SYMBOLIC COMPUTATION OF SOME NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS

A. BISWAS[1,2,a], A.H. BHRAWY[2,3,b], M.A. ABDELKAWY[3], A.A. ALSHAERY[4], E.M. HILAL[4]

[1]Department of Mathematical Sciences, Delaware State University,
Dover, DE 19901-2277, USA,
[2]Department of Mathematics, Faculty of Science,
King Abdulaziz University, Jeddah 21589, Saudi Arabia,
E-mail[a]: biswas.anjan@gmail.com
[3]Department of Mathematics, Faculty of Science,
Beni-Suef University, Beni-Suef 62511, Egypt,
E-mail[b]: alibhrawy@yahoo.co.uk
[4]Department of Mathematics, Faculty of Science for Girls,
King Abdulaziz University, Jeddah, Saudi Arabia.

Received December 27, 2013

This paper studies a few fractional nonlinear equations from mathematical physics. The fractional derivatives are in the sense of modified Riemann-Liouville fractional derivative. The $\frac{G'}{G}$-expansion method is applied to retrieve solutions to these equations. Several forms of solutions are obtained that are listed in this paper.

Key words: $\frac{G'}{G}$-expansion method, modified Riemann-Liouville fractional derivative, double sine-Poisson equation, double sinh-Poisson equation, Liouville equation.

PACS: 02.30.Gp, 02.30.Jr, 02.70.-c.

1. INTRODUCTION

Over the past few years, many fields of science and engineering can be expressed by fractional differential equations [1]. Due to their widely applications in many research areas, more attention has been paid to the study of fractional differential equations. Electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, and optics and signal processing models have been constructed by using fractional differential equations [2–5].

In this article we solve three special cases of the nonlinear fractional equation given in the form:

$$\frac{\partial^{2\alpha}u}{\partial X^{2\alpha}}+\frac{\partial^{2\beta}u}{\partial Y^{2\beta}}=F(u),\quad 0<\alpha,\quad \beta<1. \tag{1}$$

where the modified Riemann-Liouville derivative $\frac{\partial^{\alpha}u}{\partial X^{\alpha}}$ is defined as [6–9]

$$\frac{\partial^{\alpha}u}{\partial X^{\alpha}}=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dX}\int_0^X (X-\varpi)^{-\alpha}(f(\varpi)-f(X))d\varpi, \tag{2}$$

using the following transformations

$$x = \frac{X^{\alpha}}{\Gamma(1+\alpha)}, \qquad y = \frac{Y^{\beta}}{\Gamma(1+\beta)}. \tag{3}$$

Eq. (1) has been transferred to the nonlinear equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(u). \tag{4}$$

For several choices of $F(u)$, we can find the exact solution of equation (4) using many methods. In this article we use the $\frac{G'}{G}$-expansion method for a reliable treatment of the nonlinear fractional partial differential equations (4). We find exact solutions of three nonlinear fractional partial differential equations. The exact solutions of these equations are found using the $\frac{G'}{G}$ expansion method [10–14].

The study of nonlinear evolution equations (NLEEs) is an absolute necessity in theoretical physics since NLEEs form the essential element in this area of research. Although there is a plethora of results reported in this area of research, during the past few decades, it must be noted that NLEEs with fractional evolution are at their infancy. Therefore it is imperative to address these fractional evolution equations from the perspective of integrability. These fractional NLEEs lead to results that carry unprecedented novelty. In the past decades, many methods were developed for finding exact solutions of NLEEs as the exp-function method [15], inverse scattering method [16], Hirota's bilinear method [17], new similarity transformation method [18], homogeneous balance method [19], the sine-cosine method [20, 21], Jacobi and Weierstrass elliptic function method [22–25], F-expansion method [26, 27] and other very interesting methods [28]-[43].

The study of fractional evolution equations is a growing area of research during the past decade. One profound advantage of considering fractional-order evolution, as opposed to integer-order evolution, is that fractional-order evolution relates to memory and hereditary of several materials and processes. Additionally, fractional-order evolution equation is much closer to reality than integer-order evolution one. This paper is going to address three NLEEs that carry fractional evolutions.

The integrability aspect is going to be the focus of this paper with fractional evolution. There are several tools that are available during modern times to address this issue. However, this paper will employ the well-known G'/G-expansion approach to carry out the integration of these equations. This algorithm will be described in the next section and will be implemented in the subsequent sections in order to integrate the equations and list the obtained results.

2. $\frac{G'}{G}$ EXPANSION METHOD

This section is devoted to the study of implementing the $\frac{G'}{G}$ expansion method for a given partial differential equation

$$G(u, u_x, u_y, u_t, u_{xy}, ...) = 0, \tag{5}$$

where $u(x,y,t)$ is an unknown function of the independent variables x, y, and t. In order to obtain the solution of Eq. (5), we combine the independent variables x, y, and t into one particular variable through the new variable

$$\zeta = x + y + \nu t, \quad u(x,y,t) = U(\zeta), \tag{6}$$

where ν is the wave speed. Using this variable enables us to reduce Eq. (5) to the following ordinary differential equation (ODE)

$$G(U, U', U'', U''', ...) = 0. \tag{7}$$

The ODE is integrated as long as all terms contain derivatives in ζ, upon setting the constant of integration to zero. We search for the exact solutions that satisfy this ODE. $U(\zeta)$ can be expressed as a finite series of $\frac{G'}{G}$,

$$u(x,y,t) = U(\zeta) = \sum_{i=0}^{N} a_i \frac{G'}{G}^i, \quad G'' + \mu G' + \lambda G = 0. \tag{8}$$

The parameter N can be determined by balancing the linear term(s) of highest order with the nonlinear one(s), where N is a positive integer, so that an analytical solution in closed form may be investigated. We express the ODE (7) in terms of Eq. (8) and we compare the coefficients of each power of $\frac{G'}{G}$ in both sides. We next solve the over-determined system of nonlinear algebraic equations by using the Mathematica computer code.

3. DOUBLE SINE-POISSON EQUATION

Considering the choice $F(u) = \sin u + \sin 2u$, Eq. (1) turns into the fractional nonlinear double sine-Poisson equation

$$\frac{\partial^{2\alpha} u}{\partial X^{2\alpha}} + \frac{\partial^{2\beta} u}{\partial Y^{2\beta}} = \sin u + \sin 2u, \quad 0 < \alpha, \quad \beta < 1. \tag{9}$$

By taking the transformation (3), Eq. (9) tends to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sin u + \sin 2u. \tag{10}$$

In order to apply the $\frac{G'}{G}$ expansion method to Eq. (10), we first introduce the transformations

$$\sin u = \frac{e^{iu} - e^{-iu}}{2i}, \quad v = e^{iu},$$

and as a consequence, we get

$$2(v_x)^2 + 2(v_y)^2 - 2vv_{xx} - 2vv_{yy} - v^4 - v^3 + v + 1 = 0. \tag{11}$$

Using the wave variable $u(x,y) = U(\zeta)$, $\zeta = x + \nu y$, Eq. (11) is reduced to an ODE

$$2(1+\nu^2)VV'' - 2(1+\nu^2)(V')^2 + V^4 + V^3 - V - 1 = 0. \tag{12}$$

Balancing the terms $V''V$ and V^4 in Eq. (12) we obtain $N = 1$,

$$V(\zeta) = \sum_{i=0}^{1} a_i (\frac{G'}{G})^i. \tag{13}$$

Substituting Eq. (13) into Eq. (12) and comparing the coefficients of each power of $\frac{G'}{G}$ in both sides, we get an over-determined system of nonlinear algebraic equations with respect to ν and a_i; $i = 0$, 1. Solving the over-determined system of nonlinear algebraic equations using Mathematica code, we obtain two groups of constants:

(a)

$$\mu = \sqrt{\frac{8\lambda(1+\nu^2)+3}{2(1+\nu^2)}}, \quad a_0 = -\frac{1 \pm \sqrt{8\lambda a(1+\nu^2)+3}}{2}, \quad a_1 = \pm i\sqrt{\frac{1+\nu^2}{2}}, \tag{14}$$

(b)

$$\mu = \pm\sqrt{\lambda}, \quad a_0 = 0, \quad \nu = \pm i\sqrt{\frac{0.5+\lambda}{\lambda}}, \quad a_1 = \mp\frac{1}{\sqrt{\lambda}}. \tag{15}$$

The solutions of Eq. (12) reads
For case (a)

$$V_1 = -\frac{1 \pm \sqrt{8\lambda a(1+\nu^2)+3}}{2} \pm i\sqrt{\frac{3(1+\nu^2)}{4}} \frac{c_1 \sinh(\sqrt{\frac{3}{2}})\zeta + c_2 \cosh(\sqrt{\frac{3}{2}})\zeta}{c_1 \cosh(\sqrt{\frac{3}{2}})\zeta + c_2 \sinh(\sqrt{\frac{3}{2}})\zeta}, \tag{16}$$

For case (b)

(i) $\lambda < 0$

$$V_2 = \mp i\sqrt{\frac{3}{2}}(\frac{c_1 \sinh(\sqrt{\frac{-3\lambda}{2}}\zeta) + c_2 \cosh(\sqrt{\frac{-3\lambda}{2}}\zeta)}{c_1 \cosh(\sqrt{\frac{-3\lambda}{2}}\zeta) + c_2 \sinh(\sqrt{\frac{-3\lambda}{2}}\zeta)}), \tag{17}$$

(ii) $\lambda > 0$

$$V_3 = \mp i\sqrt{\frac{3}{2}}(\frac{-c_1\sin(\sqrt{\frac{-3\lambda}{2}}\zeta)+c_2\cos(\sqrt{\frac{-3\lambda}{2}}\zeta)}{c_1\cos(\sqrt{\frac{-3\lambda}{2}}\zeta)+c_2\sin(\sqrt{\frac{-3\lambda}{2}}\zeta)}), \tag{18}$$

(iii) $\lambda = 0$

$$V_4 = \mp\frac{1}{\sqrt{\lambda}}(\frac{c_1}{c_1\zeta+c_2}). \tag{19}$$

We find the following solutions of Eq. (9)

$$\begin{aligned} u_1 = i\ln[&-\frac{1\pm\sqrt{8\lambda a(1+\nu^2)+3}}{2} \\ &\pm i\sqrt{\frac{3(1+\nu^2)}{4}}(\frac{c_1\sinh(\sqrt{\frac{3}{2}})(x+\nu y)+c_2\cosh(\sqrt{\frac{3}{2}})(x+\nu y)}{c_1\cosh(\sqrt{\frac{3}{2}})(x+\nu y)+c_2\sinh(\sqrt{\frac{3}{2}})(x+\nu y)})], \end{aligned} \tag{20}$$

$$u_2 = i\ln[\mp i\sqrt{\frac{3}{2}}(\frac{c_1\sinh(\sqrt{\frac{-3\lambda}{2}}(x\pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))+c_2\cosh(\sqrt{\frac{-3\lambda}{2}}(x\pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))}{c_1\cosh(\sqrt{\frac{-3\lambda}{2}}(x\pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))+c_2\sinh(\sqrt{\frac{-3\lambda}{2}}(x\pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))})], \tag{21}$$

$$u_3 = i\ln[\mp i\sqrt{\frac{3}{2}}(\frac{-c_1\sin(\sqrt{\frac{-3\lambda}{2}}(x\pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))+c_2\cos(\sqrt{\frac{-3\lambda}{2}}(x\pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))}{c_1\cos(\sqrt{\frac{-3\lambda}{2}}(x\pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))+c_2\sin(\sqrt{\frac{-3\lambda}{2}}(x\pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))})], \tag{22}$$

$$u_4 = i\ln[\mp\frac{1}{\sqrt{\lambda}}(\frac{c_1}{c_1(x\pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y)+c_2})]. \tag{23}$$

where $x = \frac{X^\alpha}{\Gamma(1+\alpha)}, \quad y = \frac{Y^\beta}{\Gamma(1+\beta)}$.

4. DOUBLE sinh-POISSON EQUATION

Let us assume that $F(u)$ has the form $\sinh u + \sinh 2u$. The corresponding form of Eq. (1) gives the nonlinear fractional double sinh-Poisson equation

$$\frac{\partial^{2\alpha}u}{\partial X^{2\alpha}} + \frac{\partial^{2\beta}u}{\partial Y^{2\beta}} = \sinh u + \sinh 2u, \quad 0 < \alpha, \quad \beta < 1. \tag{24}$$

By taking the transformation (3), Eq. (24) carries to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sinh u + \sinh 2u. \tag{25}$$

In order to apply the $\frac{G'}{G}$ expansion method to Eq. (25), we first introduce the transformations

$$\sinh u = \frac{e^u - e^{-u}}{2}, \quad v = e^u,$$

yielding

$$2(v_x)^2 + 2(v_y)^2 - 2vv_{xx} - 2vv_{yy} + v^4 + v^3 - v - 1 = 0. \tag{26}$$

Again using the wave transformation $u(x,y) = U(\zeta)$, $\zeta = x + \nu y$, Eq. (26) is reduced to an ODE

$$2(1+\nu^2)VV'' - 2(1+\nu^2)(V')^2 - V^4 - V^3 + V + 1 = 0. \tag{27}$$

Balancing the terms $V''V$ and V^4 in Eq. (27) we obtain $N = 1$.

$$V(\zeta) = \sum_{i=0}^{1} a_i (\frac{G'}{G})^i. \tag{28}$$

Proceeding as in the previous cases we obtain

(a)

$$\mu = \pm\sqrt{\frac{8\lambda(1+\nu^2)-3}{2(1+\nu^2)}}, \quad a_0 = -\frac{1 \pm \sqrt{8\lambda a(1+\nu^2)-3}}{2}, \quad a_1 = \pm\sqrt{\frac{1+\nu^2}{2}}, \tag{29}$$

(b)

$$\mu = \pm\sqrt{\lambda}, \quad a_0 = 0, \quad \nu = \pm\sqrt{\frac{0.5-\lambda}{\lambda}} \quad \text{and} \quad a_1 = \mp\frac{1}{\sqrt{\lambda}}. \tag{30}$$

We find the following solutions of Eq. (27):

For case (a)

$$V_1 = -\frac{1 \pm \sqrt{8\lambda a(1+\nu^2)-3}}{2} \pm \sqrt{\frac{3(1+\nu^2)}{4}} (\frac{-c_1 \sin(\sqrt{\frac{3}{2}})\zeta + c_2 \cos(\sqrt{\frac{3}{2}})\zeta}{c_1 \cos(\sqrt{\frac{3}{2}})\zeta + c_2 \sin(\sqrt{\frac{3}{2}})\zeta}), \tag{31}$$

For case (b)

(i) $\lambda < 0$

$$V_2 = \mp i\sqrt{\frac{3}{2}} (\frac{c_1 \sinh(\sqrt{\frac{-3\lambda}{2}}\zeta) + c_2 \cosh(\sqrt{\frac{-3\lambda}{2}}\zeta)}{c_1 \cosh(\sqrt{\frac{-3\lambda}{2}}\zeta) + c_2 \sinh(\sqrt{\frac{-3\lambda}{2}}\zeta)}), \tag{32}$$

(ii) $\lambda > 0$

$$V_3 = \mp i\sqrt{\frac{3}{2}}\left(\frac{-c_1\sin(\sqrt{\frac{-3\lambda}{2}}\zeta) + c_2\cos(\sqrt{\frac{-3\lambda}{2}}\zeta)}{c_1\cos(\sqrt{\frac{-3\lambda}{2}}\zeta) + c_2\sin(\sqrt{\frac{-3\lambda}{2}}\zeta)}\right), \tag{33}$$

(iii) $\lambda = 0$

$$V_4 = \mp\frac{1}{\sqrt{\lambda}}\left(\frac{c_1}{c_1\zeta + c_2}\right). \tag{34}$$

Then the solutions of the Eq. (24) are:

$$u_1 = \ln\Big[-\frac{1 \pm \sqrt{8\lambda a(1+\nu^2)-3}}{2} \pm \sqrt{\frac{3(1+\nu^2)}{4}}\left(\frac{c_1\sinh(\sqrt{\frac{3}{2}})(x+\nu y) + c_2\cosh(\sqrt{\frac{3}{2}})(x+\nu y)}{c_1\cosh(\sqrt{\frac{3}{2}})(x+\nu y) + c_2\sinh(\sqrt{\frac{3}{2}})(x+\nu y)}\right)\Big], \tag{35}$$

$$u_2 = \ln\Big[\mp i\sqrt{\frac{3}{2}}\left(\frac{c_1\sinh(\sqrt{\frac{-3\lambda}{2}}(x \pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y)) + c_2\cosh(\sqrt{\frac{-3\lambda}{2}}(x \pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))}{c_1\cosh(\sqrt{\frac{-3\lambda}{2}}(x \pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y)) + c_2\sinh(\sqrt{\frac{-3\lambda}{2}}(x \pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))}\right)\Big], \tag{36}$$

$$u_3 = \ln\Big[\mp i\sqrt{\frac{3}{2}}\left(\frac{-c_1\sin(\sqrt{\frac{-3\lambda}{2}}(x \pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y)) + c_2\cos(\sqrt{\frac{-3\lambda}{2}}(x \pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))}{c_1\cos(\sqrt{\frac{-3\lambda}{2}}(x \pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y)) + c_2\sin(\sqrt{\frac{-3\lambda}{2}}(x \pm i\sqrt{\frac{0.5+\lambda}{\lambda}}y))}\right)\Big], \tag{37}$$

$$u_4 = \ln\Big[\mp\frac{1}{\sqrt{\lambda}}\left(\frac{c_1}{c_1(x \pm \sqrt{\frac{0.5-\lambda}{\lambda}}y) + c_2}\right)\Big]. \tag{38}$$

5. LIOUVILLE EQUATION

We take the choice $F(u) = e^{2u}$. This turns Eq. (1) into the nonlinear fractional Liouville equation

$$\frac{\partial^{2\alpha}u}{\partial X^{2\alpha}} + \frac{\partial^{2\beta}u}{\partial Y^{2\beta}} = e^{2u}, \quad 0 < \alpha, \quad \beta < 1. \tag{39}$$

If we use the transformation (3), Eq. (39) carries to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{2u}. \tag{40}$$

If we use $v = e^u$, it carries Eq. (40) into

$$(v_x)^2 + (v_y)^2 - vv_{xx} - vv_{yy} + v^4 = 0. \tag{41}$$

Using the wave transformation $u(x,y) = U(\zeta)$, $\zeta = x + \nu y$, Eq. (41) is reduced to an ODE

$$(1+\nu^2)VV'' - (1+\nu^2)(V')^2 - V^4 = 0. \tag{42}$$

Balancing the terms $V''V$, and V^4 in Eq. (42) we obtain $N = 1$.

$$V(\zeta) = \sum_{i=0}^{1} a_i (\frac{G'}{G})^i. \tag{43}$$

Proceeding as in the previous cases we obtain

$$\mu = \pm\sqrt{2\lambda}, \quad a_0 = \pm\sqrt{2\lambda(1+\nu^2)} \quad \text{and} \quad a_1 = \pm\sqrt{2(1+\nu^2)}, \tag{44}$$

We find the following solutions of Eq. (42):

(i) $\lambda < 0$

$$V_1 = \pm\sqrt{2\lambda(1+\nu^2)}(1 + i\frac{c_1\sinh(\sqrt{-\lambda}\zeta) + c_2\cosh(\sqrt{-\lambda}\zeta)}{c_1\cosh(\sqrt{-\lambda}\zeta) + c_2\sinh(\sqrt{-\lambda}\zeta)}), \tag{45}$$

(ii) $\lambda > 0$

$$V_2 = \pm\sqrt{2\lambda(1+\nu^2)}(1 + \frac{-c_1\sin(\sqrt{\lambda}\zeta) + c_2\cos(\sqrt{\lambda}\zeta)}{c_1\cos(\sqrt{\lambda}\zeta) + c_2\sin(\sqrt{\lambda}\zeta)}), \tag{46}$$

(iii) $\lambda = 0$

$$V_3 = \pm\sqrt{2\lambda(1+\nu^2)}(1 + \frac{\frac{c_1}{\sqrt{\lambda}}}{c_1\zeta + c_2}). \tag{47}$$

Then the solutions of the Eq. (39) are:

$$u_1 = \ln[\pm\sqrt{2\lambda(1+\nu^2)}(1 + i\frac{c_1\sinh(\sqrt{-\lambda}(x+\nu y)) + c_2\cosh(\sqrt{-\lambda}(x+\nu y))}{c_1\cosh(\sqrt{-\lambda}(x+\nu y)) + c_2\sinh(\sqrt{-\lambda}(x+\nu y))})], \tag{48}$$

$$u_2 = \ln[\pm\sqrt{2\lambda(1+\nu^2)}(1 + \frac{-c_1\sin(\sqrt{\lambda}(x+\nu y)) + c_2\cos(\sqrt{\lambda}(x+\nu y))}{c_1\cos(\sqrt{\lambda}(x+\nu y)) + c_2\sin(\sqrt{\lambda}(x+\nu y))})], \tag{49}$$

$$u_3 = \ln[\pm\sqrt{2\lambda(1+\nu^2)}(1 + \frac{\frac{c_1}{\sqrt{\lambda}}}{c_1(x+\nu y) + c_2})]. \tag{50}$$

6. CONCLUSION

This paper addressed the integrability aspects of a few NLEEs with fractional evolution. There are several results that are reported. These results serve as a basis for further investigation in this direction. In future, these NLEEs will be studied with several perturbation terms that arise from realistic physical situations. Moreover, stochastic NLEEs with fractional evolution are also in the horizon. Those results will be reported elsewhere. Finally, numerical results in this area are imperative, which is not quite the focus of this paper.

Acknowledgements. This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University under grant number (14-130/1433 HiCi). The authors therefore acknowledge technical and financial support of KAU.

REFERENCES

1. R.L. Magin, *Fractional Calculus in Bioengineering* (Begell House Publishers, 2006).
2. R. Garrappa and M. Popolizio, Math. Comput. Simulation **81**, 1045 (2011).
3. K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (John Wiley, New York, 1993).
4. I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999).
5. R. Hilfer, *Applications of Fractional Calculus in Physics* (World Scientific, New Jersey 2000).
6. G. Jumarie, Appl. Math. Lett. **23**, 1444 (2010).
7. G. Jumarie, J. Appl. Math. Comput. **24**, 31 (2007).
8. G. Jumarie, Comput. Math. Appl. **51**, 1367 (2006).
9. G.B. Whitham, Proc. Roy. Soc. Lond. A **299**, 6 (1967).
10. A.H. Bhrawy, M.M. Tharwat, and M.A. Abdelkawy, Indian J. Phys. **87**, 665 (2013).
11. A.H. Bhrawy and M.A. Abdelkawy, Cent. Eur. J. Phys. **11**, 518 (2013).
12. M.A. Abdelkawy and A.H. Bhrawy, Indian J. Phys. **87**, 555 (2013).
13. G. Ebadi, N.Y. Fard, A.H. Bhrawy, S. Kumar, H. Triki, A. Yildirim, and A. Biswas, Rom. Rep. Phys. **65**, 27 (2013).
14. G. Ebadi, A. Yildirim, and A. Biswas, Rom. Rep. Phys. **64**, 357 (2012).
15. A.H. Bhrawy, A. Biswas, M. Javidi, W.X. Ma, Z. Pinar, and A. Yildirim, Results. Math. **63**, 675 (2013).
16. S. Ghosh and S. Nandy, Nucl. Phys. B **561**, 451 (1999).
17. A.M. Wazwaz, Appl. Math. Comput. **200**, 160 (2008).
18. A.N. Beavers and E.D. Denman, Math. Biosci. **21**, 143 (1974).
19. M.L. Wang, Y.B. Zhou, and Z.B. Li, Phys. Lett. A **216**, 67 (1996).
20. M. Yaghobi Moghaddam, A. Asgari, and H. Yazdani, Appl. Math. Comput. **210**, 422 (2009).
21. S. Tang, C. Li, and K. Zhang, Commun. Nonlinear Sci. Numer. Simulat. **15**, 3358 (2010).
22. S.K. Liu *et al.*, Phys. Lett. A **289**, 69 (2001).
23. A.H. Bhrawy, M.M. Tharwat, A. Yildirim, and M.A. Abdelkawy, Indian J. Phys. **86** 1107 (2012).
24. A. H. Bhrawy, M. A. Abdelkawy, and A. Biswas, Commun. Nonlinear Sci. Numer. Simulat. **18**, 915 (2013).

25. A.H. Bhrawy, M.A. Abdelkawy, and A. Biswas, Indian J. Phys. **87**, 1125 (2013).
26. A.H. Bhrawy, M.A. Abdelkawy, S. Kumar, S. Johnson, and A. Biswas, Indian J. Phys. **87**, 455 (2013).
27. A.H. Bhrawy, M.A. Abdelkawy, S. Kumar, and A. Biswas, Rom. J. Phys. **58**, 729 (2013).
28. A. Biswas, A. Yildirim, T. Hayat, O.M. Aldossary, and R. Sassaman, Proc. Romanian Acad. A **13**, 32 (2012).
29. G. Ebadi *et al.*, Proc. Romanian Acad. A **13**, 215 (2012).
30. B. Ahmed and A. Biswas, Proc. Romanian Acad. A **14**, 111 (2013).
31. Guangye Yang *et al.*, Rom. Rep. Phys. **65**, 902 (2013).
32. Guangye Yang *et al.*, Rom. Rep. Phys. **65**, 391 (2013).
33. A. Salas, S. Kumar, A. Yildirim, and A. Biswas, Proc. Romanian Acad. A **14**, 28 (2013).
34. G. Ebadi, N. Yousefzadeh, H. Triki, A. Yildirim, and A. Biswas, Rom. Rep. Phys. **64**, 915 (2012).
35. L. Girgis, D. Milovic, S. Konar, A. Yildirim, H. Jafari, and A. Biswas, Rom. Rep. Phys. **64**, 663 (2012).
36. A.G. Johnpillai, A. Yildirim, and A. Biswas, Rom. J. Phys. **57**, 545 (2012).
37. H. Triki *et al.*, Rom. J. Phys. **57**, 1029 (2012).
38. Y.J. He and D. Mihalache, Rom. Rep. Phys. **64**, 1243 (2012).
39. H. Leblond, H. Triki, and D. Mihalache, Rom. Rep. Phys. **65**, 925 (2013).
40. G. Ebadi *et al.*, Rom. J. Phys. **58**, 3 (2013).
41. A.M. Wazwaz, Rom. J. Phys. **58**, 685 (2013).
42. H. Leblond and D. Mihalache, Phys. Reports **523**, 61 (2013).
43. Zai-Yun Zhang *et al.*, Rom. J. Phys. **58**, 766 (2013).