

# MODIFIED ADOMIAN DECOMPOSITION METHOD FOR DOUBLE SINGULAR BOUNDARY VALUE PROBLEMS

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This paper deals with the double singular boundary value problems of second order. The differential equation has two singular points and the singular points are just the boundary points. Nonlinear models are solved by this method to get more reliable and efficient numerical results. It can also be used to solve singular boundary value problems where the traditional methods fail.

*Key words:* Adomian decomposition method; singular boundary value problem; analytic solution.

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## 1. INTRODUCTION

In the present article, we consider the following double singular boundary value problem (BVP) of second-order

$$u''(t) + \frac{f(t)}{t(t-1)}u'(t) + \frac{g(t)}{t(t-1)}N(u(t)) = \frac{h(t)}{t(t-1)}, 0 < t < 1, u(0) = a, u(1) = b,$$

where  $f(t)$ ,  $g(t)$ , and  $h(t)$  are known continuous functions of  $t$  in the interval  $(0, 1)$ . Here  $N(u)$  is a nonlinear function of  $u$ . Let the above equation be singular at these two boundary value points  $t = 0, 1$ .

Scientists and engineers are interested in singular BVPs because they arise in a wide range of applications, such as in chemical engineering, mechanical engineering, nuclear industry, and nonlinear dynamical systems and solitons [1]-[8]. Therefore, this kind of problem has been studied by many researchers. For example, the existence of the solution of this type of equation has been widely studied in [9]-[10]. Numerical methods such as Adomian decomposition method (ADM) [11], fast Fourier-Galerkin methods [12], finite difference method [13], Chebyshev economization [14], differential transform method [15], and iterative shooting method [16] were investigated during the past years. Generally, it is not easy to produce a good approximation for classical numerical methods. For example, you can not use the ADM directly to study the double singular BVP because the two boundary values are just singular.

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Since Adomian in early 1980's developed the ADM concept, ADM as well as its various modified versions have been extensively used to solve many nonlinear problems, including ordinary differential equations (ODE) [17]-[20], partial differential equations [21]-[22], differential-algebraic equations [23]-[24], differential-difference equations [25]-[26], and integro-differential equations [27]-[31]. Numerical solutions can also be obtained [32]-[39]. Furthermore, Lai *et al.* [40] applied ADM to solve some eigenvalue problems. It is well known now in literature that this algorithm gets the rapidly convergent solution.

The purpose of this paper is to introduce a new reliable modification of ADM. Our new method gives better approximation of the solution than the traditional one. For those problems where the standard ADM fails, our new scheme may still converge.

## 2. MODIFIED TECHNIQUE

We first give a brief outline of the standard ADM. For this reason, the differential equation

$$Lu(t) + Ru(t) + N(u(t)) = g(t) \quad (1)$$

is considered, where  $L$  is the highest order derivative and easily or trivially invertible linear operator,  $R$  is the remaining part of the linear operator, and  $N$  is the nonlinear operator. We assume this ODE has a unique solution.

Applying the inverse operator  $L^{-1}$  of  $L$  on both sides of (1), we obtain

$$u = L^{-1}g + \varphi(t) - L^{-1}Ru - L^{-1}N(u). \quad (2)$$

Here  $\varphi(x)$  satisfies  $L\varphi = 0$ , which is normally found by the initial conditions. According to the Adomian decomposition method,  $u(t)$  is defined by the series

$$u = \sum_{i=0}^{\infty} u_i \quad (3)$$

and the nonlinear terms are defined by the series

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n). \quad (4)$$

The components of  $A_i$  are called the Adomian polynomials and are given by [41]

$$A_0 = N(u_0),$$

$$A_1 = u_1 N'(u_0),$$

$$A_2 = u_2 N'(u_0) + \frac{1}{2!} u_1^2 N''(u_0), \quad (5)$$

$$A_3 = u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 N'''(u_0),$$

$$\vdots$$

The other polynomials can be produced by the general form

$$A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}.$$

We can find that  $A_0$  depends only on  $u_0$ ,  $A_1$  depends only on  $u_0$  and  $u_1$ , and so on.

From (2), we have the standard ADM

$$u_0 = L^{-1}g + \varphi(x),$$

$$u_1 = -L^{-1}Ru_0 - L^{-1}A_0, \quad (6)$$

$$u_2 = -L^{-1}Ru_1 - L^{-1}A_1,$$

$$\vdots$$

$$u_{i+1} = -L^{-1}Ru_i - L^{-1}A_i,$$

for  $i = 1, 2, \dots$ . So the solution  $u(t)$  is decomposed into an infinite series (3) and the series is supposed to be convergent.

For the singular points  $t = 0$  or  $t = 1$ , the standard ADM can not be used directly. Now, we present our new method.

Consider the ordinary differential equation of order two

$$t(t-1)u''(t) + f(t)u'(t) + g(t)N(u(t)) = h(t) \quad (7)$$

Substituting

$$\begin{aligned} [(t-1)(tu)']' &= [(t-1)(tu' + u)]' \\ &= (t^2 - t)u'' + (3t - 2)u' + u \end{aligned} \quad (8)$$

into (7), we have

$$[(t-1)(tu)']' = (3t-2)u' + u - f(t)u'(t) - g(t)N(u(t)) + h(t). \quad (9)$$

Integrating (9) from 1 to  $t$  we get,

$$(t-1)(tu)' = d + \int_1^t \{ [3s-2-f(s)]u'(s) + u(s) - g(s)N(u(s)) + h(s) \} ds. \quad (10)$$

As  $t = 1$  in (10) we obtain,

$$(1 - 1) = d + 0 \Rightarrow d = 0.$$

Dividing by  $(t - 1)$  both sides, we have

$$(tu)' = \frac{1}{t-1} \int_1^t \{[3s - 2 - f(s)]u'(s) + u(s) - g(s)N(u(s)) + h(s)\} ds. \quad (11)$$

Integrating (11) from 0 to  $t$  we have,

$$tu = e + \int_0^t \left\{ \frac{1}{h-1} \int_1^h [(3s - 2 - f(s))u'(s) + u(s) - g(s)N(u(s)) + h(s)] ds \right\} dh. \quad (12)$$

As  $t = 0$  in (12) we get,

$$0 = e + 0 \Rightarrow e = 0$$

Dividing by  $t$  both sides, we have

$$u = \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h [(3s - 2 - f(s))u'(s) + u(s) - g(s)N(u(s)) + h(s)] ds \right\} dh. \quad (13)$$

By using ADM (6) we have

$$u_0 = \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h h(s) ds \right\} dt; \quad (14)$$

$$u_{k+1} = \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h [(3s - 2 - f(s))u'_k(s) + u_k(s) - g(s)A_k] ds \right\} dh;$$

$$A_n = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right]_{\lambda=0}.$$

**Theorem 1.** Let  $u_0$  be a polynomial, i.e.,  $u_0 = a_n t^n + \dots + a_1 t + a_0$ , then  $u_k$  is a polynomial, i.e.,  $u_k = b_n t^n + \dots + b_1 t + b_0$ .

**Proof.**

We have

$$\begin{aligned} \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h s^k ds \right\} dt &= \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \frac{1}{k+1} (h^{k+1} - 1) \right\} dt \\ &= \frac{1}{t} \int_0^t \frac{1}{k+1} (h^k + h^{k-1} + \dots + h + 1) dt \\ &= \frac{1}{k+1} \left( \frac{t^k}{k+1} + \frac{t^{k-1}}{k} + \dots + \frac{t}{2} + 1 \right). \end{aligned} \quad (15)$$

For a given polynomial  $u_0 = a_n t^n + \dots + a_1 t + a_0$ ,  $u_0'$  is also a polynomial, then the combination of polynomials is also a polynomial. So  $u_1$  will be the polynomial by (15). The same process holds to obtain  $u_k$  as a polynomial series. So the present method (14) is a nonsingular one.

In fact, if  $h(t)$  in (14) is a polynomial, then  $u_0$  will be a polynomial.

### 3. NUMERICAL EXAMPLES

Next we show some numerical experiments of our new ADM with integrating factor and compare our results with the traditional method without integrating factor. All computations are coded under the computer algebra package *Mathematica* with 32 working decimal digits.

**Example 1.** We consider the double singular BVP

$$t(t-1)u''(t) + tu(t) + u^2(t) = 2t^2 - 2t + t^3 + t^4 \quad (16)$$

with the exact solution  $u(t) = t^2$ .

Substituting (8) into (16), we have

$$[(t-1)(tu)']' = (3t-2)u' + u - tu - u^2 + 2t^2 - 2t + t^3 + t^4. \quad (17)$$

Integrating (9) from 1 to  $t$ , we get

$$(t-1)(tu)' = d + \int_1^t [(3s-2)u'(s) + (1-s)u(s) - u^2(s) + 2s^2 - 2s + s^3 + s^4] ds. \quad (18)$$

As  $t = 1$  in (18),

$$(1-1) = d + 0 \Rightarrow d = 0.$$

Dividing by  $(t-1)$  both sides, we get

$$(tu)' = \frac{1}{t-1} \int_1^t [(3s-2)u'(s) + (1-s)u(s) - u^2(s) + 2s^2 - 2s + s^3 + s^4] ds. \quad (19)$$

Integrating (19) from 0 to  $t$ , we obtain

$$tu = e + \int_0^t \left\{ \frac{1}{h-1} \int_1^h [(3s-2)u'(s) + (1-s)u(s) - u^2(s) + 2s^2 - 2s + s^3 + s^4] ds \right\} dh. \quad (20)$$

As  $t = 0$  in (20) we get,

$$0 = e + 0 \Rightarrow e = 0$$

Dividing by  $t$  both sides, we get

$$u = \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h [(3s-2)u'(s) + (1-s)u(s) - u^2(s) + 2s^2 - 2s + s^3 + s^4] ds \right\} dh. \quad (21)$$

By using ADM (6) we have

$$\begin{aligned} u_0 &= \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h [2s^2 - 2s + s^3 + s^4] ds \right\} dt; \\ u_{k+1} &= \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h [(3s-2)u'_k(s) + (1-s)u_k(s) + A_k] ds \right\} dh; \end{aligned} \quad (22)$$

$$A_0 = u_0^2,$$

$$A_1 = 2u_0u_1,$$

$$A_2 = u_1^2 + 2u_0u_2$$

$$\vdots$$

$$u_0 = 0.116667 + 0.0583333t + 0.372222t^2 + 0.1125t^3 + 0.04t^4;$$

$$u_1 = -0.00123222 + 0.00618945t + 0.235099t^2 + \dots;$$

$$u_2 = -0.0562723 - 0.0214744t + 0.138301t^2 - \dots;$$

$$\vdots$$

Table 1

The error between exact solution and numerical solution using the present method in Example 1.

$t$	$m = 15$ TERMS	$m = 16$ TERMS	$m = 17$ TERMS
0	$2.8 \cdot 10^{-4}$	$2.0 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$
0.2	$3.0 \cdot 10^{-4}$	$2.2 \cdot 10^{-4}$	$1.6 \cdot 10^{-4}$
0.4	$2.2 \cdot 10^{-4}$	$1.6 \cdot 10^{-4}$	$1.2 \cdot 10^{-4}$
0.6	$1.2 \cdot 10^{-4}$	$9.0 \cdot 10^{-5}$	$6.5 \cdot 10^{-5}$
0.8	$3.0 \cdot 10^{-5}$	$2.7 \cdot 10^{-5}$	$2.1 \cdot 10^{-5}$
1.0	$2.2 \cdot 10^{-5}$	$9.7 \cdot 10^{-5}$	$5.2 \cdot 10^{-5}$

**Example 2.** We consider the double singular BVP

$$t(t-1)u''(t) + \frac{12}{5}tu(t) + \frac{1}{10}u^2(t) = \frac{1}{10}(t^2 + \frac{7}{5}t)e^t - \frac{1}{1000}e^{2t} \quad (23)$$

with the exact solution  $u(t) = \frac{1}{10}e^t$ .

By using the same procedure (17-21), we have

$$\begin{aligned} u &= \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h [(3s-2)u'(s) + (1 - \frac{12}{5}s)u(s) - \frac{1}{10}u^2(s) \right. \\ &\quad \left. + \frac{1}{10}(s^2 + \frac{7}{5}s)e^s - \frac{1}{1000}e^{2s}] ds \right\} dh. \end{aligned} \quad (24)$$

Note that  $e^s$  and  $e^{2s}$  can not be calculated by using this method. By using Taylor series expansion

$$e^s = \sum_{j=0}^{\infty} \frac{1}{j!} s^j, e^{2s} = \sum_{j=0}^{\infty} \frac{1}{j!} (2s)^j \tag{25}$$

we see that all terms are expressed in polynomial series.

By ADM (6), we have

$$u_0 = \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h \left[ \frac{1}{10} (s^2 + \frac{7}{5}s) \left( \sum_{j=0}^m \frac{1}{j!} s^j \right) - \frac{1}{1000} \sum_{j=0}^{\infty} \frac{1}{j!} (2s)^j \right] ds \right\} dt; \tag{26}$$

$$u_{k+1} = \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h [(3s-2)u'_k(s) + (1-\frac{12}{5}s)u_k(s) - \frac{1}{10}A_k] ds \right\} dh;$$

$$u_0 = 0.214961 + 0.10698t + 0.0476535t^2 + 0.0155735t^3 + \dots;$$

$$u_1 = -0.148497 - 0.0724382t - 0.00109257t^2 + \dots;$$

$$u_2 = 0.0989162 + 0.0480764t + 0.0191612t^2 + \dots;$$

$$\vdots$$

Table 2

The error between exact solution and numerical solution using the present method in Example 2.

$t$	$m = 5$ TERMS	$m = 7$ TERMS	$m = 9$ TERMS
0	$1.9 \cdot 10^{-2}$	$7.7 \cdot 10^{-3}$	$1.1 \cdot 10^{-3}$
0.2	$6.7 \cdot 10^{-2}$	$2.1 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$
0.4	$1.4 \cdot 10^{-2}$	$6.6 \cdot 10^{-3}$	$2.4 \cdot 10^{-3}$
0.6	$2.0 \cdot 10^{-2}$	$1.2 \cdot 10^{-2}$	$7.2 \cdot 10^{-3}$
0.8	$2.5 \cdot 10^{-2}$	$1.6 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$
1.0	$3.0 \cdot 10^{-2}$	$2.0 \cdot 10^{-2}$	$1.4 \cdot 10^{-2}$

**Example 3.** We consider the double singular BVP

$$t(t-1)u''(t) + \frac{7}{5}u(t) + \frac{1}{10}u^2(t) = \sin \frac{t}{10} \left( \frac{7}{5} - \frac{1}{100}t^2 + \frac{1}{100}t \right) + \frac{1}{200} \left( 1 - \cos \frac{t}{5} \right) \tag{27}$$

with the exact solution  $u(t) = \sin \frac{t}{10}$ .

By using the same procedure (17-21), we have

$$u = \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h [(3s-2)u'(s) - \frac{7}{5}u(s) - \frac{1}{10}u^2(s) + \sin \frac{s}{10} (\frac{7}{5} - \frac{1}{100}s^2 + \frac{1}{100}s) + \frac{1}{200}(1 - \cos \frac{s}{5})] ds \right\} dh. \quad (28)$$

Note that  $\sin \frac{s}{10}$  and  $\cos \frac{s}{5}$  can not be calculated by using this method. By using Taylor series expansion

$$\sin \frac{s}{10} = \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{s}{10})^{2j+1}}{(2j+1)!}, \quad \cos \frac{s}{5} = \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{s}{5})^{2j}}{(2j)!}, \quad (29)$$

all terms are expressed in polynomial series.

By ADM (6) we have

$$u_0 = \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h \left[ \sum_{j=0}^m \frac{(-1)^j (\frac{s}{10})^{2j+1}}{(2j+1)!} \right] (\frac{7}{5} - \frac{1}{100}s^2 + \frac{1}{100}s) + \frac{1}{200} (1 - \sum_{j=0}^m \frac{(-1)^j (\frac{s}{5})^{2j}}{(2j)!}) \right\} ds \right\} dt; \quad (30)$$

$$u_{k+1} = \frac{1}{t} \int_0^t \left\{ \frac{1}{h-1} \int_1^h [(3s-2)u'_k(s) - \frac{7}{5}u_k(s) - \frac{1}{10}A_k] ds \right\} dh;$$

$$\begin{aligned} u_0 &= 0.0703576 + 0.0351788t + 0.00011921t^2 + \dots; \\ u_1 &= -0.0535799 + 0.0227079t + 0.0000564247t^2 + \dots; \\ u_2 &= 0.00625209 + 0.014741t + 0.0000152791t^2 + \dots; \\ &\vdots \end{aligned}$$

Table 3

The error between exact solution and numerical solution using the present method in Example 3.

$t$	$m = 9$ TERMS	$m = 10$ TERMS	$m = 11$ TERMS
0	$9.0 \cdot 10^{-4}$	$6.1 \cdot 10^{-4}$	$3.9 \cdot 10^{-4}$
0.2	$6.2 \cdot 10^{-4}$	$4.3 \cdot 10^{-4}$	$2.7 \cdot 10^{-4}$
0.4	$3.5 \cdot 10^{-4}$	$2.5 \cdot 10^{-4}$	$1.6 \cdot 10^{-4}$
0.6	$8.4 \cdot 10^{-5}$	$7.8 \cdot 10^{-5}$	$4.1 \cdot 10^{-5}$
0.8	$1.8 \cdot 10^{-4}$	$9.4 \cdot 10^{-5}$	$7.1 \cdot 10^{-5}$
1.0	$4.4 \cdot 10^{-4}$	$2.6 \cdot 10^{-4}$	$1.8 \cdot 10^{-4}$



#### 4. CONCLUSION

The modification of the Adomian decomposition method proposed in this paper has demonstrated that the nonlinear double singular BVP can be handled without difficulty. The numerical computation gives a more precise approximation of the solution. The reported results show a greater improvement over the traditional method. How to establish the error analysis of this method? Maybe the Adomian decomposition method with integrating factor gives some useful results, see Ref. [42].

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