Dedicated to Professor Apolodor Aristotel Răduţă's 70^{th} *Anniversary*

ON BOUNDS FOR REAL ROOTS OF POLYNOMIALS

DORU ŞTEFĂNESCU

Faculty of Physics, University of Bucharest, P.O.Box MG-11, Bucharest–Măgurele, România
E-mail: doru.stefanescu@fizica.unibuc.ro

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We give a device for computing bounds for positive roots of polynomials. Our results allow the computation of absolute values for real and complex roots.

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1. INTRODUCTION

The computation of the real roots of univariate polynomials with real coefficients is done using several algorithmic devices. Many of them are based on the isolation of the real roots, *i.e.* the computation of a finite number of intervals with the property that each of them contains exactly one root. For that one of the steps is that of computing bounds for the roots. This can be realized using classical bounds for the absolute values of complex roots, see [9]. However there exist bounds specific to real roots. We obtain a device for computing absolute positiveness bounds for the real roots. The method is based on an improvement of the results of D. Ştefănescu [12] on upper bounds for positive roots. It it useful for the isolation of these roots, *i.e.* for the computation of a finite number of intervals such that each interval contains exactly one root [1]. These methods allows us to compute an interval containing all the positive roots of the derivatives of the polynomial. Analytical properties of polynomials and the algorithmic methods for the computation of their roots are relevant for the study of many physical problems, see, for example [7], [10], [11].

2. BOUNDS FOR POSITIVE ROOTS

We construct new bounds for positive roots of polynomials and give a method for obtaining bounds of absolute positiveness. We first compute an upper bound for the for the unique positive root of a special polynomial.

Lemma 1 *Let*

$$P(X) = X^d + X^{d-1} + \cdots + X^{e+1} - \gamma X^e + b_{e-1}X^{e-1} + \cdots + b_1 X + b_0,$$

with the coefficients γ and b_j strictly positive. Then $\gamma^{1/(d-e)}$ is an upper bound for the positive roots of the polynomials $P^{(i)}$, for all $i \in \mathbb{N}$.

Proof: We observe that if $\beta > \gamma^{1/(d-e)}$ we have $\beta^{d-e} > \gamma$, therefore $\beta^{d-e} - \gamma > 0$. Therefore

$$P(\beta) = \beta^e(\beta^{d-e} - \gamma) + \beta^{d-1} + \cdots + \beta^{e+1} + b_{e-1}\beta^{e-1} + \cdots + b_d > 0,$$

On the other hand, if we consider the polynomial

$$P'(X) = dX^{d-1} + \cdots + (e+1)X^e - \gamma e X^{e-1} + (e-1)b_{e-1}X^{e-2} + \cdots + b_1$$

we observe that

$$d\beta^{d-1} - \gamma e\beta^{e-1} = d\beta^{e-1}(\beta^{d-e} - \frac{e}{d}\gamma) > d\beta^{e-1}(\beta^{d-e} - \gamma) > 0.$$

It follows that $P'(\beta) > 0$. ∎

Proposition 2 *Let $d > e \geq 0$ be integers and let's consider the polynomial*

$$R(X) = \sum_{k=e+1}^{d} \binom{k}{e} a_k X^{k-e} - \gamma, \quad \textit{where} \quad a_d = 1, \gamma > 0 \quad \textit{for all} \quad a_i \geq 0.$$

The unique positive root α of the polynomial R satisfies the inequality

$$\alpha > \left(\frac{1+\gamma}{M}\right)^{1/d} - 1,$$

with $M = \max\{a_d, a_{d-1}, \ldots, a_{e+1}\}$.

Proof: Since the sequence of the coefficients of R has exactly one change of sign, by the rule of Descartes R has a unique positive root. We put $a_e = 1$ and we have

$$\sum_{k=e+1}^{d} \binom{k}{e} a_k X^{k-e} - \gamma = \sum_{k=e}^{d} \binom{k}{e} a_k X^{k-e} - 1 - \gamma$$
$$= \sum_{j=0}^{d-e} \binom{e+j}{e} a_{j+e} X^j - 1 - \gamma.$$

On the other hand $\quad \binom{e+j}{j} \leq \binom{d}{j} \quad$ for all $\quad j = 0, \ldots, d-e,$

so we obtain

$$\sum_{k=e}^{d} \binom{k}{e} a_k \alpha^{k-e} = \sum_{j=0}^{d-e} \binom{e+j}{e} a_{j+e}\alpha^j \leq \sum_{j=0}^{d-e} \binom{d}{j} a_{j+e}\alpha^j \leq M \sum_{j=0}^{d-e} \binom{d}{j} \alpha^j.$$

Because

$$\sum_{j=0}^{d-e} \binom{d}{j} \alpha^j \leq \sum_{j=0}^{d-1} \binom{d}{j} \alpha^j = (1+\alpha)^d - \alpha^d,$$

it follows that

$$\begin{aligned} 0 = R(\alpha) &= -1-\gamma+\sum_{k=e}^{d}\binom{k}{e}a_k\alpha^{k-e} \\ &\leq -1-\gamma+M\left((1+\alpha)^d-\alpha^d\right) \\ &< -1-\gamma+M(1+\alpha)^d, \end{aligned}$$

hence the conclusion. ∎

Corollary 3 *If $d > e \geq 0$ are integers and $\gamma > 0$, $a_d = 1$, $a_i \geq 0$, with $e < i < d$, the unique positive root of the polynomial*

$$R(X) = \sum_{k=e+1}^{d}\binom{k}{e}a_k X^{k-e} - \gamma$$

lies in the interval $\left(\left(\frac{1+\gamma}{M}\right)^{1/d} - 1, \gamma^{1/(d-e)}\right)$.

Proof: It is sufficient to observe that $R = P^{(e)}$. ∎

3. ABSOLUTE POSITIVENESS

We remind that a number $B > 0$ is an *absolute positiveness bound* of the univariate polynomial $P \in \mathbb{R}[X]$ if, for any $t \in \mathbb{N}$, we have

$$P^{(t)}(x) > 0 \quad \text{for all} \quad x \geq B.$$

That means that B is an upper bound for the positive roots of P and for the positive roots of all its derivatives. As H. Hoon have noticed [3], the bounds for complex roots of univariate polynomials over the reals are also bounds for absolute positiveness, thanks to the theorem of Guaß–Lucas (s. M. Marden [8]). In fact, if P is univariate with real coefficients, the convex hull of its zeros contains also the zeros of its derivative P'. Its trace on the real line contains the real zeros of P and also all zeros of P'. However, there exist bounds for positive roots which are not absolute, see [13]. We use a new bound for positive roots of polynomials with real coefficients:

Theorem 4 *Let*

$$P(X) = a_1X^{d_1} + a_2X^{d_2} + \cdots + a_sX^{d_s} - b_1X^{e_1} - b_2X^{e_2} - \cdots - b_tX^{e_t} \in \mathbb{R}[X],$$

where $a_i > 0$, $b_j > 0$, $d_1 = \deg(P)$ and $d_1 > d_2 > \cdots > d_s$. An upper bound for the positive roots of P is given by

$$\max_{\substack{1\leq i\leq s, 1\leq j\leq t \\ d_i \geq e_j}} \left(\frac{b_j}{\beta_j a_i}\right)^{\frac{1}{d_i - e_j}}$$

for any $\beta_j > 0$ *such that*

$$\beta_1 + \cdots + \beta_t \leq 1.$$

Proof: We suppose $x \in \mathbb{R}$, $x > 0$. We have

$$\begin{aligned}
P(x) &= \sum_{i=1}^{s} a_i x^{d_i} - \sum_{j=1}^{t} b_j x^{e_j} \\
&\geq \sum_{i=1}^{s} (\beta_1 + \cdots + \beta_t) a_i x^{d_i} - \sum_{j=1}^{t} b_j x^{e_j} \\
&\sum_{i=1}^{s} \left(\sum_{j=1}^{t} \beta_j a_i x^{d_i} \right) - \sum_{j=1}^{t} b_j x^{e_j} \\
&= \sum_{j=1}^{t} \left(\left(\sum_{i=1}^{s} \beta_j a_i x^{d_i} \right) - b_j x^{e_j} \right) \\
&\geq \sum_{j=1}^{t} \left(\left(\sum_{\substack{i=1 \\ d_i \geq e_j}}^{s} \beta_j a_i x^{d_i} \right) - b_j x^{e_j} \right) \\
&= \sum_{j=1}^{t} \left(\left(\sum_{\substack{i=1 \\ d_i \geq e_j}}^{s} \beta_j a_i x^{d_i - e_j} \right) - b_j \right) x^{e_j}.
\end{aligned}$$

The last sum is positive if $\beta_j a_i x^{d_i - e_j} - b_j > 0$ for all i, j such that $d_i \geq e_j$, *i.e.* if

$$x > \left(\frac{b_j}{\beta_j a_i} \right)^{\frac{1}{d_i - e_j}},$$

which proves the result. ∎

Corollary 5 (D. Ştefănescu [12]) *Let*

$$P(X) = X^d - b_1 X^{d-m_1} - \cdots - b_t X^{d-m_t} + g(X),$$

with $b_1, \ldots, b_k > 0$ *and* $g(X) \in \mathbb{R}_+[X]$.

The number

$$S_1(P) = \max\{(tb_1)^{1/m_1}, \ldots, (tb_t)^{1/m_t}\}$$

is an upper bound for the positive roots of P.

Proof: Let $Q(X) = X^d - b_1 X^{d-m_1} - \cdots - b_t X^{d-m_t}$. We observe that a bound for the positive roots of g is also a bound for the positive roots of P. With the notation from Theorem 4 we have $s=1$ and $a_1=1$. We consider $\beta_1=\ldots\beta_t=\frac{1}{t}$ and it follows that the number

$$S_1(P)=\max\{(tb_1)^{1/m_1},\ldots,(tb_t)^{1/m_t}\}$$

is an upper bound for the positive roots of Q. But any upper bound for the positive roots of the polynomial Q is also an upper bound for the positive roots of P. ∎

Corollary 6 (J. B. Kioustelidis [4]) *Let*

$$P(X)=X^d-b_1X^{d-m_1}-\cdots-b_tX^{d-m_t}+g(X),$$

where $b_1,\ldots,b_k>0$ *and* $g\in\mathbb{R}_+[X]$.

The number

$$K(P)=2\cdot\max\{b_1^{1/m_1},\ldots,b_k^{1/m_k}\}$$

is an upper bound for the positive roots of P.

Proof: As in the proof of the previous result, it is sufficient to check that K is an upper bound for the positive roots of the polynomial $Q(X)=X^d-b_1X^{d-m_1}-\cdots-b_kX^{d-m_k}$ and we consider

$$\beta_i=\left(\frac{1}{2}\right)^{d-m_i}\qquad\text{for all}\quad i.$$

Without loss of generality we may suppose that

$$m_1<\cdots<m_t$$

and we have

$$\begin{aligned}\beta_1+\cdots+\beta_t&=\left(\frac{1}{2}\right)^{d-m_1}+\cdots+\left(\frac{1}{2}\right)^{d-e_t}\\&=\frac{1}{2^{m_1}}\sum_{m=1}^{t}\left(1+\frac{1}{2^{m_2-m_1}}+\cdots+\frac{1}{2^{m_t-m1}}\right)\\&\leq\frac{1}{2^{m_1}}\sum_{m=1}^{t}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{t-1}}\right)\\&\leq\frac{1}{2^{m_1}}\cdot\frac{1-\frac{1}{2^t}}{1-\frac{1}{2}}<\frac{1}{2^{m_1-1}}\leq 1.\end{aligned}$$

Therefore $K(P)$ is an upper bound for the positive roots of Q, then also for those of Q. ∎

Notation: Let $P \in \mathbb{R}[X] \setminus \mathbb{R}$ be such that it has an even number of sign variations and can be represented as

$$P(X) = c_1X^{d_1} - b_1X^{m_1} + c_2X^{d_2} - b_2X^{m_2} + \cdots + c_kX^{d_k} - b_kX^{m_k} + g(X),$$

with $g \in \mathbb{R}_+$, $d_1 > d_2, \ldots, d_k$, $c_i > 0$, $b_i > 0$ and $d_i > m_i$ for all i.
We put

$$S_2(P) = \max\left\{ \left(\frac{b_1}{c_1}\right)^{1/(d_1-m_1)}, \ldots, \left(\frac{b_k}{c_k}\right)^{1/(d_k-m_k)} \right\}$$

D. Ştefănescu proved in [13] that the bounds $S_1(P)$, $S_2(P)$ and $K(P)$ are absolute.

BOUNDS FOR ORTHOGONAL POLYNOMIALS

We consider now classical orthogonal polynomials. They are hyperbolic, *i.e.* they have only real roots. Because of the interlacing of the roots with those of the derivative a bound for the positive roots of an orthogonal polynomial is also a positiveness bound. We obtain new bounds using the Hessian of Laguerre.

If we consider $f(X) = \sum_{j=1}^{n} a_jX^j$, a univariate polynomial with real coefficients, its *Hessian* is $\mathrm{Hess}(f) = (n-1)^2f'^2 - n(n-1)ff'$. Laguerre [6] proved that it is positive. This gives

Theorem 7 (E. Laguerre) *Let $f \in \mathbb{R}[X]$ be a polynomial of degree $n \geq 2$, that has only real simple roots and that satisfies the second–order differential equation*

$$p(x)y'' + q(x)y' + r(x)y = 0, \tag{1}$$

with p, q and r univariate polynomials with real coefficients, $p(x) \neq 0$.

If α is a root of the polynomial f, then we have

$$4(n-1)\Big(p(\alpha)r(\alpha) + p(\alpha)q'(\alpha) - p'(\alpha)q(\alpha)\Big) - (n+2)q(\alpha)^2 \geq 0. \tag{2}$$

Using also the positivity of the Hessian we obtain another inequality:

Theorem 8 *Let $f \in \mathbb{R}[X]$ be a polynomial of degree $n \geq 2$ that satisfies the second–order differential equation*

$$p(x)y'' + q(x)y' + r(x)y = 0, \tag{3}$$

with p, q and r real real polynomials, $p(x) \neq 0$.

Let us assume that all the roots of f are real and simple. If α is a root of f, then we have $(n-3)q_2(\alpha)^2 - (n-2)q(\alpha)q_3(\alpha) \geq 0$, where

$$q_2 = q^2 + p'q - pq' - pr,$$
$$q_3 = \left(2p' + q\right)\left(-q^2 - p'q + pq' - pr\right) - pq\left(p'' + 2q' + r\right) - p^2\left(q'' + 2r'\right).$$

Proof: We differentiate twice the relation

$$g'(\alpha) = -\frac{q(\alpha)}{2p(\alpha)} \cdot g(\alpha). \tag{4}$$

and obtain

$$p(x)y''' + \big(p'(x) + q(x)\big)\, y'' + \big(q'(x) + r(x)\big)\, y' + r'(x)y = 0,$$

respectively

$$p(x)y^{(iv)} + \big(2p'(x) + q(x)\big)\, y''' + \big(p''(x) + 2q'(x) + r(x)\big)\, y'' \\ + \big(q''(x) + 2r'(x)\big)\, y' + r''(x)y = 0.$$

Therefore

$$g''(\alpha) = \frac{q_2(\alpha)}{3p(\alpha)^2} \cdot g(\alpha). \tag{5}$$

with $q_2 = q^2 + p'q - pq' - pr$ and

$$g'''(\alpha) = \frac{q_3(\alpha)}{4p(\alpha)^3} \cdot g(\alpha), \tag{6}$$

where $q_3 = (2p' + q)\left(-q^2 - p'q + pq' - pr\right) - pq\,(p'' + 2q' + r) - p^2\,(q'' + 2r')$. The Hessian of g' is $(n-3)^2 g'' - (n-2)(n-3)g'g'''$ and it is positive. So

$$(n-3)q_2(\alpha)^2 - (n-2)q(\alpha)q_3(\alpha) \geq 0. \tag{7}$$

∎ □

Example 1. The Legendre polynomial of degree n satisfies the differential equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0.$$

Using the Hessian (1) of Laguerre we obtain the following upper bound for the roots $La(n) = (n-1)\sqrt{\dfrac{n+2}{n(n^2+2)}}$.

Example 2. The Hermite polynomial H_n, which satisfies the differential equation $y'' - 2xy' + 2ny = 0$ and by (1) it follows that $He(n) = (n-1)\sqrt{\dfrac{2}{n+2}}$ is a bound for the roots of H_n.

On the other hand, by Theorem 8 we have the bound

$$\sqrt{\frac{2n^2 + n + 6 + \sqrt{(2n^2 + n + 6 + 32(n+6)(n^3 - 5n^2 + 7n - 3)}}{4(n+6)}}.$$

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