SOLVING PARTIAL q-DIFFERENTIAL EQUATIONS WITHIN REDUCED q-DIFFERENTIAL TRANSFORMATION METHOD

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In this paper, the reduced q-differential transform method is presented for solving partial differential equations. In this method, the solution is calculated in the form of convergent power series with easily computable components. Three test problems are discussed to illustrate the effectiveness and performance of the proposed method. The results show that the proposed iteration technique is very effective and convenient.

Key words: Reduced differential transform method, Initial value problem, Partial differential equation.

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1. INTRODUCTION

The q-calculus, while dating in a sense back to Euler and Jacobi, is only recently beginning to see more usefulness in quantum mechanics, having an intimate connection with commutativity relations and Lie algebra [1–3]. At the beginning of the last century, studies on q-difference equations appeared in intensive works especially by Jackson [2, 3], Carmichael [4], Mason [5], and Adams [6]. The q-difference has found many applications in various mathematical areas, such as number theory, combinatorics, orthogonal polynomials, and other sciences: quantum theory, mechanics, and theory of relativity [7]. In the last years fractional calculus has become popular as a useful tool in mathematics, physics, electronics, mechanics, etc. [8–16].

Recently, finding efficient methods to obtain exact or approximate solution of ordinary and partial q-differential equation has been an active research undertaking [17–21, 23, 22]. Wu has applied the *variational iteration method* for solving q-diffusion equations and q-difference equations of second order[17, 20]. In [21], Qin and Zeng have extended homotopy perturbation method to obtain the exact solution

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of q-diffusion equation. The one-dimensional q-differential transformation (qDTM) has been used in [19] for solving the ordinary q-differential equations. After that, El-Shahed and Gaber applied the two-dimensional q-differential transform to solve the q-diffusion and q-wave equations [18].

In recent decades, there has been a great interest in the differential transform method in order to obtain analytical approximate solutions to ordinary and partial differential equations [24–26]. In this paper, we extend reduced differential transform method for solving some linear and nonlinear partial q-differential equations. Henceforth this method is called reduced q-differential transform method (RqDTM). One can implement RqDTM on linear and nonlinear partial q-differential equations and treat then as one dimensional q-differential transform method thus reducing many of computations. One special advantage of RqDTM over to qDTM is that the RqDTM produces all the exact Poisson series coefficients of solutions, whereas the qDTM produce all the exact Taylor series coefficients of solutions.

The paper is organized as follows. In Section 2, some basic concept of q-calculus are considered. In Section 3, definitions and theoretical aspects of the method are discussed. In Section 4, several examples will be given to show the effectiveness of the proposed method. Conclusions are given in Section 5.

2. PRELIMINARIES

In this section, we recall some basic concepts of q-calculus [2–4, 27].
Let $f(x)$ be a real continuous function. The q-derivative is defined as follows

$$\begin{aligned} &\frac{d_q}{d_q x} f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0,\ 0 < q < 1, \\ &\frac{d_q}{d_q x} f(0) = \lim_{x \to 0} \frac{d_q f(x)}{dx} \end{aligned} \tag{1}$$

The q-derivative is also known as the Jackson derivative. Also the partial q-derivative of a multivariable real continuous function $f(x_1, x_2, ..., x_n)$ to a variable x_i is defined by (see [28])

$$\begin{aligned} &D_{q,x_i} f(\vec{x}) = \frac{(\varepsilon_{q,i} f)(\vec{x}) - f(\vec{x})}{(q-1)x_i}, \quad x_i \neq 0, \ \ q \in (0,1), \\ &[D_{q,x_i} f(\vec{x})]_{x_i = 0} = \lim_{x_i \to 0} D_{q,x_i} f(\vec{x}), \end{aligned}$$

where

$$(\varepsilon_{q,i} f)(\vec{x}) = f(x_1, x_2, ..., qx_i, ..., x_n).$$

We use $D^k_{q,x}$ instead of operator $\frac{\partial^k_q}{\partial_q x^k}$ for some simplification.

Lemma 2.1 *q-Leibniz formula [7]*

$$D_q^n(f(t)g(t)) = \sum_{k=0}^{n} \binom{n}{k}_q (D_q^k f)(tq^{n-k}) D_q^{n-k} g(t)$$

Theorem 2.1 *(q-Taylor formula [29]) Suppose that there exist all q-differentials of $f(\vec{x})$ in some neighborhood of $\vec{a}$. Then*

$$f(\vec{x}) = \sum_{k=0}^{\infty} \frac{d_q^k f(\vec{x},\vec{a})}{[k]_q!} \tag{2}$$

where

$$d_q^k f(\vec{x},\vec{a}) = ((x_1 - a_1)D_{q,x_1} + (x_2 - a_2)D_{q,x_2} + ... + (x_n - a_n)D_{q,x_n})^{(k)} f(\vec{a}),$$

$$(z-c)^{(0)} = 1, \quad (z-c)^{(k)} = \prod_{i=0}^{k-1}(z - cq^i) \quad (k \in N).$$

Definition 2.2 *The q-exponential function is defined by the following series representation,*

$$e_q^x = \sum_{i=0}^{\infty} \frac{x^i}{[i]_q!}$$

and have the q-derivative $\dfrac{\partial_q}{\partial_q x} e_q^x = e_q^x$.

Note that [29]

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + ... + q^{n-1}, \quad [n]_q! = [1]_q [2]_q ... [n]_q,$$

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Then for the functions of two variables we have

$$f(x,y) = \sum_{k=0}^{\infty} \frac{(x-a)^{(k)}}{[k]_q!} \left[\frac{\partial_q^k}{\partial_q x^k} f(x,y) \right]_{x=a} \tag{3}$$

For more details in q-calculus, the reader is referred to the Refs. [2–4, 7, 27].

3. REDUCED q-DIFFERENTIAL TRANSFORM METHOD

In this section, after giving some basic definitions of RqDTM, we present some theorems which we need them for solving partial q- differential equations.

Definition 3.1 *Suppose that all q-differentials of $u(x,t)$ exist in some neighborhood of $t=a$, then let*

$$U_k(x) = \frac{1}{[k]_q!}\left[\frac{\partial_q^k}{\partial_q t^k}u(x,t)\right]_{t=a}, \tag{4}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this paper, the lowercase $u(x,t)$ represents the original function while the uppercase $U_k(x)$ stands for the transformed function.

Definition 3.2 *The q-differential inverse transform of $U_k(x)$ is defined as follows:*

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-a)^{(k)}, \tag{5}$$

Substituting equation (4) in (5) we obtain

$$u(x,t) = \sum_{k=0}^{\infty}\frac{1}{[k]_q!}\left[\frac{\partial_q^k}{\partial_q t^k}u(x,t)\right]_{t=a}(t-a)^{(k)}. \tag{6}$$

In the following theorems we set $a=0$ then $(t-a)^{(k)} = (t-0)^{(k)} = t^k$. From the linearity of the q-derivative, we can get the following theorem:

Theorem 3.3 *If $w(x,t) = \alpha u(x,t) \pm v(x,t)$ then $W_k(x) = \alpha U_k(x) \pm V_k(x)$.*

Theorem 3.4 *If $w(x,y) = x^m t^n$ then $W_k(x) = x^m\delta(k-n)$ where*

$$\delta(k) = \begin{cases} 1, & k=0 \\ 0, & k\neq 0. \end{cases}$$

Proof. By definition (3.2) we have

$$\begin{aligned} W_k(x) &= \frac{1}{[k]_q!}\left[\frac{\partial_q^k(x^m t^n)}{\partial_q t^k}u(x,t)\right]_{t=0} = \frac{x^m}{[k]_q!}\left[\frac{\partial_q^k(t^n)}{\partial_q t^k}u(x,t)\right]_{t=0} \\ &= \begin{cases} x^m.\dfrac{k_q!}{k_q!} = x^m, & k=n \\ x^m.\dfrac{n_q.(n-1)_q...(n-k+1)_q}{k_q!}t^{n-k}|_{t=0} = 0, & k<n \\ x^m.0 = 0, & k>n \end{cases} \\ &= x^m\delta(k-n) \end{aligned}$$

Theorem 3.5 *If $w(x,t) = \dfrac{\partial_q}{\partial_q x}u(x,t)$ then $W_k(x) = \dfrac{\partial_q}{\partial_q x}U_k(x)$.*

$$W_k(x) = \frac{1}{[k]_q!}\left[\frac{\partial_q^k}{\partial_q t^k}(\frac{\partial_q}{\partial_q x}u(x,t))\right]_{t=0} = \frac{1}{[k]_q!}\left[\frac{\partial_q}{\partial_q x}(\frac{\partial_q^k}{\partial_q t^k}u(x,t))\right]_{t=0}$$
$$= \frac{\partial_q}{\partial_q x}\left[\frac{1}{[k]_q!}\frac{\partial_q^k}{\partial_q t^k}u(x,t))\right]_{t=0} = \frac{\partial_q}{\partial_q x}U_k(x)$$

Theorem 3.6 *If* $w(x,t) = \frac{\partial_q^r}{\partial_q t^r}u(x,t)$ *then*

$$W_k(x) = [k+1]_q[k+2]_q...[k+r]_q U_{k+r}(x).$$

Proof.

$$W_k(x) = \frac{1}{[k]_q!}\left[\frac{\partial_q^k}{\partial_q t^k}(\frac{\partial_q^r}{\partial_q t^r}u(x,t))\right]_{t=0} = \frac{[k+r]_q!}{[k]_q!}\frac{1}{[k+r]_q!}\frac{\partial_q^{k+r}}{\partial_q t^{k+r}}u(x,t))$$
$$= [k+r]_q...[k+1]_q U_{k+r}(x).$$

To prove the next theorem, we will use the following lemma [7]:

Lemma 3.1

$$f(x,tq^s) = \sum_{i=0}^{s}(-1)^i(1-q)^i\binom{s}{i}_q q^{\frac{i(i-1)}{2}}x^i D_{t,q}^i f(x,t)$$

Theorem 3.7 *If* $w(x,t) = u(x,t)\,v(x,t)$ *then* $W_k(x) = \sum_{n=0}^{k} U_{k-n}(x)\,V_n(x)$.

Proof.

$$W_k(x) = \frac{1}{[k]_q!}D_{t,q}^k\left[u(x,t).v(x,t)\right]_{t=0}$$
$$= \frac{1}{[k]_q!}\sum_{n=0}^{k}\binom{n}{k}_q\left[D_{t,q}^{k-n}u(x,tq^n)D_{t,q}^n v(x,t)\right]_{t=0}$$
$$= \sum_{n=0}^{k}\frac{1}{[k-n]_q![n]_q!}[\sum_{i=0}^{k-n}(q-1)^i\binom{k-n}{i}_q q^{\frac{i(i-1)}{2}}t^i(D_{t,q}^{k-n+i}u(x,t))D_{t,q}^n v(x,t)]_{t=0}$$

At $t = 0$ all terms of the inner series vanish, except at $i = 0$ then we have

$$\sum_{n=0}^{k}\frac{1}{[k-n]_q![n]_q!}D_{t,q}^{k-n}u(x,t)D_{t,q}^n v(x,t)$$
$$= \sum_{n=0}^{k}\frac{1}{[k-n]_q!}D_{t,q}^{k-n}u(x,t)\frac{1}{[n]_q!}D_{t,q}^n v(x,t)$$
$$= \sum_{n=0}^{k}U_{k-n}(x)V_n(x)$$

It is worth noting that a similar theorem has been advanced in [18]; however, the result of that study is different from this study.

4. SOLUTIONS OF PARTIAL q-DIFFERENTIAL EQUATIONS

In order to assess the advantages and accuracy of RqDTM, we solve some partial q-differential equations. In particular we solve q-diffusion and q-wave equations as test problems.

Example 4.1 *Consider the q-diffusion equation [30–32, 18]*

$$\frac{\partial_q}{\partial_q t}u(x,t) = \frac{\partial_q^2}{\partial_q x^2}u(x,t), \tag{7}$$

subject to the initial condition

$$u(x,0) = e_q^x, \tag{8}$$

where e_q^x is the q-exponential function.

In view of the discussion in Section 3, using the RqDTM, the Eq. (7) can be viewed as the following recursive formula

$$[k+1]_q U_{k+1}(x) = \frac{\partial_q^2}{\partial_q x^2}U_k(x) \quad k = 0,1,2,\cdots. \tag{9}$$

Using the initial conditions (8), we have

$$U_0(x) = u(x,0) = e_q^x. \tag{10}$$

Now, substituting (10) into (9), we obtain the following $U_k(x)$ values successively

$$\begin{aligned} U_1(x) =& \frac{1}{[1]_q!}e_q^x, \\ U_2(x) =& \frac{1}{[2]_q}U_1(x) = \frac{1}{[1]_q[2]_q}e_q^x = \frac{1}{[2]_q!}e_q^x, \\ U_3(x) =& \frac{1}{[3]_q!}e_q^x, \\ &\vdots \\ U_k(x) =& \frac{1}{[k]_q!}e_q^x. \end{aligned}$$

In view of (5), the differential inverse transform of $U_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{\infty} U_k t^k = \sum_{k=0}^{\infty} \frac{1}{[k]_q!}e_q^x t^k = e_q^x \sum_{k=0}^{\infty} \frac{t^k}{[k]_q!} = e_q^x e_q^t.$$

which is the analytic solution of the problem (7).

Example 4.2 *Suppose we want to solve the following nonlinear partial q-differential equation*

$$\frac{\partial_q}{\partial_q t}u(x,t) = u^2(x,t) + \frac{\partial_q}{\partial_q x}u(x,t), \tag{11}$$

with the initial condition

$$u(x,0) = 1+2x. \tag{12}$$

By taking the RqDTM of Eq. (11) and using the above theorems we obtain the following equation

$$[k+1]_q U_{k+1}(x) = \sum_{n=0}^{k} U_{k-n}(x)U_n(x) + \frac{\partial_q}{\partial_q x}U_k(x) \tag{13}$$

From the initial condition (12) we write

$$U_0(x) = 1+2x. \tag{14}$$

Solving (13) with the initial condition (14) we successively achieve value $U_k(x)$ as follows.

$U_1(x) = 3+4x+4x^2$

$U_2(x) = \frac{1}{1+q}(10+4(6+q)x+24x^2+16x^3),$

$U_3(x) = \frac{53+136x+200x^2+16q^2x^2+160x^3+80x^4+q(1+2x)^2\left(13+4x+4x^2\right)}{(1+q)\left(1+q+q^2\right)}$

$\vdots$

Substituting all $U_k(x)$ in (5) we obtain the series solution as

$u(x,t) = 1+2x+(3+4x+4x^2)t+(\frac{1}{1+q}(10+4(6+q)x+24x^2+16x^3))t^2+\cdots.$

For $q \to 1$, this result is the same as the result obtained in Refs. [33, 34].

Comment. Examples (4.1) and (4.2) have been solved by El-Shahed and Gaber [18] using the two-dimensional q-differential transform method. The method presented here is easier compared to two-dimensional q-differential transform method with relatively less computations, and gives answers with the same accuracy for example (4.1), however for the example (4.2) on account of different result in theorem 3.7, different results were obtained.

Example 4.3 *In this example, we consider the following partial q-differential equation [17, 21]*

$$\frac{\partial_q}{\partial_q t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + \frac{\partial}{\partial x}\left(xu(x,t)\right) \tag{15}$$

with the initial condition

$$u(x,0) = x^2, \tag{16}$$

By using RqDTM on this problem we have

$$[k+1]_q U_{k+1}(x) = \frac{\partial^2}{\partial x^2} U_k(x) + \frac{\partial}{\partial x}(xU_k(x)) \tag{17}$$

where

$$U_0(x) = x^2. \tag{18}$$

Solving (17) with the initial condition (18) we successively achieve the values $U_k(x)$ as follows:

$$U_1(x) = \frac{2+3x^2}{[1]_q},$$
$$U_2(x) = \frac{8+9x^2}{[1]_q[2]_q},$$
$$U_3(x) = \frac{26+27x^2}{[1]_q[2]_q[3]_q},$$
$$\vdots$$

Substituting all $U_k(x)$ in (5) we obtain the series solution as

$$\begin{aligned} u(x,t) &= x^2 + (2+3x^2)\frac{t}{[1]_q!} + (8+9x^2)\frac{t^2}{[2]_q!} + \cdots + (3^n - 1 + 3^n x^2)\frac{t^n}{[n]_q!} + \cdots \\ &= \sum_{n=0}^{\infty} (3^n - 1 + 3^n x^2)\frac{t^n}{[n]_q!}. \end{aligned}$$

This example has been solved using HPM and VIM as well [17, 21].

5. CONCLUSION

In this paper, we introduced RqDTM and proposed closed form series solution of partial q-differential equations. The results of the test examples show that the RqDTM results for two-dimensional q-differential equations are identical to the results obtained through HPM and VIM. It is worth noting that our findings in comparison with previous works on this subject are highly compatible and involve less complicated computations.

Mathematica has been used for computations and programming in this paper.

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