A NOTE ON TELEPARALLEL LIE SYMMETRIES USING NON DIAGONAL TETRAD

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This paper considers teleparallel theory of gravitation and explores Killing and proper homothetic vector fields for non diagonal tetrad of Kantowski-Sachs spacetime. For the purpose, direct integration technique has been used. It turns out that the number of Killing vector fields is either four or seven. The seven generators of teleparallel Killing vector fields, responsible for conservation of energy, linear and angular momentum are recovered and other three generators responsible for spin angular momentum are lost. The effect of torsion on Killing symmetry has also been determined for the spacetime under consideration. More interestingly, there exists no teleparallel proper homothetic vector field for the choice of non-diagonal tetrad of Kantowski-Sachs spacetime.

Key words: Teleparallel theory, Tetrad fields, Teleparallel Killing vector fields.

1. INTRODUCTION

In general relativity theory, symmetries perform a key role in discovering geometrical and physical features of the spacetime. It has been remained an interesting topic to find and analyze solutions of Einstein's field equations through different symmetries. Since, symmetries have wide range of applications in understanding and revealing the hidden mysteries of our universe, a large body of literature is available on the topic. For instance, in [1] the author obtained canonical forms of a real function which generate the metric of type-N Robinson-Trautman spacetime under the assumption that the spacetime admits at least one curvature collineation. The covariant characterization of self-similarity of second type for spherical distribution of matter is discussed in [2]. The Ricci and matter collineation are found in [3] where authors introduced mathematical description of differentiability and dimensionality for these symmetries. Various physical and mathematical properties of the spacetimes admitting kinematic self-similarity are discussed in [4]. In [5] all spherically symmetric spacetimes are classified according to their kinematic self-similar vectors of second, infinite and zeroth kinds.

Self-similar solutions of first, second, infinite and zeroth kinds for Bianchi type III and static spherically symmetric spacetimes are obtained in [6-7]. Ugur Camci *et al.* [8] classified Bianchi type II spacetime according to its matter collineation for degenerate and non-degenerate energy momentum tensor. The author in [9] presented exact cylindrically symmetric solutions of the Einstein field equations with super fluid as a source.

General relativity is not the only theory of gravity. Teleparallel theory also describes gravity and is considered an equivalent theory of gravitation to the general theory of relativity. The general theory of relativity is based upon curvature of the spacetime, that is, gravitational interaction is dependent upon curvature within spacetime. In teleparallel theory the spacetime does not have curvature rather it has torsion which compels the particles to feel gravitation. Moreover, teleparallel theory uses Weitzenböck connections instead of torsion less Christoffel symbols.

The role of symmetries in teleparallel theory may not be neglected in determining the results for gravitational energy, momentum and angular momentum. A deep insight of the effect of torsion on symmetries of the spacetime is also needed. In order to address these issues the authors in [10–14] explored Killing, homothetic, and conformal vector fields for different spacetimes in context of teleparallel theory of gravitation and argued that torsion has a strong effect on symmetries. The main problem one faces in teleparallel theory is the choice of an appropriate tetrad field for different applications. Two different types of tetrads for the same spacetime may give different results, for example see [15] where authors obtained two different equations of motion for different choices of tetrads for the same spacetime. In such scenario symmetries in teleparallel theory may help us in choosing right tetrad.

The role of symmetries in determination of right tetrad for different applications can be seen in [16]. In this paper the author showed that a non diagonal tetrad produces more generators for teleparallel Killing vector fields than a diagonal tetrad. Moreover, the same non diagonal tetrad produces better results for energy, momentum, irreducible mass and angular momentum. Keeping in mind the advantage of non diagonal tetrad over diagonal one, we are interested to find Killing and proper teleparallel homothetic vector fields for the choice of non diagonal tetrad for Kantowski-Sachs spacetime. With the help of produced outcome, the effect of torsion on Killing symmetry will be determined.

2. TELEPARALLEL THEORY (AN OVERVIEW)

The teleparallel covariant derivative ∇_ρ of a covariant tensor of rank 2 is defined as [17]

$$\nabla_\rho F_{\mu\nu} = F_{\mu\,\nu,\rho} - \Gamma^\theta{}_{\rho\nu} F_{\mu\,\theta} - \Gamma^\theta{}_{\mu\,\rho} F_{\nu\,\theta}, \tag{2.1}$$

where comma denote partial derivative and $\Gamma^\theta{}_{\rho\nu}$ are Weitzenböck connections defined as [17]

$$\Gamma^\theta{}_{\mu\,\nu} = W_a{}^\theta \partial_\nu W^a{}_\mu, \tag{2.2}$$

where $W^a{}_\mu$ is the non-trivial tetrad field. Its inverse field is denoted by $W_a{}^\nu$ and satisfies the relations

$$W^a{}_\mu W_a{}^\nu = \delta_\mu{}^\nu, \qquad W^a{}_\mu W_b{}^\mu = \delta_b{}^a \tag{2.3}$$

The Riemannian metric can be generated from the tetrad field as

$$g_{\mu\,\nu} = \eta_{a\,b} W^a{}_\mu W^b{}_\nu, \tag{2.4}$$

where η_{ab} is the Minkowski metric given by $\eta_{ab} = \mathrm{diag}(-1,\ 1,\ 1,\ 1)$. The Weitzenböck and Levi-Civita connections have the relation

$$\Gamma^\theta{}_{\mu\,\nu} = \Gamma^{0^\theta}{}_{\mu\,\nu} \; + S^\theta{}_{\mu\,\nu}, \tag{2.5}$$

where

$$S^\theta{}_{\mu\,\nu} = \frac{1}{2}[T_\mu{}^\theta{}_\nu + T_\nu{}^\theta{}_\mu - T^\theta{}_{\mu\,\nu}] \tag{2.6}$$

is tensor quantity called the contortion tensor and $\Gamma^{0^\theta}{}_{\mu\nu}$ is the Levi-Civita connection defined as

$$\Gamma^{0^\theta}{}_{\mu\,\nu} \; = \frac{1}{2} g^{\theta\,\sigma}(g_{\sigma\,\nu,\mu} + g_{\sigma\,\mu,\nu} - g_{\mu\,\nu,\sigma}). \tag{2.7}$$

Torsion in terms of Weitzenböck connections is defined as

$$T^\theta{}_{\mu\nu} = \Gamma^\theta{}_{\nu\mu} - \Gamma^\theta{}_{\mu\,\nu}, \tag{2.8}$$

this is anti symmetric with respect to its lower indices. The Riemann curvature tensor in terms of Weitzenböck connection in teleparallel theory is given as

$$R^\theta{}_{\sigma\mu\nu} = \Gamma^\theta{}_{\sigma\nu,\mu} - \Gamma^\theta{}_{\sigma\mu,\nu} + \Gamma^\theta{}_{\lambda\mu}\Gamma^\lambda{}_{\sigma\nu} - \Gamma^\theta{}_{\lambda\nu}\Gamma^\lambda{}_{\sigma\mu}. \tag{2.9}$$

Now using equation (2.5) in (2.9), the teleparallel Riemann curvature tensor vanishes, *i.e.*

$$R^\sigma{}_{\theta\mu\nu} = R^{0^\sigma}{}_{\theta\mu\nu} + D^\sigma{}_{\theta\mu\nu} = 0, \tag{2.10}$$

where $R^{0^\sigma}_{\theta\mu\nu}$ represents Riemann curvature tensor in general relativity and

$$D^{\sigma}{}_{\theta\mu\nu} = \nabla_{\mu} S^{\sigma}{}_{\theta\nu} - \nabla_{\nu} S^{\sigma}{}_{\theta\mu} - S^{\lambda}{}_{\theta\nu} S^{\sigma}{}_{\lambda\mu} + S^{\lambda}{}_{\theta\mu} S^{\sigma}{}_{\lambda\nu}, \tag{2.11}$$

is a tensor quantity based on Weitzenböck connection only. In [18] the authors defined Killing equation in teleparallel theory for the vector field $\boldsymbol{X}$ as

$$\underset{X}{L^{T}}\, g_{\alpha\beta} = g_{\alpha\beta,\rho} X^{\rho} + g_{\rho\beta} X^{\rho}{}_{,\alpha} + g_{\alpha\rho} X^{\rho}{}_{,\beta} + X^{\rho} (g_{\theta\beta} T^{\theta}{}_{\alpha\rho} + g_{\alpha\,\theta} T^{\theta}{}_{\beta\rho}) = 0, \tag{2.12}$$

where $\underset{X}{L^{T}}$ represents Lie derivative in teleparallel theory. For finding teleparallel proper homothetic vector fields we shall use the above definition in the extended form as

$$\underset{X}{L^{T}}\, g_{\mu\nu} = 2\alpha g_{\mu\nu}, \quad \alpha \in R. \tag{2.13}$$

3. KILLING VECTOR FIELDS OF KANTOWSKI-SACHS SPACETIME

Kantowski-Sachs spacetime in spherical coordinates (t, r, θ, ϕ) is defined as [19]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + M^2(t)\mathrm{d}r^2 + N^2(t)\ (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{3.1}$$

where M and N are functions of t only which are nowhere zero. The four independent Killing vector fields of (3.1) in general relativity are given as [19]

$$\frac{\partial}{\partial r},\ \frac{\partial}{\partial \phi},\ \cos\phi \frac{\partial}{\partial \theta} - \cot\theta \sin\phi \frac{\partial}{\partial \phi},\ \sin\phi \frac{\partial}{\partial \theta} + \cot\theta\cos\phi \frac{\partial}{\partial \phi}. \tag{3.2}$$

Following a well-known procedure, given in [20] the tetrad $W^{a}{}_{\mu}$, its inverse $W_{a}{}^{\mu}$, Weitzenböck connections $\Pi^{a}{}_{bc}$ and torsion components $T^{\theta}{}_{\alpha\beta}$ for Kantowski-Sachs spacetime are obtained as

$$W^{a}{}_{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & M\cos\phi\sin\theta & N\cos\theta\cos\phi & -N\sin\theta\sin\phi \\ 0 & M\sin\theta\sin\phi & N\cos\theta\cos\phi & N\cos\phi\sin\theta \\ 0 & M\cos\theta & -N\sin\theta & 0 \end{pmatrix} \tag{3.3}$$

$$W_{a}{}^{\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{M}\cos\phi\sin\theta & \frac{1}{M}\sin\theta\sin\phi & \frac{1}{M}\cos\theta \\ 0 & \frac{1}{N}\cos\theta\cos\phi & \frac{1}{N}\cos\theta\sin\phi & -\frac{1}{N}\sin\theta \\ 0 & -\frac{1}{N}\cos\theta\sin\phi & \frac{1}{N}\cos\theta\cos\phi & 0 \end{pmatrix} \tag{3.4}$$

$$\Pi^{1}{}_{10}=\frac{\dot{M}}{M},\quad \Pi^{2}{}_{12}=\frac{M}{N},\quad \Pi^{3}{}_{13}=\frac{M}{N},\quad \Pi^{1}{}_{22}=-\frac{N}{M},$$
$$\Pi^{2}{}_{20}=\frac{\dot{N}}{N},\quad \Pi^{3}{}_{23}=\cot\theta,\quad \Pi^{1}{}_{33}=-\frac{N}{M}\sin^{2}\theta, \tag{3.5}$$
$$\Pi^{2}{}_{33}=-\cos\theta\sin\theta,\quad \Pi^{3}{}_{30}=\frac{\dot{N}}{N},$$

where dot represents derivative with respect to t.

$$T^{1}{}_{01}=\frac{\dot{M}}{M},\quad T^{2}{}_{12}=-\frac{M}{N},\quad T^{3}{}_{13}=-\frac{M}{N},\quad T^{2}{}_{20}=\frac{\dot{N}}{N},\quad T^{3}{}_{30}=\frac{\dot{N}}{N}. \tag{3.6}$$

A vector field X is called a teleparallel homothetic vector field if it satisfies equation (2.13). Expanding equation (2.13) with the help of equations (3.1) and (3.6), the following ten non linear, coupled partial differential equations are obtained

$$X^{0}{}_{,0}=X^{1}{}_{,1}=\alpha \tag{3.7}$$

$$M^{2}X^{1}{}_{,0}-X^{0}{}_{,1}+M\dot{M}X^{1}=0, \tag{3.8}$$

$$N^{2}X^{2}{}_{,0}-X^{0}{}_{,2}+N\dot{N}X^{2}=0, \tag{3.9}$$

$$N^{2}\sin^{2}\theta X^{3}{}_{,0}-X^{0}{}_{,3}+N\dot{N}\sin^{2}\theta X^{3}=0, \tag{3.10}$$

$$N^{2}X^{2}{}_{,1}+M^{2}X^{1}{}_{,2}-MNX^{2}=0, \tag{3.11}$$

$$M^{2}X^{1}{}_{,3}+N^{2}\sin^{2}\theta X^{3}{}_{,1}-MN\sin^{2}\theta X^{3}=0, \tag{3.12}$$

$$NX^{2}{}_{,2}+MX^{1}=\alpha N, \tag{3.13}$$

$$X^{2}{}_{,3}+\sin^{2}\theta X^{3}{}_{,2}=0, \tag{3.14}$$

$$N\cot\theta X^{2}+NX^{3}{}_{,3}+MX^{1}=\alpha N, \tag{3.15}$$

First, solving equations (3.7)-(3.15) for teleparallel Killing vector fields by taking $\alpha=0$. Integrating equation (3.7) and taking $\alpha=0$ we get

$$X^{0}=E^{1}(r,\theta,\phi),\quad X^{1}=E^{2}_{\theta\phi}(t,\theta,\phi) \tag{3.16}$$

Also equation (3.13) gives

$$X^{2}=-\frac{M}{N}E^{2}_{\phi}(t,\theta,\phi)+E^{3}_{\phi}(t,r,\phi). \tag{3.17}$$

Now using equations (3.16) and (3.17) in equation (3.15), we get

$$X^3 = -\frac{M}{N} E^2_\theta(t,\theta,\phi) + \frac{M}{N}\cot\theta E^2(t,\theta,\phi) - \cot\theta E^3(t,r,\phi) + E^4(t,r,\theta) \qquad (3.18)$$

The above functions $E^1(r,\theta,\phi)$, $E^2(t,\theta,\phi)$, $E^3(t,r,\phi)$ and $E^4(t,r,\theta)$, are functions of integration. The values of these functions will be determined by using the remaining six equations. In the following results have been given directly just to avoid extensive details of simplification.

Case I: In this case the two metric functions are different *i.e.* $M(t) \neq N(t)$. The resulting line element for Kantowski-Sachs spacetime is given in equation (3.1). Teleparallel Killing vector fields for spacetime (3.1) becomes

$$\begin{aligned} X^0 &= c_1, \quad X^1 = \frac{1}{M(t)} c_2 \cos\theta + \frac{1}{M(t)} \sin\theta(-c_3 \sin\phi + c_4 \cos\phi), \\ X^2 &= -\frac{1}{N(t)} c_2 \sin\theta + \frac{1}{N(t)} \cos\theta(-c_3 \sin\phi + c_4 \cos\phi), \\ X^3 &= -\frac{1}{N(t)} \text{cosec}\theta(c_3 \cos\phi + c_4 \sin\phi) \;, \end{aligned} \qquad (3.19)$$

where $c_1, c_2, c_3, c_4 \in R$. The four generators of the teleparallel Killing vectors for the spacetime (3.1) are $\zeta_0 = \frac{\partial}{\partial t}$, $\zeta_1 = -\frac{1}{M(t)}\sin\theta\sin\phi\frac{\partial}{\partial r} - \frac{1}{N(t)}\cos\theta\sin\phi\frac{\partial}{\partial\theta} - -\frac{1}{N(t)}\text{cosec}\theta\cos\phi\frac{\partial}{\partial\phi}$, $\zeta_2 = \frac{1}{M(t)}\cos\theta\frac{\partial}{\partial r} - \frac{1}{N(t)}\sin\theta\frac{\partial}{\partial\theta}$ and $\zeta_3 = \frac{1}{M(t)}\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{N(t)}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{1}{N(t)}\text{cosec}\theta\sin\phi\frac{\partial}{\partial\phi}$. Comparing the obtained results with the generators for Poincare symmetry algebra of Minkowski spacetime given in [1], the four generators responsible for conservation of energy and linear momentum are recovered and the generators which yield conservation of angular and spin momentum are lost.

Case II: In this case the metric functions take the form $M = \text{constant}$ and $N = N(t)$. After a suitable rescaling of r equation (3.1) becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + N^2(t)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \qquad (3.20)$$

Teleparallel Killing vectors for (3.20) are given as

$$X^0 = c_1, \quad X^1 = c_2 \cos\theta + \sin\theta(-c_3 \sin\phi + c_4 \cos\phi),$$
$$X^2 = -\frac{1}{N(t)} c_2 \sin\theta + \frac{1}{N(t)} \cos\theta(-c_3 \sin\phi + c_4 \cos\phi), \tag{3.21}$$
$$X^3 = -\frac{1}{N(t)} \text{cosec}\theta(c_3 \cos\phi + c_4 \sin\phi) \;,$$

where $c_1, c_2, c_3, c_4 \in R$. The four generators of the teleparallel Killing vectors for the spacetime (3.20) are $\zeta_0 = \frac{\partial}{\partial t}$, $\zeta_1 = -\sin\theta\sin\phi\frac{\partial}{\partial r} - \frac{1}{N(t)}\cos\theta\sin\phi\frac{\partial}{\partial\theta} - -\frac{1}{N(t)}\text{cosec}\theta\cos\phi\frac{\partial}{\partial\phi}$, $\zeta_2 = \cos\theta\frac{\partial}{\partial r} - \frac{1}{N(t)}\sin\theta\frac{\partial}{\partial\theta}$ and $\zeta_3 = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{N(t)}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{1}{N(t)}\text{cosec}\theta\sin\phi\frac{\partial}{\partial\phi}$. Results show that generators yielding angular and spin angular momentum are lost for the metric (3.20) and other generators are recovered as in Case I.

Case III: In this case the spacetime takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + M^2(t)\mathrm{d}r^2 + \eta^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{3.22}$$

Teleparallel Killing vector fields for the spacetime (3.22) are given as

$$X^0 = c_1, \qquad X^1 = \frac{1}{M(t)} c_2 \cos\theta + \frac{1}{M(t)} \sin\theta(-c_3 \sin\phi + c_4 \cos\phi),$$
$$X^2 = -\frac{1}{\eta} c_2 \sin\theta + \frac{1}{\eta} \cos\theta(-c_3 \sin\phi + c_4 \cos\phi), \tag{3.23}$$
$$X^3 = -\frac{1}{\eta} \text{cosec}\theta(c_3 \cos\phi + c_4 \sin\phi) \;,$$

where $c_1, c_2, c_3, c_4 \in R$. The four generators of the teleparallel Killing vectors for the spacetime (3.22) are $\zeta_0 = \frac{\partial}{\partial t}$, $\zeta_1 = -\frac{1}{M(t)}\sin\theta\sin\phi\frac{\partial}{\partial r} - \frac{1}{\eta}\cos\theta\sin\phi\frac{\partial}{\partial\theta} - -\frac{1}{\eta}\text{cosec}\theta\cos\phi\frac{\partial}{\partial\phi}$, $\zeta_2 = \frac{1}{M(t)}\cos\theta\frac{\partial}{\partial r} - \frac{1}{\eta}\sin\theta\frac{\partial}{\partial\theta}$ and $\zeta_3 = \frac{1}{M(t)}\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{\eta}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{1}{\eta}\cos ec\theta\sin\phi\frac{\partial}{\partial\phi}$. Same generators like Case I for the teleparallel Killing vector fields are obtained with a slight change that $N=\eta$ in this case.

Case IV: In this case the metric functions take the form $M=\beta, \quad \beta\in R\setminus\{0\}$ and $N=\eta\ , \quad \eta\in R\setminus\{0\},\beta\neq\eta.$ The resulting line element after suitable rescaling of r is given as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + \eta^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{3.24}$$

Teleparallel Killing vector fields for spacetime (3.24) are given as

$$\begin{aligned}
&X^0 = c_1, \qquad X^1 = c_2\cos\theta + \sin\theta(-c_3\sin\phi + c_4\cos\phi),\\
&X^2 = -\frac{1}{\eta}c_2\sin\theta + \frac{1}{\eta}\cos\theta(-c_3\sin\phi + c_4\cos\phi) + e^{\frac{r}{\eta}}(c_5\cos\phi + c_6\sin\phi),\\
&X^3 = -\frac{1}{\eta}\mathrm{cosec}\theta(c_3\cos\phi + c_4\sin\phi) + \cot\theta \qquad e^{\frac{r}{\eta}}(-c_5\sin\phi + c_6\cos\phi) + \eta c_7 e^{\frac{r}{\eta}}
\end{aligned} \tag{3.25}$$

where $c_1,c_2,c_3,c_4,c_5,c_6,c_7\in R.$ The seven generators of the teleparallel Killing vectors for spacetime (3.24) are $\zeta_0=\frac{\partial}{\partial t},\ \zeta_1=\cos\theta\frac{\partial}{\partial r}-\frac{1}{\eta}\sin\theta\frac{\partial}{\partial\theta},$

$\zeta_2=-\sin\theta\sin\phi\frac{\partial}{\partial r}-\frac{1}{\eta}\cos\theta\sin\phi\frac{\partial}{\partial\theta}-\frac{1}{\eta}\mathrm{cosec}\theta\cos\phi\frac{\partial}{\partial\phi},$

$\zeta_3=\sin\theta\cos\phi\frac{\partial}{\partial r}+\frac{1}{\eta}\cos\theta\cos\phi\frac{\partial}{\partial\theta}-\frac{1}{\eta}\mathrm{cosec}\theta\sin\phi\frac{\partial}{\partial\phi},$

$\zeta_4=e^{\frac{r}{\eta}}(\cos\phi\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}),\quad \zeta_5=e^{\frac{r}{\eta}}(\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi})$ and $\zeta_6=\eta\ e^{\frac{r}{\eta}}\frac{\partial}{\partial\phi}.$

Comparing these results with the generators for Poincare symmetry algebra of Minkowski spacetime given in [1], it comes out that the four generators $\zeta_0,\zeta_1,\zeta_2,\zeta_3$ responsible for conservation of energy and linear momentum and three generators ζ_4,ζ_5,ζ_6 yielding conservation of angular momentum are recovered and the generators responsible for conservation of spin angular momentum are lost for the spacetime (3.24).

Case V: In this case the resulting metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + M^2(t)(\mathrm{d}r^2 + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{3.26}$$

Teleparallel Killing vector fields for the metric (3.26) becomes

$$X^0 = c_1, \quad X^1 = \frac{1}{M(t)} c_2 \cos\theta + \frac{1}{M(t)} \sin\theta(-c_3 \sin\phi + c_4 \cos\phi),$$

$$X^2 = -\frac{1}{M(t)} c_2 \sin\theta + \frac{1}{M(t)} \cos\theta(-c_3 \sin\phi + c_4 \cos\phi) +$$

$$+\frac{e^r}{M(t)}(-c_5 \sin\phi + c_6 \cos\phi), \tag{3.27}$$

$$X^3 = -\frac{1}{M(t)} \cos ec\theta(c_3 \cos\phi + c_4 \sin\phi) + \frac{e^r}{M(t)} \cot\theta(-c_5 \cos\phi - c_6 \sin\phi) +$$

$$+\frac{1}{M(t)} c_7 e^r,$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7 \in R$. The seven generators of the teleparallel Killing vectors for spacetime (3.26) are $\zeta_0 = \frac{\partial}{\partial t}$,

$$\zeta_1 = \frac{1}{M(t)}(\sin\theta\cos\phi\frac{\partial}{\partial r} + \cos\theta\cos\phi\frac{\partial}{\partial\theta} - \text{cosec}\theta\sin\phi\frac{\partial}{\partial\phi}),$$

$$\zeta_2 = \frac{1}{M(t)}(\cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{\partial}{\partial\theta}),$$

$$\zeta_3 = \frac{1}{M(t)}(\sin\theta\sin\phi\frac{\partial}{\partial r} + \cos\theta\sin\phi\frac{\partial}{\partial\theta} + \text{cosec}\theta\cos\phi\frac{\partial}{\partial\phi}),$$

$\zeta_4 = -\frac{e^r}{M(t)}(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi})$, $\zeta_5 = \frac{e^r}{M(t)}(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi})$ and $\zeta_6 = \frac{e^r}{M(t)}\frac{\partial}{\partial\phi}$. Results show that for the metric (3.26), the generators yielding only spin angular momentum are lost and all other generators are recovered.

It is also important to note that the case when $M(t) = N(t) = \eta$ where $\eta \in R \setminus \{0\}$. The resulting metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \eta^2(\mathrm{d}r^2 + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2), \tag{3.28}$$

The generators of teleparallel Killing vectors for the metric (3.28) are the same as in Case V with a slight change that $M(t) = \eta$.

4. PROPER TELEPARALLEL HOMOTHETIC VECTOR FIELDS OF KANTOWSKI-SACHS SPACETIME

In this section equations (3.7) – (3.15) are solved completely when $\alpha \neq 0$. Solving equation (3.7) we get

$$X^0 = \alpha t + E^1(r,\theta,\phi), \quad X^1 = \alpha r + E^2_\theta(t,\theta,\phi) \tag{4.1}$$

Using equation (4.1) in equation (3.13) and solving, we get

$$X^2 = \alpha\theta - \alpha\frac{M}{N} r\theta - \frac{M}{N} E^2(t,\theta,\phi) + E^3(t,r,\phi) \tag{4.2}$$

Now using equations (4.1) and (4.2) in equation (3.10) and solving, we get

$$X^3 = \frac{\operatorname{cosec}^2\theta}{N} E^1_\phi(r,\theta,\phi)\int \frac{1}{B}\mathrm{d}t + \frac{1}{N} E^4(r,\theta,\phi). \tag{4.3}$$

The above functions $E^1(r,\theta,\phi)$, $E^2(t,\theta,\phi)$, $E^3(t,r,\phi)$ and $E^4(r,\theta,\phi)$, are integration functions. We will determine the values of these functions by using the remaining six equations. Thus, we need to determine the unknown functions involved in the following system of equations:

$$\begin{aligned} &X^0 = \alpha t + E^1(r,\theta,\phi), \quad X^1 = \alpha r + E^2_\theta(t,\theta,\phi), \\ &X^2 = \alpha\theta - \alpha\frac{M}{N} r\theta - \frac{M}{N} E^2(t,\theta,\phi) + E^3(t,r,\phi), \\ &X^3 = \frac{\operatorname{cosec}^2\theta}{N} E^1_\phi(r,\theta,\phi)\int \frac{1}{B}\mathrm{d}t + \frac{1}{N} E^4(r,\theta,\phi). \end{aligned} \tag{4.4}$$

Using system of equations (4.4) in equation (3.8), implies

$$M^2 E^2_{t\theta}(t,\theta,\phi) - E^1_r(r,\theta,\phi) + \alpha M\dot{M} r + M\dot{M} E^2_\theta(t,\theta,\phi) = 0 \tag{4.5}$$

Solving equation (3.23) tactfully by first differentiating with respect to r and then integrating the resulting equation, gives $E^1(r,\theta,\phi) = \frac{c_1}{2} r^2 + rF^1_\theta(\theta,\phi) + F^2(\theta,\phi)$ and $M(t) = \left(\frac{2c_1}{\alpha} t + c_2\right)^{\frac{1}{2}}$, where $c_1, c_2 \in R$, $c_1 \neq 0$. Substituting back these values in equation (4.5) and solving we get $E^2(t,\theta,\phi) = \frac{1}{M} F^1(\theta,\phi)\int \frac{1}{M}\mathrm{d}t +$ $+ \frac{1}{M} F^3(\theta,\phi) + F^4(t,\phi)$, where $F^1(\theta,\phi), F^2(\theta,\phi), F^3(\theta,\phi)$ and $F^4(t,\phi)$ are functions of integration. Refreshing the system of equations (4.4) by using the above information, we get

$$
\begin{aligned}
X^0 &= \alpha t + \frac{c_1}{2} r^2 + r F^1_\theta(\theta,\phi) + F^2(\theta,\phi),\\
X^1 &= \alpha r + \frac{1}{M} F^1_\theta(\theta,\phi)\int \frac{1}{M}\mathrm{d}t + \frac{1}{M} F^3_\theta(\theta,\phi),\\
X^2 &= \alpha\theta - \alpha\frac{M}{N} r\theta - \frac{1}{N} F^1(\theta,\phi)\int\frac{1}{M}\mathrm{d}t - \frac{1}{N}F^3(\theta,\phi) -\\
&\quad - \frac{M}{N} F^4(t,\phi) + E^3(t,r,\phi),\\
X^3 &= \frac{\operatorname{cosec}^2\theta}{N}[rF^1_{\theta\phi}(\theta,\phi) + F^2_\phi(\theta,\phi)]\int\frac{1}{N}\mathrm{d}t + \frac{1}{N}E^4(r,\theta,\phi).
\end{aligned} \tag{4.6}
$$

Substituting equation (4.6) in equation (3.11) and then differentiating the resulting equation with respect to r and θ, we reach to an equation $\alpha\ \ M^2(t) = 0$ which simply implies that $\alpha = 0$. Thus we conclude that Kantowski-Sachs spacetime do not admit proper teleparallel homothetic vector fields for the choice of non-diagonal tetrad.

5. SUMMARY AND DISCUSSION

In this paper a non diagonal tetrad is formed for Kantowski-Sachs spacetime. Teleparallel Lie derivative has been applied to obtain teleparallel Killing vector fields along with their respective metrics. Investigation is also extended to proper teleparallel homothetic vector fields. It has shown that dimension of the teleparallel Killing vector field is either four or seven. The generators of teleparallel Killing vectors have been compared to the generators of Poincare symmetry algebra of Minkowski spacetime. This comparison reveals, when Kantowski-Sachs spacetime admit four teleparallel Killing vector fields, it loses six generators responsible for conservation of angular and spin angular momentum. Moreover, when Kantowski-Sachs spacetime admit seven teleparallel Killing vector fields, it also loses three generators responsible for conservation of spin angular momentum.

The purpose of this study was also to see that the choice of non-diagonal tetrad allows Kantowski-Sachs spacetime to admit any extra symmetry other than teleparallel Killing symmetry. Interestingly, the choice of non-diagonal tetrad restricted the spacetime to exhibit only Killing symmetry and there exist no proper teleparallel homothetic vector fields.

It is important to note that in teleparallel theory of gravity, the geometry of spacetime remains flat. It was expected that Kantowski-Sachs spacetime with flat spacetime structure would, exhibit ten teleparallel Killing vector fields. To our surprise, it admits only four or seven teleparallel Killing vector fields. Thus, presence of torsion reduced the number of Killing symmetries. It has also been observed that the presence of torsion in Kantowski-Sachs spacetime have an effect upon the angular and spin angular momentum only, while conservation of energy and linear momentum stays unaffected.

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