v is given by (5).

When $K=1$, we have the following 1-soliton solution

$$q(x,t)=\left\{\frac{(k^2-v^2)(1+n)}{4bn^2}\mathrm{sech}^2\left[\frac{1}{2}(x-vt)\right]\right\}^{\frac{1}{2n}} \times e^{i\left(-\kappa x\pm\left\{\frac{\sqrt{4n^2(\kappa^2k^2-a)+v^2-k^2}}{2n}\right\}t+\theta\right)}, \quad (59)$$

where v is given by (5).

When $K=-1$, we obtain the following 1-soliton solution

$$q(x,t)=\left\{\frac{(v^2-k^2)(1+n)}{4bn^2}\mathrm{csch}^2\left[\frac{1}{2}(x-vt)\right]\right\}^{\frac{1}{2n}} \times e^{i\left(-\kappa x\pm\left\{\frac{\sqrt{4n^2(\kappa^2k^2-a)+v^2-k^2}}{2n}\right\}t+\theta\right)}, \quad (60)$$

where v is given by (5).

5. SINE-COSINE FUNCTION METHOD

This is the third integration tool that will be implemented to extract soliton and other solutions to cKGE. Once again, cKGE with cubic as well as with power law nonlinearity will be considered. The integration scheme will be first overview and subsequently applied to cKGE in the following subsections.

5.1. PREVIEW OF THE ALGORITHM

This integration technique is a mathematical mechanism to extract soliton and other solutions to any NLEE [16]. A NLEE

$$P(u,u_t,u_x,u_{xx},u_{xt},u_{tt},...)=0, \quad (61)$$

can be converted to an ODE

$$Q\left(U,U',U'',...\right)=0, \quad (62)$$

upon using a travelling wave variable $u(x,t)=U(z)$, $z=x-vt$. If possible, integrate Eq. (62) term by term one or more times. This will reduce the order of Eq. (62). For simplicity, the integration constants can be set to zero. The solutions of the reduced ODE can be expressed in the form

$$U(z)=\lambda\cos^{\beta}(\mu z), \quad |z|\le\frac{\pi}{2\mu}, \quad (63)$$

or in the form

$$U(z)=\lambda\sin^{\beta}(\mu z), \quad |z|\le\frac{\pi}{\mu}, \quad (64)$$

where λ, μ, and β are parameters that will be determined, μ and v are the wave number and the wave speed respectively. These assumptions give

$$(U^n)'' = -n^2\mu^2\beta^2\lambda^n\cos^{n\beta}(\mu z) + n\mu^2\lambda^n\beta(n\beta-1)\cos^{n\beta-2}(\mu z), \tag{65}$$

and

$$(U^n)'' = -n^2\mu^2\beta^2\lambda^n\sin^{n\beta}(\mu z) + n\mu^2\lambda^n\beta(n\beta-1)\sin^{n\beta-2}(\mu z). \tag{66}$$

Using (65)-(66) in the reduced ODE gives a trigonometric equation in $\cos^K(z)$ or $\sin^K(z)$ terms. The parameters are then determined by first balancing the exponents of each pair of cosines or sines to determine K. We next collect all coefficients of the same power in $\cos^K(z)$ or $\sin^K(z)$, where these coefficients have to vanish. This gives a system of algebraic equations among the unknowns β, λ, v, and μ that will be determined. The solutions proposed in (63) and (64) follow immediately.

5.2. APPLICATION TO cKGE

The sine-cosine algorithm will be applied to integrate cKGE with same two nonlinear forms. The following two subsections will enumerate the details for these two laws as an application of the integration tool to cKGE.

5.2.1. Cubic law nonlinearity

Using the assumption

$$U(z) = \lambda\cos^{\beta}(\mu z), \tag{67}$$

in Eq. (6) we obtain

$$U_{zz} = -\mu^2\beta^2\lambda\cos^{\beta}(\mu z) + \mu^2\lambda\beta(\beta-1)\cos^{\beta-2}(\mu z). \tag{68}$$

Substituting Eqs. (67) and (68) into Eq. (6), we have

$$\lambda\left\{(k^2-v^2)\mu^2\beta^2 - \left(a+\omega^2-\kappa^2k^2\right)\right\}\cos^{\beta}(\mu z) + \lambda\mu^2\beta(\beta-1)(v^2-k^2)\cos^{\beta-2}(\mu z) - b\lambda^3\cos^{3\beta}(\mu z) = 0. \tag{69}$$

Using the balance method, by equating the exponents and the coefficients of $\cos^K$, we get

$$\beta(\beta-1) \neq 0 \tag{70}$$

$$3\beta = \beta - 2, \tag{71}$$

$$\mu^2\beta(\beta-1)(v^2-k^2) - b\lambda^2 = 0, \tag{72}$$

$$(k^2-v^2)\mu^2\beta^2 - \left(a+\omega^2-\kappa^2k^2\right) = 0. \tag{73}$$

Solving the system (Eqs (71)-(73)) simultaneously, we get the solution set

$$\beta=-1,\ \mu=\pm\sqrt{\frac{a+\omega^2-\kappa^2k^2}{k^2-v^2}},\ \lambda=\pm\sqrt{-\frac{2\,(a+\omega^2-\kappa^2k^2)}{b}}. \tag{74}$$

Consequently, we obtain the following periodic solutions:

$$q(x,t)=\pm\sqrt{-\frac{2(a+\omega^2-\kappa^2k^2)}{b}}\sec\left[\sqrt{-\frac{a+\omega^2-\kappa^2k^2}{v^2-k^2}}\,(x-vt)\right]\times e^{i(-\kappa x+\omega t+\theta)}, \tag{75}$$

and

$$q(x,t)=\pm\sqrt{-\frac{2(a+\omega^2-\kappa^2k^2)}{b}}\csc\left[\sqrt{-\frac{a+\omega^2-\kappa^2k^2}{v^2-k^2}}\,(x-vt)\right]\times e^{i(-\kappa x+\omega t+\theta)}, \tag{76}$$

where v is given by (5).

It is easy to see that solutions (75) and (76) can reduce to hyperbolic solutions as follows:

$$q(x,t)=\pm\sqrt{-\frac{2(a+\omega^2-\kappa^2k^2)}{b}}\operatorname{sech}\left[\sqrt{\frac{a+\omega^2-\kappa^2k^2}{v^2-k^2}}\,(x-vt)\right]\times e^{i(-\kappa x+\omega t+\theta)}, \tag{77}$$

and

$$q(x,t)=\pm\sqrt{\frac{2(a+\omega^2-\kappa^2k^2)}{b}}\operatorname{csch}\left[\sqrt{\frac{a+\omega^2-\kappa^2k^2}{v^2-k^2}}\,(x-vt)\right]\times e^{i(-\kappa x+\omega t+\theta)}, \tag{78}$$

where v is given by (5).

5.2.2. Power law nonlinearity

Using the assumption

$$U(z)=\lambda\cos^{\beta}(\mu z), \tag{79}$$

in Eq. (6) we obtain

$$U_{zz}=-\mu^2\beta^2\lambda\cos^{\beta}(\mu z)+\mu^2\lambda\beta(\beta-1)\cos^{\beta-2}(\mu z). \tag{80}$$

Substituting Eqs. (79) and (80) into Eq. (6), we have

$$\lambda\left\{(k^2-v^2)\mu^2\beta^2-\left(a+\omega^2-\kappa^2k^2\right)\right\}\cos^{\beta}(\mu z)$$
$$+\lambda\mu^2\beta(\beta-1)(v^2-k^2)\cos^{\beta-2}(\mu z)-b\lambda^{2n+1}\cos^{(2n+1)\beta}(\mu z)=0. \quad (81)$$

Using the balance method, by equating the exponents and the coefficients of $\cos^K$, we obtain

$$\beta(\beta-1)\neq 0 \quad (82)$$
$$(2n+1)\beta=\beta-2, \quad (83)$$
$$\mu^2\beta(\beta-1)(v^2-k^2)-b\lambda^{2n+1}=0, \quad (84)$$
$$(k^2-v^2)\mu^2\beta^2-\left(a+\omega^2-\kappa^2k^2\right)=0. \quad (85)$$

Solving the system (Eqs (83)-(85)) simultaneously, we recover the solution set

$$\beta=-\frac{1}{n},\ \mu=\pm n\sqrt{\frac{a+\omega^2-\kappa^2k^2}{k^2-v^2}},\ \lambda=\left\{-\frac{(n+1)\left(a+\omega^2-\kappa^2k^2\right)}{b}\right\}^{\frac{1}{2n}}.$$

Consequently, the following periodic solutions fall out:

$$q(x,t)=\left\{-\frac{(n+1)(a+\omega^2-\kappa^2k^2)}{b}\sec^2\left[n\sqrt{-\frac{a+\omega^2-\kappa^2k^2}{v^2-k^2}}\,(x-vt)\right]\right\}^{\frac{1}{2n}}$$
$$\times e^{i(-\kappa x+\omega t+\theta)}, \quad (86)$$

and

$$q(x,t)=\left\{-\frac{(n+1)(a+\omega^2-\kappa^2k^2)}{b}\csc^2\left[n\sqrt{-\frac{a+\omega^2-\kappa^2k^2}{v^2-k^2}}\,(x-vt)\right]\right\}^{\frac{1}{2n}}$$
$$\times e^{i(-\kappa x+\omega t+\theta)}, \quad (87)$$

where v is given by (5).

It is easy to see that solutions (86) and (87), respectively, revert to soliton solutions as follows:

$$q(x,t)=\left\{-\frac{(n+1)(a+\omega^2-\kappa^2k^2)}{b}\operatorname{sech}^2\left[n\sqrt{\frac{a+\omega^2-\kappa^2k^2}{v^2-k^2}}\,(x-vt)\right]\right\}^{\frac{1}{2n}}$$
$$\times e^{i(-\kappa x+\omega t+\theta)}, \quad (88)$$

and

$$q(x,t) = \left\{ \frac{(n+1)(a+\omega^2-\kappa^2 k^2)}{b} \mathrm{csch}^2 \left[n\sqrt{\frac{a+\omega^2-\kappa^2 k^2}{v^2-k^2}} (x-vt) \right] \right\}^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \quad (89)$$

after adjusting the complex arguement, where v is given by (5).

6. CONCLUSIONS

This paper addressed the cKGE that appears in meson physics. There are three integration algorithms that are applied to obtain soliton and other solutions to this equation. They are G'/G-expansion scheme, Kudryashov's method and finally the sine-cosine algorithm. There are two types of nonlinearity that are considered. They are cubic and power law. All of these integration algorithms were applied to both of these forms of nonlinearity. These three integration techniques lead to several forms of solution to cKGE. These are solitons, singular solitons as well as singular periodic solutions. The results of this paper will be later applied for further research down the road.

In future, numerical simulations are going to be addressed. Perturbation terms will be included. This will lead to an extended version of cKGE, which will be studied further along and the results of those research activities will be reported elsewhere. Additionally, more integration tools will be implemented to study cKGE as well as perturbed cKGE. One such promising tool is Lie symmetry analysis. This is just a tip of the iceberg.

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