A STUDY ON COUPLINGS OF THE FIFTH-ORDER INTEGRABLE SAWADA-KOTERA AND LAX EQUATIONS

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We construct nonlinear integrable couplings of the fifth-order nonlinear integrable Sawada-Kotera (SK) equation and Lax equation. We use the algebra of coupled scalars to construct the two classes of couplings. We study the constructed couplings by using the simplified Hirota's method. We show that these classes of couplings possess multiple soliton solutions the same as the multiple soliton solutions of the SK and the Lax equations, with one change in the sign of the transformation used. This change of signs exhibits soliton solution then anti-soliton solution, consecutively.

Key words: Couplings of integrable equations, multiple soliton solutions.

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1. INTRODUCTION

The theory of nonlinear integrable couplings of ordinary soliton systems attracted researchers for more works on this topic. The couplings of nonlinear equations was presented in [1]-[3] and further studied in [4]-[7] and by some of the references therein. Integrable couplings can be defined as coupled systems of integrable equations. Many powerful methods for constructing integrable couplings have been developed, such as the perturbation method used by Ma *et al.* in Refs. [1]-[3], the enlarged Lie algebra method used by Zhang *et al.* in [4], the non-semi simple Lie algebras [5, 6], the algebra of coupled scalars used in [7] and by other methods as well. It is known that for any integrable couplings, it must include the given integrable equation as a sub-system.

The well-known fifth-order Korteweg-de Vries (fKdV) equation in its standard form reads [8]

$$u_t + \alpha u^2 u_x + \beta u_x u_{xx} + \gamma u u_{3x} + u_{5x} = 0, \qquad (1)$$

where α, β, and γ are arbitrary nonzero and real parameters, and $u = u(x,t)$ is a sufficiently smooth function. The fKdV equation (1) involves two dispersive terms u_{3x} and u_{5x}. Because the parameters α, β, and γ are arbitrary and take different values, this will drastically change the characteristics of the fKdV equation (1). A variety of the fKdV equations can be developed by changing the real values of the parameters α, β, and γ.

However, five well known forms of the fKdV that are of particular interest given by [8]:
(i) The Sawada-Kotera (SK) equation is given by

$$u_t + 5u^2 u_x + 5u_x u_{xx} + 5uu_{3x} + u_{5x} = 0, \tag{2}$$

characterized by

$$\beta = \gamma, \alpha = \frac{1}{5}\gamma^2, \tag{3}$$

where $\gamma = 5$ is selected.
(ii) The Caudrey-Dodd-Gibbon equation (CDG) is given by

$$u_t + 180u^2 u_x + 30u_x u_{xx} + 30uu_{xxx} + u_{xxxxx} = 0, \tag{4}$$

characterized by

$$\beta = \gamma, \alpha = \frac{1}{5}\gamma^2, \tag{5}$$

where $\gamma = 30$ is selected.
(iii) The Lax equation reads

$$u_t + 30u^2 u_x + 20u_x u_{xx} + 10uu_{3x} + u_{5x} = 0, \tag{6}$$

characterized by

$$\beta = 2\gamma, \alpha = \frac{3}{10}\gamma^2, \tag{7}$$

where $\gamma = 10$ is selected.
(iv) The Kaup-Kuperschmidt (KK) equation reads

$$u_t + 20u^2 u_x + 25u_x u_{xx} + 10uu_{3x} + u_{5x} = 0, \tag{8}$$

characterized by

$$\beta = \frac{5}{2}\gamma, \alpha = \frac{1}{5}\gamma^2, \tag{9}$$

where $\gamma = 10$ is selected.
(v) The Ito equation is given as

$$u_t + 2u^2 u_x + 6u_x u_{xx} + 3uu_{3x} + u_{5x} = 0, \tag{10}$$

characterized by

$$\beta = 2\gamma, \alpha = \frac{2}{9}\gamma^2, \tag{11}$$

where $\gamma = 3$ is selected.

The first four aforementioned equations SK, CDG, Lax, and KK equations are completely integrable equations that have infinite sets of conserved quantities and give multiple soliton solutions. However, the Ito equation is not completely integrable but has a limited number of conserved quantities.

Recently, two other fifth-order integrable equations were established by Wazwaz in [9, 10], given by

$$u_{ttt} - u_{txxxx} - 4(u_x u_t)_{xx} - 4(u_x u_{xt})_x = 0, \tag{12}$$

and

$$u_{ttt} - u_{txxxx} - \alpha(u_x u_t)_{xx} - \beta(u_x u_{xt})_x = 0, \tag{13}$$

that give multiple kink solutions. It is of particular noteworthy the third-order derivative with respect to the temporal variable t and the mixed fifth-order derivative compared to the first order derivative with respect to t and the fifth-order derivative with respect to the spatial variable x of the aforementioned fifth-order equations SK, CDG, Lax, KK, and Ito equations.

It is interesting to point out that for more ideas about solitons beyond the standard slowly varying envelope approximation and for a series of emerging applications in areas such as nonlinear optics, Bose-Einstein condensates, plasmas, and nonlinear plasmonics, see *e.g.*, Refs. [11]-[18].

Our aim from this work is two fold. The first goal is to employ the newly developed algebra of coupled scalars [7] to construct nonlinear integrable couplings for the fifth-order SK equation and the Lax equation, hence we will use first the generalized form (1). We aim second to study the developed classes of the couplings of the SK equation (2) and the couplings of the Lax equation (6) respectively. We aim to show that each couplings possesses the same features as the fifth-order standard equation, but differ only in the signs of the transformations used, see Refs. [10] and [19]-[25]. This difference exhibits soliton solutions for some equations and anti-soliton solutions for others.

2. CONSTRUCTING NONLINEAR INTEGRABLE EQUATIONS

In [7], a practical method was developed to construct couplings of the integrable equations. To summarize the approach, it was assumed that if

$$\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i, \tag{14}$$

where $\mathbf{e_i}$ are the basis vectors, then

$$\mathbf{a}\cdot\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \tag{15}$$

where

$$c_i = a_i b_i + a_i \left(\sum_{k=1}^{i-1} b_k \right) + \left(\sum_{k=1}^{i-1} a_k \right) b_i. \tag{16}$$

This means that the value of the coefficient c_i is given by $a_i b_i$ plus terms depending on lower order elements [7], a_k, b_k with $k < i$. This method was called the algebra of coupled scalars, and was found to be unital, commutative and associative. For $n = 3$, we find

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_2 b_2 + a_2 b_1 + a_1 b_2 \\ a_3 b_3 + a_3(b_1 + b_2) + (a_1 + a_2) b_3 \end{pmatrix}. \tag{17}$$

For more details about the algebra of coupled scalars and its properties, see Ref. [7].

Using the algebra of coupled scalars developed in [7], we introduce a one field soliton system

$$u_t = K[u] \equiv K[u, u_x, u_{xx}, u_{xxx}, \cdots], \tag{18}$$

that can be extended to the system of coupled PDEs of the form

$$\mathbf{u}_t = K[\mathbf{u}] \equiv K[\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \cdots], \tag{19}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}. \tag{20}$$

Accordingly, the system (19) takes the form [7]

$$\begin{aligned} (u_1)_t &= K[u_1], \\ (u_k)_t &= K\left[\sum_{i=1}^{k} u_i\right] - K\left[\sum_{i=1}^{k-1} u_i\right], k = 2, 3, \cdots n. \end{aligned} \tag{21}$$

Using (1), we can set

$$\mathbf{u}_t = -\mathbf{u}_{xxxxx} - \alpha \mathbf{u}^2 \mathbf{u}_x - \beta \mathbf{u}_x \mathbf{u}_{xx} - \gamma \mathbf{u} \mathbf{u}_{\mathbf{xxx}}. \tag{22}$$

Inserting (22) into (21), we develop the n-coupled fifth-order equation, given by

$$\begin{aligned}
(u_1)_t &= -(u_1)_{xxxxx} - \alpha u_1^2 (u_1)_x - \beta (u_1)_x (u_1)_{xx} - \gamma (u_1)(u_1)_{xxx},\\
(u_2)_t &= -(u_2)_{xxxxx} - \alpha u_1^2 (u_2)_x - \alpha (2u_1 u_2 + u_2^2)(u_1+u_2)_x\\
&- \beta (u_1)_x (u_2)_{xx} - \beta (u_2)_x (u_1+u_2)_{xx} - \gamma (u_1)(u_2)_{xxx}\\
&- \gamma u_2 (u_1+u_2)_{xxx},\\
(u_3)_t &= -(u_3)_{xxxxx} - \alpha (u_1+u_2)^2 (u_3)_x\\
&- \alpha \left[2(u_1+u_2)u_3 + u_3^2\right]\left[(u_1+u_2)_x + (u_3)_x\right]\\
&- \beta (u_1+u_2)_x (u_3)_{xx} - \beta (u_3)_x (u_1+u_2+u_3)_{xx}\\
&- \gamma (u_1+u_2)(u_3)_{xxx} - \gamma u_3 (u_1+u_2+u_3)_{xxx},\\
(u_4)_t &= -(u_4)_{xxxxx} - \alpha (u_1+u_2+u_3)^2 (u_4)_x\\
&- \alpha \left[2(u_1+u_2+u_3)u_4 + u_4^2\right]\left[(u_1+u_2+u_3)_x + (u_4)_x\right]\\
&- \beta (u_1+u_2+u_3)_x (u_4)_{xx} - \beta (u_4)_x (u_1+u_2+u_3+u_4)_{xx}\\
&- \gamma (u_1+u_2+u_3)(u_4)_{xxx} - \gamma u_4 (u_1+u_2+u_3+u_4)_{xxx}, \\
&\vdots \quad ,\\
(u_n)_t &= -(u_n)_{xxxxx} - \alpha \left(\sum_{k=1}^{n-1} u_k\right)^2 (u_n)_x\\
&- \alpha \left[2\left(\sum_{k=1}^{n-1} u_k\right)(u_n) + (u_n)^2\right]\left(\sum_{k=1}^{n} u_k\right)_x\\
&- \beta \left[\left(\sum_{k=1}^{n-1} u_k\right)_x (u_n)_{xx}\right] - \beta (u_n)_x \left(\sum_{k=1}^{n} u_k\right)_{xx}\\
&- \gamma \left[\left(\sum_{k=1}^{n-1} u_k\right)(u_n)_{xxx}\right] - \gamma (u_n)\left(\sum_{k=1}^{n} u_k\right)_{xxx}, \quad n \geq 2.
\end{aligned} \tag{23}$$

We first substitute the transformation

$$u(x,t) = R\frac{\partial^2 \ln f(x,t)}{\partial x^2} = R\frac{f f_{2x} - (f_x)^2}{f^2}, \tag{24}$$

into (1), where the auxiliary function, as assumed by the simplified Hirota's method reads [25]-[33]

$$f = 1 + e^{\theta}, \tag{25}$$

where the wave variable is given by

$$\theta = kx - ct, \tag{26}$$

and solving the outcome we get

$$\begin{aligned} \alpha &= \frac{\gamma^2+\gamma\beta}{10}, \\ R &= \frac{60}{\gamma+\beta}, \end{aligned} \tag{27}$$

that works for the afore mentioned fifth-order equations.

3. COUPLINGS OF THE SAWADA–KOTERA EQUATION

In this section we will examine the couplings of the Sawada-Kotera equation given by

$$\begin{aligned}
(u_1)_t &= -(u_1)_{xxxxx} - \frac{\gamma^2}{5}u_1^2(u_1)_x - \gamma(u_1)_x(u_1)_{xx} - \gamma(u_1)(u_1)_{xxx}, \\
(u_2)_t &= -(u_2)_{xxxxx} - \frac{\gamma^2}{5}u_1^2(u_2)_x - \frac{\gamma^2}{5}(2u_1u_2+u_2^2)(u_1+u_2)_x \\
&\quad - \gamma(u_1)_x(u_2)_{xx} - \gamma(u_2)_x(u_1+u_2)_{xx} - \gamma(u_1)(u_2)_{xxx} \\
&\quad - \gamma u_2(u_1+u_2)_{xxx}, \\
(u_3)_t &= -(u_3)_{xxxxx} - \frac{\gamma^2}{5}(u_1+u_2)^2(u_3)_x \\
&\quad - \frac{\gamma^2}{5}\left[2(u_1+u_2)u_3+u_3^2\right]\left[(u_1+u_2)_x+(u_3)_x\right] \\
&\quad - \gamma(u_1+u_2)_x(u_3)_{xx} - \gamma(u_3)_x(u_1+u_2+u_3)_{xx} \\
&\quad - \gamma(u_1+u_2)(u_3)_{xxx} - \gamma u_3(u_1+u_2+u_3)_{xxx}, \\
&\quad \vdots, \\
(u_n)_t &= -(u_n)_{xxxxx} - \frac{\gamma^2}{5}\left(\sum_{k=1}^{n-1}u_k\right)^2(u_n)_x \\
&\quad - \frac{\gamma^2}{5}\left[2\left(\sum_{k=1}^{n-1}u_k\right)(u_n)+(u_n)^2\right]\left(\sum_{k=1}^{n}u_k\right)_x \\
&\quad - \gamma\left[\left(\sum_{k=1}^{n-1}u_k\right)_x(u_n)_{xx}\right] - \gamma(u_n)_x\left(\sum_{k=1}^{n}u_k\right)_{xx} \\
&\quad - \gamma\left[\left(\sum_{k=1}^{n-1}u_k\right)(u_n)_{xxx}\right] - \gamma(u_n)\left(\sum_{k=1}^{n}u_k\right)_{xxx}, \quad n \geq 2,
\end{aligned} \tag{28}$$

where we replaced $\alpha = \frac{\gamma^2}{5}$ and $\beta = \gamma$ as given by the characterization of the Sawada-Kotera equation given earlier. Using the transformation

$$u_i(x,t) = R_i(\ln f(x,t))_{xx}, 1 \le i \le n, \tag{29}$$

into the members of the couplings (28) and solving we find that

$$R_i = (-1)^{i+1}\frac{30}{\gamma}, 1 \le i \le n. \tag{30}$$

We next substitute the transformation

$$u_i(x,t) = (-1)^{i+1}\frac{30}{\gamma}(\ln f(x,t))_{xx}, \tag{31}$$

into the linear terms of each member of the couplings (28), where the auxiliary function, as assumed by the simplified Hirota's method reads

$$f = 1 + e^{\theta}, \tag{32}$$

and the wave variable is given by

$$\theta = kx - ct, \tag{33}$$

and solving the outcome we get the dispersion relation by

$$c = k^5, \tag{34}$$

and in view of this result we obtain

$$\theta_i = k_i x - k_i^5 t. \tag{35}$$

Consequently, for the one-soliton solution, we set

$$f(x,t) = 1 + e^{k_1 x - k_1^5 t}. \tag{36}$$

The one soliton solution is therefore given by

$$u_i(x,t) = (-1)^{i+1}\frac{30k_1^2 e^{k_1 x - k_1^5 t}}{\gamma(1 + e^{k_1 x - k_1^5 t})^2}. \tag{37}$$

This clearly shows that equations of the couplings give soliton solution for i is odd, and anti-soliton solution for i is even, but with the same amplitude.

To determine the two-soliton solution, we set the auxiliary function by

$$f(x,t) = 1 + e^{k_1 x - k_1^5 t} + e^{k_2 x - k_2^5 t} + a_{12}e^{(k_1+k_2)x - (k_1^5 + k_2^5)t}, \tag{38}$$

into (28) and proceed as before to obtain the phase factor a_{12} by

$$a_{12} = \frac{(k_1 - k_2)^2(k_1^2 - k_1 k_2 + k_2^2)}{(k_1 + k_2)^2(k_1^2 + k_1 k_2 + k_2^2)}, \tag{39}$$

and hence can be generalized to

$$a_{ij} = \frac{(k_i - k_j)^2(k_i^2 - k_i k_j + k_j^2)}{(k_i + k_j)^2(k_i^2 - k_i k_j + k_j^2)}, 1 \leq i < j \leq 3. \tag{40}$$

This in turn gives

$$f(x,t) = 1 + e^{k_1 x - k_1^5 t} + e^{k_2 x - k_2^5 t} + \frac{(k_1 - k_2)^2(k_1^2 - k_1 k_2 + k_2^2)}{(k_1 + k_2)^2(k_1^2 + k_1 k_2 + k_2^2)} e^{(k_1+k_2)x - (k_1^5 + k_2^5)t}. \tag{41}$$

The two-soliton solutions, and the two anti-soliton solutions can be obtained by using (29) for the function f in (41).

To determine the three soliton solutions we proceed as before and set

$$\begin{aligned} f(x,t) = 1 &+ e^{k_1 x - k_1^5 t} + e^{k_2 x - k_2^5 t} + e^{k_3 x - k_3^5 t} \\ &+ a_{12} e^{(k_1+k_2)x - (k_1^5 + k_2^5)t} + a_{13} e^{(k_1+k_3)x - (k_1^5 + k_3^5)t} \\ &+ a_{23} e^{(k_2+k_3)x - (k_2^5 + k_3^5)t} + b_{123} e^{(k_1+k_2+k_3)x - (k_1^5 + k_2^5 + k_3^5)t}, \end{aligned} \tag{42}$$

and proceeding as before to find that

$$b_{123} = a_{12} a_{13} a_{23}. \tag{43}$$

This shows that the couplings (28) gives three soliton solutions, and hence multiple soliton solutions. To determine the three-solitons solution explicitly, we proceed as before.

4. COUPLINGS OF THE LAX EQUATION

In this section we will examine the couplings of the Lax equation given by

$$\begin{aligned}
(u_1)_t &= -(u_1)_{xxxxx} - \frac{3}{10}\gamma^2 u_1^2 (u_1)_x - 2\gamma (u_1)_x (u_1)_{xx} - \gamma (u_1)(u_1)_{xxx},\\
(u_2)_t &= -(u_2)_{xxxxx} - \frac{3}{10}\gamma^2 u_1^2 (u_2)_x - \frac{3}{10}\gamma^2 (2u_1u_2 + u_2^2)(u_1+u_2)_x\\
&\quad - 2\gamma (u_1)_x (u_2)_{xx} - 2\gamma (u_2)_x (u_1+u_2)_{xx} - \gamma (u_1)(u_2)_{xxx}\\
&\quad - \gamma u_2 (u_1+u_2)_{xxx},\\
(u_3)_t &= -(u_3)_{xxxxx} - \frac{3}{10}\gamma^2 (u_1+u_2)^2 (u_3)_x\\
&\quad - \frac{3}{10}\gamma^2 \left[2(u_1+u_2)u_3 + u_3^2\right]\left[(u_1+u_2)_x + (u_3)_x\right]\\
&\quad - 2\gamma (u_1+u_2)_x (u_3)_{xx} - 2\gamma (u_3)_x (u_1+u_2+u_3)_{xx}\\
&\quad - \gamma (u_1+u_2)(u_3)_{xxx} - \gamma u_3 (u_1+u_2+u_3)_{xxx},\\
(u_4)_t &= -(u_4)_{xxxxx} - \frac{3}{10}\gamma^2 (u_1+u_2+u_3)^2 (u_4)_x\\
&\quad - \frac{3}{10}\gamma^2 \left[2(u_1+u_2+u_3)u_4 + u_4^2\right]\left[(u_1+u_2+u_3)_x + (u_4)_x\right]\\
&\quad - 2\gamma (u_1+u_2+u_3)_x (u_4)_{xx} - 2\gamma (u_4)_x (u_1+u_2+u_3+u_4)_{xx}\\
&\quad - \gamma (u_1+u_2+u_3)(u_4)_{xxx} - \gamma u_4 (u_1+u_2+u_3+u_4)_{xxx},\\
&\vdots,\\
(u_n)_t &= -(u_n)_{xxxxx} - \frac{3}{10}\gamma^2 \left(\sum_{k=1}^{n-1} u_k\right)^2 (u_n)_x\\
&\quad - \frac{3}{10}\gamma^2 \left[2\left(\sum_{k=1}^{n-1} u_k\right)(u_n) + (u_n)^2\right]\left(\sum_{k=1}^{n} u_k\right)_x\\
&\quad - 2\gamma \left[\left(\sum_{k=1}^{n-1} u_k\right)_x (u_n)_{xx}\right] - 2\gamma (u_n)_x \left(\sum_{k=1}^{n} u_k\right)_{xx}\\
&\quad - \gamma \left[\left(\sum_{k=1}^{n-1} u_k\right)(u_n)_{xxx}\right] - \gamma (u_n)\left(\sum_{k=1}^{n} u_k\right)_{xxx}, \quad n \geq 2.
\end{aligned} \tag{44}$$

where we replaced $\alpha = \frac{3}{10}\gamma^2$ and $\beta = 2\gamma$ as given by the characterization of the Lax equation given earlier. Following the simplified Hirota's method we use the transformation

$$u_i(x,t) = R_i(\ln f(x,t))_{xx}, 1 \leq i \leq n, \tag{45}$$

into the members of the couplings (44) and solving we find that

$$R_i = (-1)^{i+1}\frac{20}{\gamma}, 1 \le i \le n. \tag{46}$$

We next substitute the transformation

$$u_i(x,t) = (-1)^{i+1}\frac{20}{\gamma}(\ln f(x,t))_{xx}, \tag{47}$$

into the linear terms of each member of the couplings (44), where the auxiliary function, as assumed by the simplified Hirota's method reads

$$f = 1 + e^{\theta}, \tag{48}$$

where the wave variable is given by

$$\theta = kx - ct, \tag{49}$$

and by solving the outcome we get the dispersion relation by

$$c = k^5, \tag{50}$$

and in view of this result we obtain

$$\theta_i = k_i x - k_i^5 t. \tag{51}$$

Consequently, for the one-soliton solution, we set

$$f(x,t) = 1 + e^{k_1 x - k_1^5 t}. \tag{52}$$

The one soliton solution and the one anti-soliton solution are therefore given by

$$u_i(x,t) = (-1)^{i+1}\frac{20k_1^2 e^{k_1 x - k_1^5 t}}{\gamma(1 + e^{k_1 x - k_1^5 t})^2}. \tag{53}$$

This clearly shows that equations of the couplings give soliton solution for i is odd, and anti-soliton solution for i is even, but with the same amplitude.

To determine the two-soliton solution, we set the auxiliary function by

$$f(x,t) = 1 + e^{k_1 x - k_1^5 t} + e^{k_2 x - k_2^5 t} + a_{12}e^{(k_1+k_2)x - (k_1^5 + k_2^5)t}, \tag{54}$$

into (46) and proceed as before to obtain the phase factor a_{12} by

$$a_{12} = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{55}$$

and hence

$$a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, 1 \le i < j \le 3. \tag{56}$$

This in turn gives

$$f(x,t) = 1 + e^{k_1 x - k_1^5 t} + e^{k_2 x - k_2^5 t} + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}e^{(k_1+k_2)x - (k_1^5 + k_2^5)t}. \tag{57}$$

The two-soliton solutions, and the two anti-soliton solutions can be obtained by using (45) for the function f in (57).

To determine the three soliton solutions we proceed as before and set

$$\begin{aligned} f(x,t) \;=\; & 1+e^{k_1x-k_1^5t}+e^{k_2x-k_2^5t}+e^{k_3x-k_3^5t} \\ & +a_{12}e^{(k_1+k_2)x-(k_1^5+k_2^5)t}+a_{13}e^{(k_1+k_3)x-(k_1^5+k_3^5)t} \\ & +a_{23}e^{(k_2+k_3)x-(k_2^5+k_3^5)t}+b_{123}e^{(k_1+k_2+k_3)x-(k_1^5+k_2^5+k_3^5)t}, \end{aligned} \tag{58}$$

and proceeding as before to find that

$$b_{123} = a_{12}a_{13}a_{23}. \tag{59}$$

This shows that the couplings (44) gives three soliton solutions, and hence multiple soliton solutions. To determine the three-solitons solution explicitly, we proceed as in the previous section.

5. CONCLUDING REMARKS

In this work we constructed couplings of the fifth-order Sawada-Kotera equation and the fifth-order Lax equation. We used the algebra of coupled scalars for constructing the two classes of couplings. We derived multiple soliton and multiple anti-soliton solutions for the couplings of the fifth-order Sawada-Kotera equation and Lax equation. We showed that the derived couplings of the Sawada-Kotera equation possess the same properties as the fifth-order Sawada-Kotera equation: the same phase variable, the same phase shift, and the same amplitude. However, the only difference is that some equations give soliton solutions for i is odd, whereas others give anti-soliton solutions for i even. The same conclusion holds for the couplings of the Lax equation. The algebra of coupled scalars is reliable and can be used for constructing other couplings of other integrable equations.

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