

SOLITONS AND PERIODIC SOLUTIONS TO THE DISSIPATION-MODIFIED KdV EQUATION WITH TIME-DEPENDENT COEFFICIENTS

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In this work, a dissipation-modified Korteweg-de Vries (KdV) equation is considered. The equation is relevant to describe, for instance, nonlinear Bénard-Marangoni oscillatory instability in a liquid layer heated from above. In particular, the model with time-dependent coefficients required for the case of inhomogeneous media is studied. Periodic solutions and soliton solutions of this equation are obtained using the sine-cosine and solitary wave ansatz methods. Parametric conditions for the existence and uniqueness of exact solutions are presented. These analytical findings may be of significant importance for the explanation of physical phenomena arising in nonlinear systems described by the dissipation-modified KdV-type equations.

Key words: Solitons; periodic solutions; dissipation; KdV equation.

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1. INTRODUCTION

Recently, envelope solitons have attracted much interest owing to their extensive applications in a broad range of physical settings. A soliton is a self-localized solution of a nonlinear evolution equation (NLEE) describing the evolution of a nonlinear dynamical system with an infinite number of degrees of freedom [1]. The distinction between solitary wave and soliton solutions is that when any number of solitons interact they do not change form, and the only outcome of the interaction is a phase shift [2]. Originally, the term "soliton" was reserved for a particular set of integrable solutions existing as a result of the delicate balance between dispersion or diffraction and nonlinearity [3]. These objects have been studied extensively, both experimentally and theoretically [4]-[28].

The problem of searching exact analytic solutions for NLEEs has been a subject of considerable interest for many years. The finding of such solutions is relevant for a better understanding of the mechanism of the complicated physical phenomena and dynamical processes modeled by wave equations. In addition, exact solutions allow one to calculate certain important physical quantities analytically as well as serving as diagnostics for simulations [29]. In particular, soliton type solutions become more interesting and more important for various branches of nonlinear sciences.

Recently, many powerful methods such as the sine-cosine methods [30]-[32], the subsidiary ordinary differential equation method [33]-[35], Hirota's method [36], the Petrov-Galerkin method [37], the collocation method [38], the solitary wave ansatz method [39, 40], Exp-function method [41], and many others, have been successfully applied to exactly solve NLEEs and their generalizations.

The well known KdV equation [42]

$$u_t + \alpha uu_x + \gamma u_{xxx} = 0, \quad (1)$$

where α and γ are the nonlinear and dispersion coefficients, respectively, is the generic model for the study of weakly nonlinear long waves [43]. It arises in physical systems which involve a balance between nonlinearity and dispersion at leading-order [44]. For example, it describes surface waves of long wavelength and small amplitude on shallow water and internal waves in a shallow density-stratified fluid [44].

The dissipation-modified KdV evolution equation for solitary excitations in a liquid layer [45],

$$u_t + \beta u_x + \alpha_1 uu_x + \alpha_2 u_{xxx} + \alpha_3 u_{xx} + \alpha_4 u_{xxxx} + \alpha_5 (uu_x)_x = 0, \quad (2)$$

is a combination of the original KdV equation [46] and the appropriate Kuramoto-Sivashinsky equation for Bénard-Marangoni convection [47]-[49]. Such equations have been associated with long wave oscillatory instability in problems where an input-output energy balance exists besides the usual KdV balance between nonlinearity and dispersion [50]. In [51], the group classification for the dissipation-modified KdV equation (2) by means of the Lie method of the infinitesimals has been given.

Considering variable coefficients in a given NLEE, we can accurately describe the nonlinear wave dynamics in inhomogeneous systems. Due to the inhomogeneities of media and nonuniformities of boundaries, the variable-coefficient NLEEs can be used to describe the real physical backgrounds [52]. It is remarked that the existence of the inhomogeneities in the media influences the accompanied physical effects giving rise to spatial or temporal dispersion and nonlinearity variations.

In this work, we establish families of soliton solutions and periodic solutions to the dissipation-modified KdV equation having time-dependent coefficients as follows

$$u_t + \beta(t)u_x + \alpha_1(t)uu_x + \alpha_2(t)u_{xxx} + \alpha_3(t)u_{xx} + \alpha_4(t)u_{xxxx} + \alpha_5(t)(uu_x)_x = 0, \quad (3)$$

where $\alpha_i(t)$ (with $i = 1, \dots, 5$) and $\beta(t)$ are the corresponding time-dependent coefficients. As a matter of fact, this model is much more general since nonlinear wave equations with variable coefficients can be considered as generalizations of the constant coefficient ones. To our knowledge, the study of the dissipation-modified KdV equation with time-dependent coefficients has not been widespread.

The purpose of the present work is to establish fundamental analytical results regarding Eq. (2). Interestingly, many new families of exact travelling wave solutions of the considered model are successfully obtained using the sine-cosine and the solitary wave ansatz methods.

2. SOLITONS AND PERIODIC SOLUTIONS

2.1. THE SINE-COSINE METHOD

In what follows we will use the modified sine-cosine method to develop travelling wave solutions to this equation. The modified sine-cosine method admits the use of solutions [32, 53]

$$u(x, t) = \lambda(t) \cos^m(\mu\xi), \quad \xi = x - c(t)t \quad (4)$$

and

$$u(x, t) = \lambda(t) \sin^m(\mu\xi), \quad \xi = x - c(t)t \quad (5)$$

for some parameters $\lambda(t)$, μ , and m that are to be determined. Here, μ is the wave number and $c(t)$ is the wave speed. The assumption (4) gives

$$u_t = \frac{d\lambda(t)}{dt} \cos^m(\mu\xi) - \lambda(t)m\mu \left(-c(t) - \frac{dc(t)}{dt}t \right) \cos^{m-1}(\mu\xi) \sin(\mu\xi), \quad (6)$$

$$u_x = -\lambda(t)m\mu \cos^{m-1}(\mu\xi) \sin(\mu\xi), \quad (7)$$

$$u_{xx} = -\lambda(t)\mu^2 m^2 \cos^m(\mu\xi) + \lambda(t)\mu^2 m(m-1) \cos^{m-2}(\mu\xi), \quad (8)$$

$$uu_x = -\lambda^2(t)m\mu \cos^{2m-1}(\mu\xi) \sin(\mu\xi), \quad (9)$$

$$(uu_x)_x = \lambda^2(t)m(2m-1)\mu^2 \cos^{2m-2}(\mu\xi) - 2\lambda^2(t)m^2\mu^2 \cos^{2m}(\mu\xi), \quad (10)$$

$$\begin{aligned} u_{xxx} &= -\lambda(t)\mu^3 m(m-1)(m-2) \cos^{m-3}(\mu\xi) \sin(\mu\xi) \\ &\quad + \lambda(t)\mu^3 m^3 \cos^{m-1}(\mu\xi) \sin(\mu\xi), \end{aligned} \quad (11)$$

$$\begin{aligned} u_{xxxx} &= \lambda(t)\mu^4 m^4 \cos^m(\mu\xi) - 2\lambda(t)\mu^4 m(m-1)(m^2 - 2m + 2) \cos^{m-2}(\mu\xi) \\ &\quad + \lambda(t)\mu^4 m(m-1)(m-2)(m-3) \cos^{m-4}(\mu\xi), \end{aligned} \quad (12)$$

and the assumption (5) gives

$$u_t = \frac{d\lambda(t)}{dt} \sin^m(\mu\xi) + \lambda(t)m\mu \left(-c(t) - \frac{dc(t)}{dt}t \right) \sin^{m-1}(\mu\xi) \cos(\mu\xi), \quad (13)$$

$$u_x = \lambda(t)m\mu \sin^{m-1}(\mu\xi) \cos(\mu\xi), \quad (14)$$

$$u_{xx} = -\lambda(t)\mu^2 m^2 \sin^m(\mu\xi) + \lambda(t)\mu^2 m(m-1) \sin^{m-2}(\mu\xi), \quad (15)$$

$$uu_x = \lambda^2(t)m\mu \sin^{2m-1}(\mu\xi) \cos(\mu\xi), \quad (16)$$

$$(uu_x)_x = \lambda^2(t)m(2m-1)\mu^2 \sin^{2m-2}(\mu\xi) - 2\lambda^2(t)m^2\mu^2 \sin^{2m}(\mu\xi), \quad (17)$$

$$u_{xxx} = \lambda(t)\mu^3 m(m-1)(m-2) \sin^{m-3}(\mu\xi) \cos(\mu\xi) \\ - \lambda(t)\mu^3 m^3 \sin^{m-1}(\mu\xi) \cos(\mu\xi), \quad (18)$$

$$u_{xxxx} = \lambda(t)\mu^4 m^4 \sin^m(\mu\xi) - 2\lambda(t)\mu^4 m(m-1)(m^2 - 2m + 2) \\ \times \sin^{m-2}(\mu\xi) + \lambda(t)\mu^4 m(m-1)(m-2)(m-3) \sin^{m-4}(\mu\xi), \quad (19)$$

Substituting (6)-(12) into the wave equation (3) gives

$$\begin{aligned} & \frac{d\lambda(t)}{dt} \cos^m(\mu\xi) - \lambda(t)m\mu \left(-c(t) - \frac{dc(t)}{dt}t \right) \cos^{m-1}(\mu\xi) \sin(\mu\xi) \\ & - \beta(t)\lambda(t)m\mu \cos^{m-1}(\mu\xi) \sin(\mu\xi) - \alpha_1(t)\lambda^2(t)m\mu \cos^{2m-1}(\mu\xi) \\ & \cdot \sin(\mu\xi) - \alpha_2(t)\lambda(t)\mu^3 m(m-1)(m-2) \cos^{m-3}(\mu\xi) \sin(\mu\xi) \\ & + \alpha_2(t)\lambda(t)\mu^3 m^3 \cos^{m-1}(\mu\xi) \sin(\mu\xi) - \alpha_3(t)\lambda(t)\mu^2 m^2 \\ & \cdot \cos^m(\mu\xi) + \alpha_3(t)\lambda(t)\mu^2 m(m-1) \cos^{m-2}(\mu\xi) \\ & + \alpha_4(t)\lambda(t)\mu^4 m^4 \cos^m(\mu\xi) - 2\alpha_4(t)\lambda(t)\mu^4 m(m-1)(m^2 - 2m + 2) \\ & \cdot \cos^{m-2}(\mu\xi) + \alpha_4(t)\lambda(t)\mu^4 m(m-1)(m-2)(m-3) \cos^{m-4}(\mu\xi) \\ & + \alpha_5(t)\lambda^2(t)m(2m-1)\mu^2 \cos^{2m-2}(\mu\xi) - 2\alpha_5(t)\lambda^2(t)m^2\mu^2 \cos^{2m}(\mu\xi) = 0 \end{aligned} \quad (20)$$

Equating the exponents and the coefficients of like powers of cosine function in (20) leads to

$$m(m-1)(m-2)(m-3) \neq 0, \quad (21)$$

$$m-4 = 2m-2, \quad (22)$$

$$\frac{d\lambda(t)}{dt} - \alpha_3(t)\lambda(t)\mu^2 m^2 + \alpha_4(t)\lambda(t)\mu^4 m^4 = 0, \quad (23)$$

$$-\lambda(t)m\mu \left(-c(t) - \frac{dc(t)}{dt}t \right) - \beta(t)\lambda(t)m\mu + \alpha_2(t)\lambda(t)\mu^3 m^3 = 0, \quad (24)$$

$$-\alpha_1(t)\lambda^2(t)m\mu - \alpha_2(t)\lambda(t)\mu^3 m(m-1)(m-2) = 0, \quad (25)$$

$$m(m-1)\lambda(t)\mu^2 [\alpha_3(t) - 2\alpha_4(t)\mu^2(m^2 - 2m + 2)] - 2\alpha_5(t)\lambda^2(t)m^2\mu^2 = 0, \quad (26)$$

$$\alpha_4(t)\lambda(t)\mu^4 m(m-1)(m-2)(m-3) + \alpha_5(t)\lambda^2(t)m(2m-1)\mu^2 = 0, \quad (27)$$

Solving this system yields

$$m \neq 0, 1, 2, 3, \quad (28)$$

$$m = -2, \quad (29)$$

$$\lambda(t) = \lambda_0 \quad (30)$$

$$c(t) = \frac{1}{t} \int \left[\beta(t) - \frac{\alpha_2(t)\alpha_3(t)}{\alpha_4(t)} \right] dt, \quad (31)$$

$$\lambda(t) = -\frac{3\alpha_2(t)\alpha_3(t)}{\alpha_1(t)\alpha_4(t)}, \quad (32)$$

$$\mu = \frac{1}{2} \sqrt{\frac{\alpha_3(t)}{\alpha_4(t)}}, \frac{\alpha_3(t)}{\alpha_4(t)} > 0, \quad (33)$$

where λ_0 is an integral constant related to the initial pulse amplitude. Now equating the two values of $\lambda(t)$ from (30) and (32) yields the relation:

$$\frac{\alpha_2(t)\alpha_3(t)}{\alpha_1(t)\alpha_4(t)} = K_1, \quad (34)$$

which serves as a constraint relation between the model coefficients. Note that K_1 is a constant value given by: $K_1 = -\lambda_0/3$. It should be pointed out that here μ as shown in (4) and (5) is a constant parameter which describes the wave number of the wave. Therefore, the ratio $\alpha_3(t)/\alpha_4(t)$ in (33) should be a positive constant.

Similar results are also obtained by using the sine method (5). This leads to the periodic solutions

$$u_1(x, t) = -\frac{3\alpha_2(t)\alpha_3(t)}{\alpha_1(t)\alpha_4(t)} \sec^2 \left[\frac{1}{2} \sqrt{\frac{\alpha_3(t)}{\alpha_4(t)}} \xi \right], \quad (35)$$

and

$$u_2(x, t) = -\frac{3\alpha_2(t)\alpha_3(t)}{\alpha_1(t)\alpha_4(t)} \csc^2 \left[\frac{1}{2} \sqrt{\frac{\alpha_3(t)}{\alpha_4(t)}} \xi \right], \quad (36)$$

where

$$\xi = x - \int \left\{ \beta(t) - \frac{\alpha_2(t)\alpha_3(t)}{\alpha_4(t)} \right\} dt. \quad (37)$$

However, for $\alpha_3(t)/\alpha_4(t) < 0$ we obtain the soliton solutions

$$u_3(x, t) = -\frac{3\alpha_2(t)\alpha_3(t)}{\alpha_1(t)\alpha_4(t)} \operatorname{sech}^2 \left[\frac{1}{2} \sqrt{-\frac{\alpha_3(t)}{\alpha_4(t)}} \xi \right], \quad (38)$$

and

$$u_4(x, t) = -\frac{3\alpha_2(t)\alpha_3(t)}{\alpha_1(t)\alpha_4(t)} \operatorname{csch}^2 \left[\frac{1}{2} \sqrt{-\frac{\alpha_3(t)}{\alpha_4(t)}} \xi \right], \quad (39)$$

where ξ is the same as in (37).

2.2. THE SOLITARY WAVE ANSATZ METHOD

Here we are interested in finding bright and dark soliton solutions of Eq. (3) by means of the solitary wave ansatz method.

2.2.1. Bright solitons

To get bright soliton solutions to (3), one starts with a solitary wave ansatz of the form [39, 40]

$$u(x, t) = \frac{A(t)}{\cosh^p \tau}, \quad (40)$$

where

$$\tau = \mu(t)(x - v(t)t). \quad (41)$$

Here $A(t)$, $\mu(t)$, and $v(t)$ are unknown time-dependent parameters to be determined as functions of the varying model coefficients $\alpha_i(t)$ (with $i = 1, \dots, 5$) and $\beta(t)$. Here $A(t)$, $\mu(t)$ and $v(t)$ are, respectively, the amplitude, the inverse width, and the velocity of the soliton. From (40) and (41), one obtains

$$u_t = \frac{dA}{dt} \frac{1}{\cosh^p \tau} - pA \frac{\tanh \tau}{\cosh^p \tau} \left\{ \frac{d\mu}{dt} (x - vt) - \mu \left(v + t \frac{dv}{dt} \right) \right\}, \quad (42)$$

$$u_x = -pA\mu \frac{\tanh \tau}{\cosh^p \tau}, \quad (43)$$

$$uu_x = -pA^2\mu \frac{\tanh \tau}{\cosh^{2p} \tau}, \quad (44)$$

$$(uu_x)_x = \frac{2p^2 A^2 \mu^2}{\cosh^{2p} \tau} - \frac{p(2p+1)A^2 \mu^2}{\cosh^{2p+2} \tau}, \quad (45)$$

$$u_{xx} = \frac{p^2 A \mu^2}{\cosh^p \tau} - \frac{p(p+1)A\mu^2}{\cosh^{p+2} \tau}, \quad (46)$$

$$u_{xxx} = -p^3 A \mu^3 \frac{\tanh \tau}{\cosh^p \tau} + p(p+1)(p+2)A\mu^3 \frac{\tanh \tau}{\cosh^{p+2} \tau}, \quad (47)$$

$$u_{xxxx} = \frac{Ap^4 \mu^4}{\cosh^p \tau} - \frac{A\mu^4 p(p+1) \{p^2 + (p+2)^2\}}{\cosh^{p+2} \tau} + \frac{A\mu^4 p(p+1)(p+2)(p+3)}{\cosh^{p+4} \tau}, \quad (48)$$

Substituting (42)-(48) into (3), gives

$$\begin{aligned} & \frac{dA}{dt} \frac{1}{\cosh^p \tau} - pA \frac{\tanh \tau}{\cosh^p \tau} \left\{ \frac{d\mu}{dt} (x - vt) - \mu \left(v + t \frac{dv}{dt} \right) \right\} - \beta p A \mu \frac{\tanh \tau}{\cosh^p \tau} \\ & - \alpha_1 p A^2 \mu \frac{\tanh \tau}{\cosh^{2p} \tau} - \alpha_2 p^3 A \mu^3 \frac{\tanh \tau}{\cosh^p \tau} + \alpha_2 p (p+1)(p+2) A \mu^3 \frac{\tanh \tau}{\cosh^{p+2} \tau} \\ & + \frac{\alpha_3 p^2 A \mu^2}{\cosh^p \tau} - \frac{\alpha_3 p (p+1) A \mu^2}{\cosh^{p+2} \tau} + \frac{\alpha_4 A p^4 \mu^4}{\cosh^p \tau} - \frac{\alpha_4 A \mu^4 p (p+1) \{p^2 + (p+2)^2\}}{\cosh^{p+2} \tau} \\ & + \frac{\alpha_4 A \mu^4 p (p+1)(p+2)(p+3)}{\cosh^{p+4} \tau} + \frac{2\alpha_5 p^2 A^2 \mu^2}{\cosh^{2p} \tau} - \frac{\alpha_5 p (2p+1) A^2 \mu^2}{\cosh^{2p+2} \tau} = 0, \end{aligned} \quad (49)$$

Now, from (49), equating the exponents $2p+2$ and $p+4$ leads to

$$2p+2 = p+4 \quad (50)$$

which gives

$$p = 2 \quad (51)$$

From (49) setting the coefficients of $1/\cosh^{p+i} \tau$ and $\tanh \tau / \cosh^{p+j} \tau$ to zero, where $i = 0, 2, 4$ and $j = 0, 2$, since these are linearly independent functions, gives

$$\frac{dA}{dt} + \alpha_3 p^2 A \mu^2 + \alpha_4 A p^4 \mu^4 = 0, \quad (52)$$

$$-pA \left\{ \frac{d\mu}{dt} (x - vt) - \mu \left(v + t \frac{dv}{dt} \right) \right\} - \beta p A \mu - \alpha_2 p^3 A \mu^3 = 0, \quad (53)$$

$$-\alpha_1 p A^2 \mu + \alpha_2 p (p+1)(p+2) A \mu^3 = 0, \quad (54)$$

$$-\alpha_3 p (p+1) A \mu^2 - \alpha_4 A \mu^4 p (p+1) \{p^2 + (p+2)^2\} + 2\alpha_5 p^2 A^2 \mu^2 \quad (55)$$

$$\alpha_4 A \mu^4 p (p+1)(p+2)(p+3) - \alpha_5 p (2p+1) A^2 \mu^2 = 0, \quad (56)$$

Solving the above system yields

$$A(t) = A_0, \quad (57)$$

$$\mu(t) = \mu_0, \quad (58)$$

$$v(t) = \frac{1}{t} \int \left[\beta(t) - \frac{\alpha_2(t)\alpha_3(t)}{\alpha_4(t)} \right] dt, \quad (59)$$

$$A(t) = -\frac{3\alpha_2(t)\alpha_3(t)}{\alpha_1(t)\alpha_4(t)}, \quad (60)$$

$$\mu(t) = \frac{1}{2} \sqrt{-\frac{\alpha_3(t)}{\alpha_4(t)}}, \quad (61)$$

where A_0 and μ_0 are integral constants related to the initial pulse amplitude and inverse width, respectively. Equating the two values of the amplitude $A(t)$ from (57)

and (60) gives

$$\frac{\alpha_2(t)\alpha_3(t)}{\alpha_1(t)\alpha_4(t)} = K_2, \quad (62)$$

which serves as a constraint relation between the model coefficients. Here K_2 is a constant given by: $K_2 = -A_0/3$. Furthermore, from (58) and (61), it is apparent that the inverse width of the solitons remains constant satisfying the condition: $\mu_0^2 = -\alpha_3(t)/4\alpha_4(t)$. Further, it follows from (61) that the solitons will exist for

$$\frac{\alpha_3(t)}{\alpha_4(t)} < 0. \quad (63)$$

Thus, finally, the 1-soliton solution to the dissipation-modified KdV equation with t -dependent coefficients (3) is given by

$$u(x, t) = -\frac{3\alpha_2(t)\alpha_3(t)}{\alpha_1(t)\alpha_4(t)} \frac{1}{\cosh^2 \tau}, \quad (64)$$

where

$$\tau = \frac{1}{2} \sqrt{-\frac{\alpha_3(t)}{\alpha_4(t)}} \left\{ x - \int \left[\beta(t) - \frac{\alpha_2(t)\alpha_3(t)}{\alpha_4(t)} \right] dt \right\} \quad (65)$$

It is interesting to see that the soliton solution (64) obtained by the solitary wave method is consistent with the solution (38) obtained by using the sine-cosine method.

2.2.2. Dark solitons

Now the search is going to be for shock wave solution or topological 1-soliton solution to the dissipation-modified KdV equation with varying coefficients (3). To this end, we adopt a solitary wave ansatz given by [40]

$$u(x, t) = A(t) \tanh^p \tau \quad (66)$$

where

$$\tau = \mu(t)(x - v(t)t) \quad (67)$$

Here in (66) and (67), $A(t)$ and $\mu(t)$ are free parameters while $v(t)$ is the velocity of the wave. Also, the exponent p is unknown and will be determined later. From (66) we get

$$u_t = \frac{dA}{dt} \tanh^p \tau + pA \left\{ \frac{d\mu}{dt}(x - vt) - \mu \left(v + t \frac{dv}{dt} \right) \right\} \times (\tanh^{p-1} \tau - \tanh^{p+1} \tau), \quad (68)$$

$$u_x = pA\mu(\tanh^{p-1}\tau - \tanh^{p+1}\tau), \quad (69)$$

$$uu_x = pA^2\mu(\tanh^{2p-1}\tau - \tanh^{2p+1}\tau), \quad (70)$$

$$(uu_x)_x = pA^2\mu^2 [(2p-1)\tanh^{2p-2}\tau + (2p+1)\tanh^{2p+2}\tau - 4p\tanh^{2p}\tau], \quad (71)$$

$$u_{xx} = pA\mu^2 [(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau], \quad (72)$$

$$u_{xxx} = pA\mu^3 [(p-1)(p-2)\tanh^{p-3}\tau - \{2p^2 + (p-1)(p-2)\}\tanh^{p-1}\tau \\ + \{2p^2 + (p+1)(p+2)\}\tanh^{p+1}\tau - (p+1)(p+2)\tanh^{p+3}\tau], \quad (73)$$

$$u_{xxxx} = pA\mu^4 [(p-1)(p-2)(p-3)\tanh^{p-4}\tau + (p+1)(p+2)(p+3) \\ \cdot \tanh^{p+4}\tau - 2\{p^2 + (p-2)^2\}(p-1)\tanh^{p-2}\tau - 2\{p^2 + (p+2)^2\} \\ \cdot (p+1)\tanh^{p+2}\tau + \{4p^3 + (p-1)^2(p-2) + (p+1)^2(p+2)\}\tanh^p\tau]. \quad (74)$$

Now substituting these relations into (3) gives

$$\begin{aligned} & \frac{dA}{dt}\tanh^p\tau + pA\left\{\frac{d\mu}{dt}(x-vt) - \mu\left(v + t\frac{dv}{dt}\right)\right\}(\tanh^{p-1}\tau - \tanh^{p+1}\tau) \\ & + \beta(t)pA\mu(\tanh^{p-1}\tau - \tanh^{p+1}\tau) + \alpha_1(t)pA^2\mu(\tanh^{2p-1}\tau - \tanh^{2p+1}\tau) \\ & + \alpha_2(t)pA\mu^3[(p-1)(p-2)\tanh^{p-3}\tau - \{2p^2 + (p-1)(p-2)\}\tanh^{p-1}\tau \\ & \quad + \{2p^2 + (p+1)(p+2)\}\tanh^{p+1}\tau - (p+1)(p+2)\tanh^{p+3}\tau] \\ & + \alpha_3(t)pA\mu^2[(p-1)\tanh^{p-2}\tau - 2p\tanh^p\tau + (p+1)\tanh^{p+2}\tau] \\ & + \alpha_4(t)pA\mu^4[(p-1)(p-2)(p-3)\tanh^{p-4}\tau + (p+1)(p+2)(p+3)\tanh^{p+4}\tau \\ & \quad - 2\{p^2 + (p-2)^2\}\{p-1\}\tanh^{p-2}\tau - 2\{p^2 + (p+2)^2\}\{p+1\}\tanh^{p+2}\tau \\ & \quad + \{4p^3 + (p-1)^2(p-2) + (p+1)^2(p+2)\}\tanh^p\tau] \\ & + \alpha_5(t)pA^2\mu^2[(2p-1)\tanh^{2p-2}\tau + (2p+1)\tanh^{2p+2}\tau - 4p\tanh^{2p}\tau] = 0, \end{aligned} \quad (75)$$

From (75), equating the exponents $2p+2$ and $p+4$ gives

$$2p+2 = p+4, \quad (76)$$

which gives

$$p = 2. \quad (77)$$

It should be noted that this same value of p is obtained on setting the coefficient of the stand-alone linearly independent functions $\tanh^{p-3}\tau$ and $\tanh^{p-4}\tau$ to zero. Now, setting the coefficients of the remaining linearly independent functions in (75) to zero gives

$$\begin{aligned} & \frac{dA}{dt} + \alpha_4(t)pA\mu^4\{4p^3 + (p-1)^2(p-2) + (p+1)^2(p+2)\} \\ & + \alpha_5(t)pA^2\mu^2(2p-1) - 2p\alpha_3(t)pA\mu^2 = 0, \end{aligned} \quad (78)$$

$$pA \left\{ \frac{d\mu}{dt}(x-vt) - \mu \left(v + t \frac{dv}{dt} \right) \right\} + \beta(t)pA\mu \quad (79)$$

$$- \alpha_2(t)pA\mu^3 \{2p^2 + (p-1)(p-2)\} = 0,$$

$$-pA \left\{ \frac{d\mu}{dt}(x-vt) - \mu \left(v + t \frac{dv}{dt} \right) \right\} - \beta(t)pA\mu \quad (80)$$

$$+ \alpha_2(t)pA\mu^3 \{2p^2 + (p+1)(p+2)\} + \alpha_1(t)pA^2\mu = 0,$$

$$- \alpha_1(t)pA^2\mu - \alpha_2(t)pA\mu^3(p+1)(p+2) = 0, \quad (81)$$

$$\alpha_3(t)pA\mu^2(p-1) - 2\alpha_4(t)pA\mu^4 \{p^2 + (p-2)^2\} \{p-1\} = 0, \quad (82)$$

$$\alpha_4(t)pA\mu^4(p+1)(p+2)(p+3) + \alpha_5(t)pA^2\mu^2(2p+1) = 0, \quad (83)$$

$$\alpha_3(t)pA\mu^2(p+1) - 2\alpha_4(t)pA\mu^4 \{p^2 + (p+2)^2\} (p+1) \quad (84)$$

$$- 4\alpha_5(t)pA^2\mu^2p = 0,$$

If we put $p = 2$ in Eqs. (78)-(84), we can determine the soliton parameters as

$$A(t) = A_0, \quad (85)$$

$$\mu(t) = \mu_0, \quad (86)$$

$$v(t) = \frac{1}{t} \int \left\{ \beta(t) - \frac{\alpha_2(t)\alpha_3(t)}{\alpha_4(t)} \right\} dt, \quad (87)$$

$$A(t) = -\frac{3\alpha_2(t)\alpha_3(t)}{2\alpha_1(t)\alpha_4(t)}, \quad (88)$$

$$\mu(t) = \frac{1}{2} \sqrt{\frac{\alpha_3(t)}{2\alpha_4(t)}}. \quad (89)$$

Equating the two values of $A(t)$ from (85) and (88) gives the condition:

$$\frac{\alpha_2(t)\alpha_3(t)}{\alpha_1(t)\alpha_4(t)} = K_3 \quad (90)$$

where K_3 is a constant: $K_3 = -2A_0/3$. We also remark from (86) that the free parameter μ should be a constant. Accordingly, the ratio $\alpha_3(t)/\alpha_4(t)$ in (89) should be a constant too and dark solitons will exist for

$$\frac{\alpha_3(t)}{\alpha_4(t)} > 0 \quad (91)$$

Having obtained the expressions for the pulse parameters A , μ and v , we construct a family of topological 1-soliton solutions to Eq. (3) given by

$$u(x, t) = -\frac{3\alpha_2(t)\alpha_3(t)}{2\alpha_1(t)\alpha_4(t)} \tanh^2 \tau \quad (92)$$

where

$$\tau = \frac{1}{2} \sqrt{\frac{\alpha_3(t)}{2\alpha_4(t)}} \left\{ x - \int \left[\beta(t) - \frac{\alpha_2(t)\alpha_3(t)}{\alpha_4(t)} \right] dt \right\}, \quad (93)$$

which exist provided that the conditions (90) and (91) are satisfied.

3. CONCLUSION

We have considered a dissipation-modified KdV evolution equation for solitary excitations in a liquid layer which is a combination of the original KdV equation and the appropriate Kuramoto-Sivashinsky equation for Bénard-Marangoni convection. In particular, the model with time-dependent coefficients which is more general than the simplest case when all coefficients are constant is studied. Solitons and periodic solutions were formally determined by means of sine-cosine method. The solitary wave ansatz method is also used to obtain bright and dark soliton solutions. Parametric conditions for the existence of exact solutions have been reported. The study highlights the power of the used methods for the determination of exact solutions to nonlinear evolution equations with variable coefficients.

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